Control Systems And Their Components (EE391)

Lec. 7: Discrete SS model, Controllability and Observability

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Lecture Outline

- Discrete SS model
- Solution of discrete SS model (Time evolution)
- Controllability and observability concepts
- Mathematical conditions of controllability and observability

Discrete SS model

- Importance is because most analog systems are controlled via digital controller
- We will find discrete SS model from the already derived continuous time SS model (Differential Eqs → Difference Eqs)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

State Transition Matrix

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t}$$

Discrete SS model

$$\mathbf{x}(t)=\mathbf{\Phi}(t)\mathbf{x}(0)+\int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

put $t=t_0$

$$\mathbf{x}(t_0)=\mathbf{\Phi}(t_0)\mathbf{x}(0)+\int_0^{t_0} \mathbf{\Phi}(t_0-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

Multiply by $\mathbf{\Phi}^{-1}(t_0)=\mathbf{\Phi}(-t_0)$ and solve for $\mathbf{x}(0)$

$$\mathbf{x}(0)=\mathbf{\Phi}(-t_0)\mathbf{x}(t_0)-\mathbf{\Phi}(-t_0)\int_0^{t_0} \mathbf{\Phi}(t_0-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{x}(0)=\mathbf{\Phi}(-t_0)\mathbf{x}(t_0)-\int_0^{t_0} \mathbf{\Phi}(-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

Back substitute $\mathbf{x}(0)$

Discrete SS model

$$\mathbf{x}(t) = \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0) - \int_0^{t_0} \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau + \int_0^{t} \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$= \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

Now set $t_0 = kT$ and $t = t_0 + T$ where T is the sample duration (discrete) to see transition of state variables from one sample to the next

$$\mathbf{x}(kT + T) = \mathbf{\Phi}(T)\mathbf{x}(kT) + \int_{kT}^{kT+T} \mathbf{\Phi}(kT + T - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

Now assume input **u** is held constant from kT to $kT+T$
which is called zero order hold (ZOH) $\rightarrow$ $\mathbf{u}(kT+T) = \mathbf{u}(kT) = \mathbf{u}(k)$
and use discrete time index by omitting T

Discrete SS model

$$\mathbf{x}(k+1) = \mathbf{\Phi}(T)\mathbf{x}(k) + \left[\int_{kT}^{kT+T} \mathbf{\Phi}(kT+T-\tau)d\tau\right]\mathbf{Bu}(k)$$

$$= \mathbf{\Phi}(T)\mathbf{x}(k) + \left[\int_{0}^{T} \mathbf{\Phi}(v)dv\right]\mathbf{Bu}(k)$$

$$= \mathbf{A}_d\,\mathbf{x}(k) + \mathbf{B}_d\,\mathbf{u}(k)$$

where

$$\mathbf{A}_d = \mathbf{\Phi}(T) = e^{\mathbf{A}T}$$

$$\mathbf{B}_d = \left[\int_{0}^{T} \mathbf{\Phi}(v)dv\right]\mathbf{B}$$

For small T →

$$\mathbf{A}_d \simeq \mathbf{I} + \mathbf{A}T$$

$$\mathbf{B}_d \simeq T\mathbf{B}$$

Discrete SS model

$n\times1 \quad n\times1 \quad p\times1$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$n\times n \quad n\times p$

$m\times1 \quad n\times1 \quad p\times1$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$m\times n \quad m\times p$

Continuous

$n\times1 \quad n\times1 \quad p\times1$

$$\mathbf{x}(k+1) = \mathbf{A}_d\,\mathbf{x}(k) + \mathbf{B}_d\,\mathbf{u}(k)$$

$n\times n \quad n\times p$

$m\times1 \quad n\times1 \quad p\times1$

$$\mathbf{y}(k) = \mathbf{C}_d\,\mathbf{x}(k) + \mathbf{D}_d\,\mathbf{u}(k)$$

$m\times n \quad m\times p$

Discrete

MATLAB

$[A_d\ B_d\ C_d\ D_d]$ = c2dm(A,B,C,D,*T*,'zoh')

