Control Systems And Their Components (EE391)

Lec. 9: Closed loop SS Control (Full state feedback)

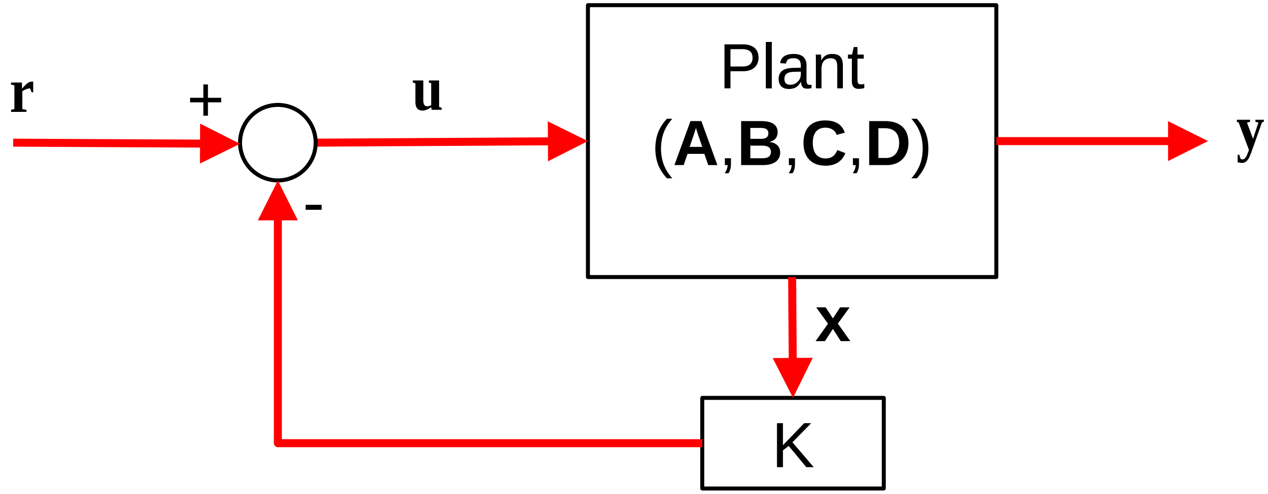
Thu. April 21st, 2016

Dr. Mohamed Hamdy Osman

Lecture Outline

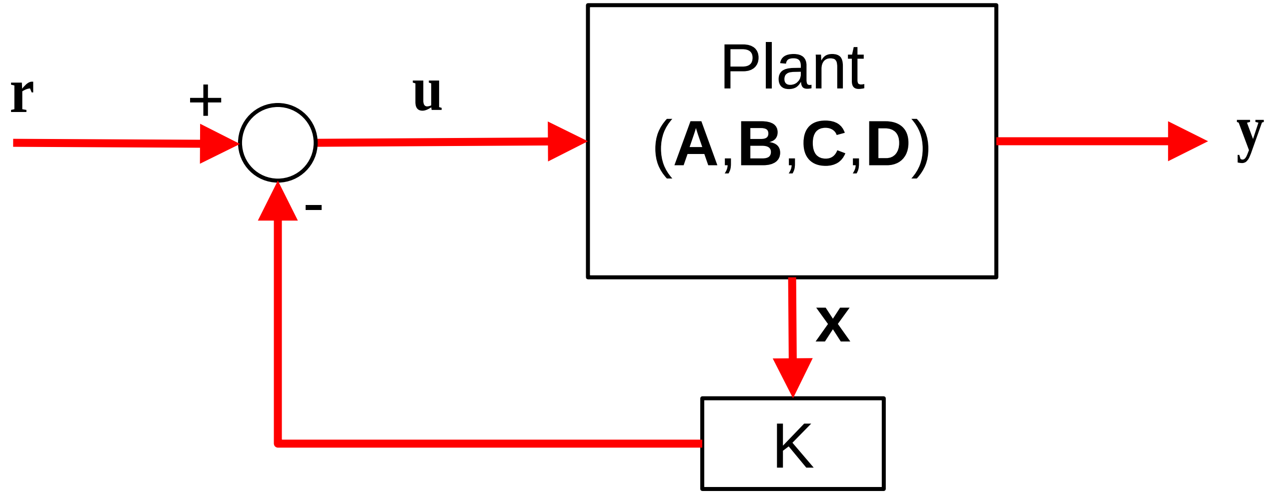
- Closed loop SS control (why feedback?)
- Full State feedback (Pole placement technique)
- Akermann’s formula
- Where to place the poles of closed loop system (Two MATLAB examples)

