Alexandria University
Faculty of Engineering
Electrical Engineering Department

Midterm Exam, 2015-2016
Time: 45 Minutes
April 2nd, 2016

Control Systems and Their Components (EE 391)

Name MODEL ANSWER **Seat Number**

1. For the following single-input single-output LTI system whose State Space (SS) representation is given by

$\boxed{\text{8 Marks}}$

$$\dot{\mathbf{x}}(t)=\underbrace{\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}}_{A}\mathbf{x}(t)+\underbrace{\begin{bmatrix}3\\0\end{bmatrix}}_{B}u(t),$$

$$y(t)=\underbrace{\begin{bmatrix}2 & 3\end{bmatrix}}_{C}\mathbf{x}(t) \qquad D=0$$

2

a. Find the output $y(t)$ for a unit step input, $u(t)=\begin{cases}1, & t\ge 0\\ 0, & t<0\end{cases}$, and initial state vector $\mathbf{x}(0)=\begin{bmatrix}1 & -1\end{bmatrix}^T$. Identify the zero-input and zero-state responses. Find the steady state value of $y(t)$.

$$x(t)=e^{At}x(0)+\underbrace{e^{At} * Bu(t)}_{\int_0 \phi(t-\tau)Bu(\tau)\,d\tau}$$

$$e^{At}=\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{bmatrix}$$

$$\therefore x(t)=\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}+\int^{t}\begin{bmatrix}e^{-(t-\tau)} & 0\\ 0 & e^{-2(t-\tau)}\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix}\cdot \underset{u(\tau)}{1}\,d\tau$$

$$=\begin{bmatrix}e^{-t}\\ -e^{-2t}\end{bmatrix}+\int_0^{t}\begin{bmatrix}3e^{-(t-\tau)}\\ 0\end{bmatrix}d\tau$$

$$e^{-t}\cdot\int_0^t e^{\tau}d\tau = e^{-t}(e^{t}-1)=1-e^{-t}$$

$$=\underbrace{\begin{bmatrix}e^{-t}\\ -e^{-2t}\end{bmatrix}}_{\text{Zero i/p resp.}}+\underbrace{\begin{bmatrix}3(1-e^{-t})\\ 0\end{bmatrix}}_{\text{Zero state resp}}=\begin{bmatrix}3-2e^{-t}\\ -e^{-2t}\end{bmatrix}$$

$$y(t)=Cx(t)+\cancel{D}u$$

$$=\begin{bmatrix}2 & 3\end{bmatrix}\begin{bmatrix}3-2e^{-t}\\ -e^{-2t}\end{bmatrix}$$

$$\boxed{y(t)=6-4e^{-t}-3e^{-2t}}$$

At SS

$$\boxed{y(\infty)=6}$$

b. Find the system's transfer function $Y(s)/U(s)$. Assume zero initial state vector in obtaining your result. 2

$$H(s)=\frac{Y(s)}{U(s)}=C(sI-A)^{-1}B+\cancel{D}=\begin{bmatrix}2 & 3\end{bmatrix}\begin{bmatrix}\frac{1}{s+1} & 0\\ 0 & \frac{1}{s+2}\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 3\end{bmatrix}\begin{bmatrix}\frac{3}{s+1}\\ 0\end{bmatrix}=\boxed{\frac{6}{s+1}}$$

(1)

c. Is this system stable? Explain why. yes pole at $s=-1<0$

(1)

d. Use the final value theorem to find the Steady state value of the output for a unit step input How does it compare to the steady state value obtained in (a)?

$$\boxed{\mathcal{L}\{u(t)\}=\frac{1}{s}}$$

$$Y(s)=H(s)U(s)=\frac{6}{s(s+1)}$$

$$y(\infty)=\lim_{s\to 0} sY(s)=\underline{\underline{6}}$$ same as (a)

(2)

e. Find the eigenvalues of the matrix A of the system. How do they compare to the poles found in (b)? Explain your answer.