$$\mathbf{A}_d = e^{\mathbf{A}T} \simeq \mathbf{I} + \mathbf{A}T$$

$$\mathbf{B}_d = \left[\int_0^T e^{\mathbf{A}v}\,dv\right]\mathbf{B} \simeq T\mathbf{B}$$

$$\mathbf{C}_d = \mathbf{C}$$

$$\mathbf{D}_d = \mathbf{D}$$

Discrete SS model

MATLAB problem

- Find the continuous time SS model of the following TF on MATLAB

$$G(s) = \frac{2s^2 + 8s + 6}{s^3 + 8s^2 + 16s + 6}$$

- Find discrete time equivalent SS model of the above system

[A B C D] = tf2ss([2 8 6],[1 8 16 6])
[A_d B_d C_d D_d] = c2dm(A,B,C,D,1,'zoh')

Solution of discrete SS model

$$\underset{n\times 1}{\mathbf{x}(k+1)} = \underset{n\times n}{\mathbf{A}_d}\,\underset{n\times 1}{\mathbf{x}(k)} + \underset{n\times p}{\mathbf{B}_d}\,\underset{p\times 1}{\mathbf{u}(k)}$$

$$\underset{m\times 1}{\mathbf{y}(k)} = \mathbf{C}_d\,\underset{n\times 1}{\mathbf{x}(k)} + \mathbf{D}_d\,\underset{p\times 1}{\mathbf{u}(k)}$$

- We are going to call them A, B, C, D for simplicity but it is implicitly known that they are the discrete equivalent of the cont. time A,B,C,D
- We need to solve this difference equation to obtain x(k) in terms of **x**(0) and **u**
- We can use Z transform similar to what we did with Laplace transform in the continuous case, or

Solution of discrete SS model

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

$$\mathbf{x}(1) = \mathbf{A}\mathbf{x}(0) + \mathbf{B}\mathbf{u}(0)$$

$$\mathbf{x}(2) = \mathbf{A}\mathbf{x}(1) + \mathbf{B}\mathbf{u}(1) = \mathbf{A}\left[\mathbf{A}\mathbf{x}(0) + \mathbf{B}\mathbf{u}(0)\right] + \mathbf{B}\mathbf{u}(1) = \mathbf{A}^2\mathbf{x}(0) + \mathbf{A}\mathbf{B}\mathbf{u}(0) + \mathbf{B}\mathbf{u}(1)$$

$$\mathbf{x}(3) = \mathbf{A}\mathbf{x}(2) + \mathbf{B}\mathbf{u}(2) = \mathbf{A}\left[\mathbf{A}^2\mathbf{x}(0) + \mathbf{A}\mathbf{B}\mathbf{u}(0) + \mathbf{B}\mathbf{u}(1)\right] + \mathbf{B}\mathbf{u}(2)$$

$$= \mathbf{A}^3\mathbf{x}(0) + \mathbf{A}^2\mathbf{B}\mathbf{u}(0) + \mathbf{A}\mathbf{B}\mathbf{u}(1) + \mathbf{B}\mathbf{u}(2)$$

$$\mathbf{x}(k) = \mathbf{A}^k\mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j}\mathbf{B}\mathbf{u}(j)$$

Similar to state Transition Matrix in discrete case

Solution of discrete SS model

Problem

- Try to obtain the same formula using Z transform

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

$$\Downarrow$$

$$\mathbf{x}(k) = \mathbf{A}^{k}\mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j}\mathbf{B}\mathbf{u}(j)$$

Problem

- Prove that the TF in discrete case is (similar to continuous)

$$\mathbf{Y}(z) = \left[\mathbf{C}\left(z\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(z)$$

Controllability concept

Illustrative Example

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} \mathbf{x}(t)$$

- Eigenvalues of **A** are -1, -2 (poles of the system)
- Let's find the TF

$$TF = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$= \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$= \frac{6}{s+1}$$

Where is the other pole at -2 ?