Closed loop SS control



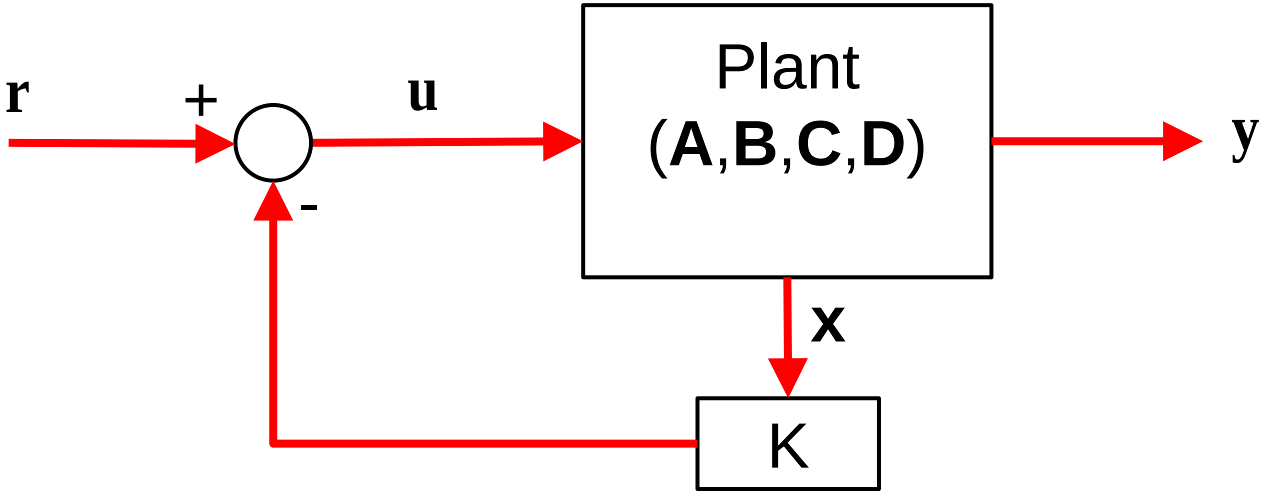
- Problems with open loop control
  - Impacted by variations of plant model (Not robust)
  - Impacted by external disturbance or noise
  - No control over the transient behavior of system (determined by eigenvalues of A → poles of TF)
- With closed loop SS control, we will control the transient behavior of system by modifying poles of TF or eigenvalues of A through **<u>pole placement technique</u>**
- Full state feedback assumes input is linear combination of state variables that are fed back → assumes **x** is fully measured

Closed loop SS control



- Full state feedback assumes input is linear combination of state variables that are fed back → assumes **x** is fully measured
- This means I have access over state variables of system
- This is not true!! since the state variables are internal and I only have access to the outputs which I can observe
- We will deal with drawback of full state feedback later by using **state estimators/observers**

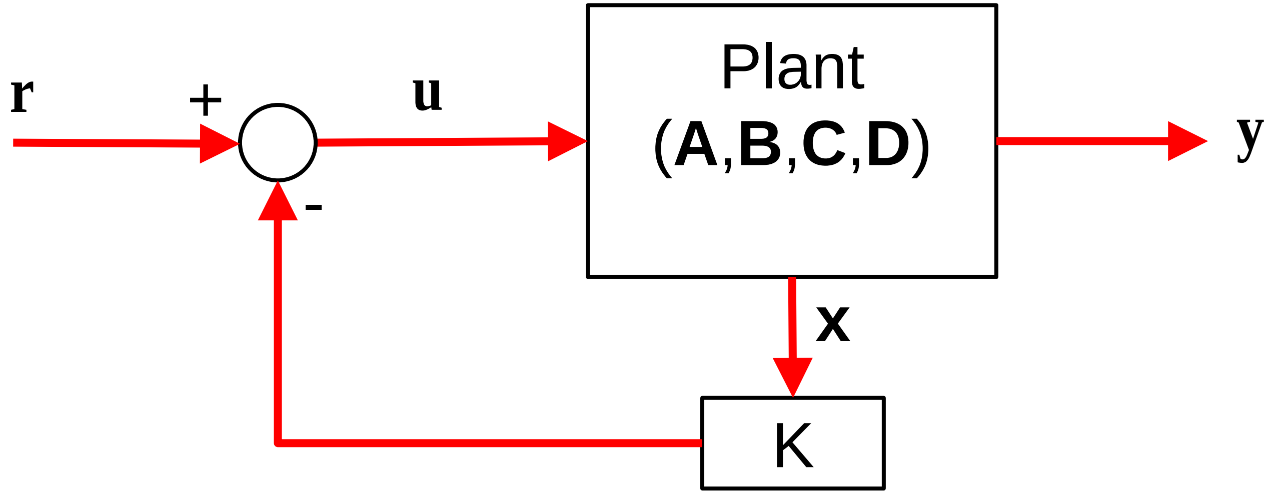
Closed loop SS control



Important Note

<u>In order to be able to perform full state feedback control, the system must be controllable</u> (makes sense but will show up mathematically later on)