$\lambda(A)=-1$ & -2

-1 ↓ same as pole in (c)

-2 ↓ got cancelled by a zero at $s=-2$

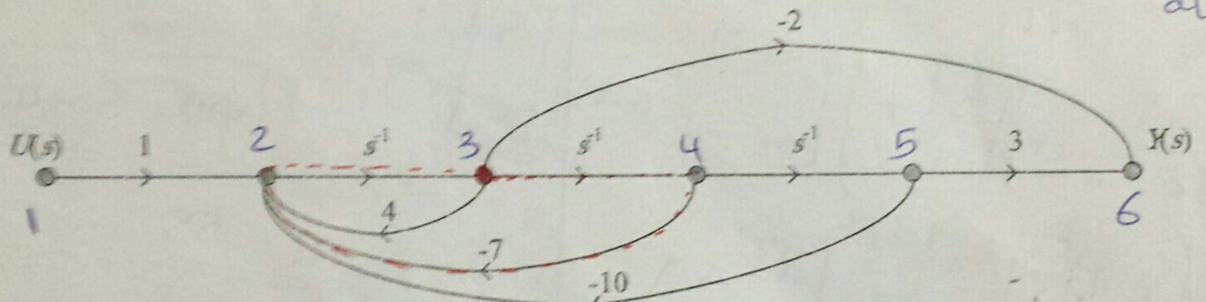
$$\begin{vmatrix} zI-A & -B \\ C & D \end{vmatrix} = \begin{vmatrix} z+1 & 0 & -3 \\ 0 & z+2 & 0 \\ 2 & 3 & 0 \end{vmatrix}$$

$$= -3 * -2\,(z+2) = 6(z+2)$$

$\boxed{z=-2}$ ⇒ Zero of system that gets cancelled with pole at $s=-2$

6 Marks

2. For the signal flow graph (SFG) below:



(3)

a. Find the transfer function $\frac{Y(s)}{U(s)}$ using Mason's gain formula.

Forward paths

$123456 \to G_1 = 3s^{-3}, \Delta_1 = 1$

$1236 \to G_2 = -2s^{-1}, \Delta_2 = 1$

Loops

$232 \to l_1 = 4s^{-1}$

$2342 \to l_2 = -7s^{-2}$

$23452 \to l_3 = -10s^{-3}$

$$\boxed{\Delta = 1-4s^{-1}+7s^{-2}+10s^{-3}}$$

$$\frac{Y(s)}{U(s)} = \frac{3s^{-3}-2s^{-1}}{1-4s^{-1}+7s^{-2}+10s^{-3}}$$

$-2s^2+3$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{3-2s^2}{s^3-4s^2+7s+10}}$$

$e^{-t}+3-3e^{-t}$

$3-2e^{-t}$

b. From the transfer function you found in (a), evaluate the state space representation realized in the controller canonical form (CCF). You should make the state variable assignment as well as the matrices **A, B, C** and **D** clear in your answer.

CCF realization

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -7 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 0 & -2 \end{bmatrix}, \quad D = 0$$

6 Marks

3. Consider the following State equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

and assume an equivalent SS vector $\tilde{\mathbf{x}}$ is obtained from the original SS vector **x** via the similarity transformation **P** as $\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x}$, where **P** is nonsingular, to produce the equivalent State equation

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{u}(t)$$

$P\dot{x} = \tilde{A}Px + \tilde{B}u$

$\dot{x} = \underbrace{P^{-1}\tilde{A}P}\, x + P^{-1}\tilde{B}u$

$\boxed{A = P^{-1}\tilde{A}P}$

$\boxed{B = P^{-1}\tilde{B}}$

$\tilde{A} = PAP^{-1}$

3

a. Prove that **A** and $\tilde{\mathbf{A}}$ have the same eigenvalues.

$|\lambda I - A| = 0$

$|\tilde{\lambda} I - \tilde{A}| = 0$

$|\tilde{\lambda}\cancel{P}P^{-1} - PAP^{-1}| = 0$

$\cancel{|P|}\,|\tilde{\lambda} I - A|\,\cancel{|P^{-1}|} = 0$

$\boxed{|\tilde{\lambda} I - A| = 0}$

so $\tilde{\lambda}$ are also eigenvalues of A

No

b. Are the eigenvectors of **A** and $\tilde{\mathbf{A}}$ the same? If not, find the relationship between the eigenvectors of **A** and $\tilde{\mathbf{A}}$ in terms of the transformation **P**.

3

$Av = \lambda v$

$P^{-1}\tilde{A}Pv = \lambda v$

$\tilde{A}(Pv) = \lambda(Pv)$

$\therefore \boxed{\tilde{v} = Pv}$

eigenvector of $\tilde{A}$

eigenvector of A

Best of Luck

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$\lambda v = 0$