Controllability concept

Illustrative Example

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} \mathbf{x}(t)$$

- Mathematically, the other pole at -2 got canceled because of the 0 in the vector **B** together with **A** being diagonal which basically means that the dynamics of the second state <u>cannot be controlled</u> by the input (the input has not control over x_2) or we say x_2 is not controllable
- If you dig deep, you can discover what happened to the eigenvalue at -2 and why it disappeared in TF
- It is because the system has a zero also at -2 that got canceled with the pole at -2 (How can you check zeros??)

$$\begin{vmatrix} z_0\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = 0$$

Controllability definition

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

$$\mathbf{x}(k) = \mathbf{A}^{k}\mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j}\mathbf{B}\mathbf{u}(j)$$

<u>Assume zero initial state vector, **x**(0) = 0 (will get back to remove this assumption later)</u>

$$\underset{n\times 1}{\mathbf{x}(k)} = \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j}\mathbf{B}\mathbf{u}(j)$$

$$= \underbrace{\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{k-1}\mathbf{B} \end{bmatrix}}_{n\times kp} \underbrace{\begin{bmatrix} \mathbf{u}(k-1) \\ \mathbf{u}(k-2) \\ \mathbf{u}(k-3) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix}}_{kp\times 1}$$

$n\times kp$

Controllability matrix $\mathbf{C}_k$

$kp\times 1$

Input vector **U** *is a concatenation of k input vectors each is p×1*

Controllability definition

$$\mathbf{x}(k) = \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j}\mathbf{B}\mathbf{u}(j)$$

$n \times 1$

$$= \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{k-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}(k-1) \\ \mathbf{u}(k-2) \\ \mathbf{u}(k-3) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix}$$

$n \times kp$

Controllability matrix $\mathbf{C}_k$

$kp \times 1$

Input vector is a concatenation of k input vectors each is p×1

$$\mathbf{x}(k) = \mathbf{C}_k \mathbf{U}$$

Controllability Definition

- The system is said to be *controllable* if there exists a succession of inputs that can steer the system from an initial state vector to any desired state vector at time k (Important as will be related to controller design)
- In other words, if there is a solution for U to get any $\mathbf{x}(k)$ in the above equation ($\mathbf{U}$ exists for any left hand side target $\mathbf{x}(k)$)

Controllability mathematical condition

$$\mathbf{x}(k) = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{k-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}(k-1) \\ \mathbf{u}(k-2) \\ \mathbf{u}(k-3) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix} \Rightarrow \boxed{\mathbf{x}(k) = \mathbf{C}_k \mathbf{U}}$$

Controllability condition

- For the system to be controllable, $\mathbf{C}_k$ must be full rank, i.e. rank($\mathbf{C}_k$) = n (why?)
- When $\mathbf{C}_k$ is full rank, the range of $\mathbf{C}_k$ or its column space is the whole space $\mathbb{R}^n$ and not just a subspace in it
- Usually $\mathbf{C}_k$ is a fat matrix (columns > rows)
- Need to check if there are n columns of $\mathbf{C}_k$ that are independent or not (pivot columns)
- If they are independent, this means that a linear combination of the columns of $\mathbf{C}_k$ spans the entire $\mathbb{R}^n$ and the system is controllable

Controllability mathematical condition

illustrative example

- Check controllability of system whose $\mathbf{C}_k$ is

$$\mathbf{C}_k = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

rank{$\mathbf{C}_k$}= 1 (why?) → col.2 is 2*col. 1 and col.3 is 3*col.1

Formal way is to do Gaussian elimination and find the number of pivot columns (to get reduced row echelon form)

$$\mathbf{C}_k \mathbf{U} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} u(1) \\ u(2) \\ u(3) \end{bmatrix}$$

$$= u(1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + u(2)\begin{bmatrix} 2 \\ 4 \end{bmatrix} + u(3)\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= \left(u(1) + 2u(2) + 3u(3)\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

In this case the column space of this rank deficient matrix is a subspace (line) in $\mathbf{R}^2$

Controllability mathematical condition

Note

- When $\mathbf{C}_k$ is fat, i.e. when $k>n$, you do not need to check the rank of C_k but you can only check the rank of C_n (why?)

$$\mathbf{C}_k = \underbrace{\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B}} & \mathbf{A}^n\mathbf{B} & \cdots & \mathbf{A}^{k-1}\mathbf{B} \end{bmatrix}$$

Check only these columns because the rest of the columns will be dependent on them (why?)