Full state feedback



$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Assume full state feedback of the form

$$\mathbf{u}(t) = \mathbf{r}(t) - \mathbf{K}\mathbf{x}(t)$$

where **r** is a reference input and $\mathbf{K} \in \mathbf{R}^{1\times n}$ (assume a single input for simplicity)

Full state feedback

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t) + \mathbf{Br}(t)$$

Now let us also assume a zero reference input for the moment **r** = 0 (called regulator)

$$\dot{\mathbf{x}}(t) = \underbrace{(\mathbf{A} - \mathbf{BK})}_{\mathbf{A}_{cl}}\mathbf{x}(t)$$

This is the matrix **A** for the overall closed loop system with feedback

$$\mathbf{x}(t) = e^{\mathbf{A}_{cl}t}\mathbf{x}(0)$$

The dynamics of the closed loop system is determined by the eigenvalues of $\mathbf{A}_{cl}$ (same as before) and we can choose **K** to set them anywhere we want

Full state feedback

$$\left| s\mathbf{I} - \mathbf{A}_{cl} \right| = \left| s\mathbf{I} - (\mathbf{A} - \mathbf{BK}) \right| = \prod_{j=1}^{n} \left(s - s_j \right) = 0$$

Desired pole locations of closed loop system

- We start with examples where the desired pole locations are given
- We will then address the issue of how to choose the desired pole locations of closed loop system from required transient specifications given (rise time, maximum overshoot, …)

Full state feedback

Example

Find the feedback gains **K** of the following SS system such that the closed loop poles become -2+2i and -2-2i

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t)$$

Solution

$$\left| sI - \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right| = (s + 2 - 2i)(s + 2 + 2i)$$

$$\left\| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \right\| = s^2 + 4s + 8$$

Full state feedback

Example

Find the feedback gains **K** of the following SS system such that the closed loop poles become -2+2i and -2-2i

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t)$$

Solution

$$\left| \begin{bmatrix} s & -1 \\ 1 + k_1 & s + 1 + k_2 \end{bmatrix} \right| = s^2 + 4s + 8$$

$$s^2 + (1 + k_2)s + (1 + k_1) = s^2 + 4s + 8$$

$$1 + k_2 = 4 \quad \Rightarrow \quad k_2 = 3$$

$$1 + k_1 = 8 \quad \Rightarrow \quad k_1 = 7$$

MATLAB

K = place(A,B,desired poles)

Full state feedback

Example

Find the feedback gains **K** of the following SS system such that the closed loop poles become -1, -2, -4

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 2 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{u}(t)$$

Solution

Eigenvalues of **A** of open loop system (without feedback) are -5.467, 0.23+0.7i, 0.23-0.7i → The system is not stable

We will make the system stable by placing the poles at desired locations -1, -2, -4 using feedback

Full state feedback

Example

Find the feedback gains **K** of the following SS system such that the closed loop poles become -1, -2, -4

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 2 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{u}(t)$$

Solution

$$\left| sI - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \right| = (s+1)(s+2)(s+4)$$

$$\left\| \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 3+k_1 & -2+k_2 & s+5+k_3 \end{bmatrix} \right\| = s^3 + 7s^2 + 14s + 8$$

Full state feedback

Example

Find the feedback gains **K** of the following SS system such that the closed loop poles become -1, -2, -4

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 2 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{u}(t)$$

Solution

$$\left\| \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 3+k_1 & -2+k_2 & s+5+k_3 \end{bmatrix} \right\| = s^3 + 7s^2 + 14s + 8$$

$$s^3 + (5+k_3)s^2 + (k_2-2)s + (3+k_1) = s^3 + 7s^2 + 14s + 8$$

$$\Rightarrow \boxed{k_1 = 5, \quad k_2 = 16, \quad k_3 = 2}$$

Full state feedback

Note

Last two examples were easy because **A**,**B** were in controller canonical form. Let's see another example not in CCF and see how things will become a bit complicated

Example

Desired closed loop poles are -2+2i and -2-2i, Find **K**

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}(t)$$

Solution

$$\left| s\mathbf{I} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right| = (s+2-2i)(s+2+2i)$$

$$\left\| \begin{bmatrix} s-1+k_1 & -1+k_2 \\ -1 & s-2 \end{bmatrix} \right\| = s^2 + 4s + 8$$

Full state feedback

Example

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}(t)$$

Solution

$$s^2 + (k_1 - 3)s + (1 - 2k_1 + k_2) = s^2 + 4s + 8$$

$$\boxed{k_1 = 7, \quad k_2 = 21}$$

We cannot find *k*'s easily as before by setting each of them to obtain each of the coefficients in the desired polynomial like in CCF (This is one benefit of CCF)

In a system with larger number of state variables we will need to do some algebra to solve for *k*'s

Full state feedback

Note

We said that in order to be able to do full state feedback, the system must be controllable → Let's see with an example

Example

Desired closed loop poles are -2+2i and -2-2i, find **K**

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}(t)$$

Solution

$$\left| s\mathbf{I} - \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right| = (s + 2 - 2i)(s + 2 + 2i)$$

$$\left\| \begin{bmatrix} s - 1 + k_1 & -1 + k_2 \\ 0 & s - 2 \end{bmatrix} \right\| = s^2 + 4s + 8$$

Full state feedback

Example

Desired closed loop poles are -2+2i and -2-2i, find **K**

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}(t)$$

Solution

$$\left\| \begin{bmatrix} s-1+k_1 & -1+k_2 \\ 0 & s-2 \end{bmatrix} \right\| = s^2 + 4s + 8$$

$$(s-1+k_1)(s-2) = s^2 + 4s + 8$$

We can only control the eigenvalue (pole) at *s* = -1 by setting k_1 but we cannot control the eigenvalue (pole) at *s* = 2

Check controllability → you will find system **<u>uncontrollable</u>**

Akermann's formula

- It gives a formal way to obtain **K** in full state feedback control all in one step
- Without proof

$$\mathbf{K} = [0 \;\; 0 \;\; \cdots \;\; 1]\,\mathbf{C}_n^{-1}\,\Phi_{cl}(\mathbf{A})$$

Controllability matrix

Characteristic equation of closed loop system for $s = \mathbf{A}$

- Clearly $\mathbf{C}_n$ needs to be invertible, hence full rank, hence the system must be **<u>controllable</u>**

Akermann's formula

Example

Use Akermann's formula to find **K** of the following SS system → closed loop poles required are -2+2i and -2-2i

$$\dot{\mathbf{x}}(t)=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}0\\ 1\end{bmatrix}\mathbf{u}(t)$$

Solution

$$\mathbf{C}_n=\left[\mathbf{B}\quad \mathbf{AB}\right]=\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}\quad\Rightarrow\quad \mathbf{C}_n^{-1}=\frac{1}{-1}\begin{bmatrix}-1 & -1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$$

$$\mathbf{\Phi}_{cl}(s)=s^2+4s+8\quad\Rightarrow\quad \mathbf{\Phi}_{cl}(\mathbf{A})=\mathbf{A}^2+4\mathbf{A}+8=\begin{bmatrix}7 & 3\\ -3 & 4\end{bmatrix}$$

$$\mathbf{K}=\left[0\quad 1\right]\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}7 & 3\\ -3 & 4\end{bmatrix}=\left[0\quad 1\right]\begin{bmatrix}4 & 7\\ 7 & 3\end{bmatrix}=\left[7\quad 3\right]$$

Same as slide 10!!

Where to put the closed loop poles?

$$\left|s\mathbf{I}-\mathbf{A}_{cl}\right| = \left|s\mathbf{I}-(\mathbf{A}-\mathbf{BK})\right| = \prod_{j=1}^{n}\left(s-s_j\right) = 0$$

???

- Desired pole locations are found from required transient specifications (Max. overshoot, settling time, rise time, …)
- Closed form expressions for these specifications in terms of the pole locations are available for 2nd order system
- If our system is second order, we can use these equations directly to find the poles from the specifications
- If our system is larger, we still use the equations of a second order prototype system to find two poles, and place the remaining poles far away from the jw axis (at 5-10 times larger than real part of desired poles of 2nd order system) to make sure they do not impact the system transient

2^{nd} order prototype system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ω_n : natural undamped frequency, ζ : damping ratio

two poles are

$$\zeta\omega_n \pm i\,\omega_n\sqrt{1-\zeta^2}$$

Maximum overshoot $= e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$

2% settling time $\approx \frac{4}{\zeta\omega_n}$, 5% settling time $\approx \frac{3.2}{\zeta\omega_n}$

MATLAB example 1 (2nd order system)

$$G(s) = \frac{8}{(s-5)(s-10)}$$

$$\text{Maximum overshoot} = 5\%$$

$$2\% \text{ settling time} = 4 \text{ sec}$$

- Find the desired two poles
- Find K that achieves so
- Make sure closed loop system satisfy the specifications

MATLAB example 2 (3^{rd} order system)

$$G(s) = \frac{8}{(s-5)(s-10)(s-7)}$$

$$\text{Maximum overshoot} = 5\%$$

$$2\% \text{ settling time} = 4 \text{ sec}$$

- Find the desired two poles
- Place the remaining pole far away
- Find K that achieves so
- Make sure closed loop system satisfy the specifications