- Because from Cayley-Hamilton theorem, every square matrix satisfies its own characteristic equation and hence $\mathbf{A}^n\mathbf{B}$ will depend on previous columns

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_1\lambda + a_0 = 0$$

$$\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + a_{n-2}\mathbf{A}^{n-2} + \ldots + a_1\mathbf{A} + a_0\mathbf{I} = 0$$

$$\therefore \mathbf{A}^n = -a_{n-1}\mathbf{A}^{n-1} - a_{n-2}\mathbf{A}^{n-2} - \ldots - a_1\mathbf{A} + a_0\mathbf{I}$$

Controllability mathematical condition

Summary

- A system with matrices **A**,**B** is said to be controllable if its controllability matrix is full rank (same for continuous and discrete)

$$\text{rank}\left\{\mathbf{C}_n\right\} = n$$

$$\mathbf{C}_n = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

Final Note

- What if initial state vector is not zero?

$$\mathbf{x}(k) = \mathbf{A}^k \mathbf{x}(0) + \mathbf{C}_k \mathbf{U} \quad \Rightarrow \boxed{\mathbf{x}(k) - \mathbf{A}^k \mathbf{x}(0) = \mathbf{C}_k \mathbf{U}}$$

- Condition of controllability stays the same since going from non-zero **x**(0) to **x**(**k**) is just equivalent to going from zero initial state vector to $\mathbf{x}(k)\text{-}\mathbf{A}^k\mathbf{x}(0)$

Observability concept

Illustrative Example

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 0 \end{bmatrix} \mathbf{x}(t)$$

- Eigenvalues of **A** are -1, -2 (poles of the system)
- Let's find the TF

$$TF = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$= \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s+2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \frac{6}{s+1}$$

Same TF as example in slide 12
Where is the other pole at -2 ?

Observability concept

Illustrative Example

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 0 \end{bmatrix} \mathbf{x}(t)$$

- Mathematically, the other pole at -2 got canceled because of the 0 in **C** together with **A** being diagonal which basically means that the dynamics of the second state <u>cannot be observed</u> at the output or we say x_2 is not observable
- If you dig deep, you can discover what happened to the eigenvalue at -2 and why it disappeared in TF
- It is because the system has a zero also at -2 that got canceled with the pole at -2 (How can you check zeros??)

$$\begin{vmatrix} z_0\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = 0$$

Observability definition

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

$$\mathbf{x}(k) = \mathbf{A}^{k}\mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j}\mathbf{B}\mathbf{u}(j)$$

Output Equation

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$

$$= \mathbf{C}\mathbf{A}^{k}\mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{C}\mathbf{A}^{k-1-j}\mathbf{B}\mathbf{u}(j) + \mathbf{D}\mathbf{u}(k)$$

Observability Definition

- The system is said to be <u>*observable*</u> if I can uniquely know the initial state variables with the knowledge of the succession of inputs and outputs over finite period of time
- Very important concept as it will be related to <u>State observers</u> that will estimate the state variables from the knowledge of input and output

Observability mathematical condition

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$

$$= \mathbf{C}\mathbf{A}^{k}\mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{C}\mathbf{A}^{k-1-j}\mathbf{B}\mathbf{u}(j) + \mathbf{D}\mathbf{u}(k)$$

$$\mathbf{y}(0) = \mathbf{C}\mathbf{x}(0) + \mathbf{D}\mathbf{u}(0)$$

$$\mathbf{y}(1) = \mathbf{C}\mathbf{A}\mathbf{x}(0) + \mathbf{C}\mathbf{B}\mathbf{u}(0) + \mathbf{D}\mathbf{u}(1)$$

$$\mathbf{y}(2) = \mathbf{C}\mathbf{A}^{2}\mathbf{x}(0) + \mathbf{C}\mathbf{A}\mathbf{B}\mathbf{u}(0) + \mathbf{C}\mathbf{B}\mathbf{u}(1) + \mathbf{D}\mathbf{u}(2)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Observability mathematical condition

$$\mathbf{y}(0) = \mathbf{C}\mathbf{x}(0) + \mathbf{D}\mathbf{u}(0)$$

$$\mathbf{y}(1) = \mathbf{C}\mathbf{A}\mathbf{x}(0) + \mathbf{C}\mathbf{B}\mathbf{u}(0) + \mathbf{D}\mathbf{u}(1)$$

$$\mathbf{y}(2) = \mathbf{C}\mathbf{A}^2\mathbf{x}(0) + \mathbf{C}\mathbf{A}\mathbf{B}\mathbf{u}(0) + \mathbf{C}\mathbf{B}\mathbf{u}(1) + \mathbf{D}\mathbf{u}(2)$$

$$\underbrace{\begin{bmatrix} \mathbf{y}(0) \\ \mathbf{y}(1) \\ \mathbf{y}(2) \\ \vdots \\ \mathbf{y}(k-1) \end{bmatrix}}_{mk\times 1} = \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{k-1} \end{bmatrix}}_{mk\times n} \underbrace{\mathbf{x}(0)}_{n\times 1} + \underbrace{\begin{bmatrix} \mathbf{D} & 0 & 0 & 0 & \cdots & 0 \\ \mathbf{C}\mathbf{B} & \mathbf{D} & 0 & 0 & \cdots & 0 \\ \mathbf{C}\mathbf{A}\mathbf{B} & \mathbf{C}\mathbf{B} & \mathbf{D} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{C}\mathbf{A}^{k-2}\mathbf{B} & \mathbf{C}\mathbf{A}^{k-3}\mathbf{B} & \cdots & \mathbf{C}\mathbf{A}\mathbf{B} & \mathbf{C}\mathbf{B} & \mathbf{D} \end{bmatrix}}_{mk\times pk} \underbrace{\begin{bmatrix} \mathbf{u}(0) \\ \mathbf{u}(1) \\ \mathbf{u}(2) \\ \vdots \\ \mathbf{u}(k-1) \end{bmatrix}}_{pk\times 1}$$

$$\boxed{\mathbf{Y} = \mathbf{O}_k\,\mathbf{x}(0) + \mathbf{V}_k\,\mathbf{U}}$$

Observability mathematical condition

$$\mathbf{Y} = \mathbf{O}_k\,\mathbf{x}(0) + \mathbf{V}_k\,\mathbf{U}$$

$$\mathbf{O}_k\,\mathbf{x}(0) = \mathbf{Y} - \mathbf{V}_k\,\mathbf{U}$$

- If I know the inputs and outputs, I know the right hand side of the above equation
- $\mathbf{x}(0)$ is uniquely defined only if rank$\{\mathbf{O}_k\} = n$ (Why?)
- If $\mathbf{O}_k$ is rank deficient then its nullspace is not empty → say $\mathbf{v} \in N\left(\mathbf{O}_k\right)$

$$\mathbf{O}_k\,\mathbf{x}(0) = \mathbf{O}_k\left[\mathbf{x}(0) + \mathbf{v}\right] = \mathbf{Y} - \mathbf{V}_k\,\mathbf{U}$$

- If $\mathbf{O}_k$ is a full rank matrix, its nullspace is empty other than zero vector hence if LHS is known, $\mathbf{x}(0)$ is uniquely determined
- Usually $\mathbf{O}_k$ is a tall matrix

Observability mathematical condition

Summary

- A system with matrices **A**,**C** is said to be observable if its observability matrix is full rank (check only rank of $\mathbf{O}_n$ if $k>n$)

$$\operatorname{rank}\{\mathbf{O}_n\} = n$$

$$\mathbf{O}_n = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

Minimal realization

Illustrative Example

$$\dot{\mathbf{x}}(t) = -2\mathbf{x}(t) + 3u(t)$$

$$y(t) = 2\mathbf{x}(t)$$

$$\begin{aligned} TF &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \\ &= 2 \cdot \frac{1}{s+2} \cdot 3 \\ &= \frac{6}{s+1} \end{aligned}$$

Same TF as example in slides 12 and 20

- Both previous examples led to the same TF but one was uncontrollable and the second was unobservable
- This realization of the same TF is both controllable and observable because it is the **minimal realization** (only 1 state variable not 2)

A minimal realization is both controllable and observable (without proof)