Control Systems And Their Components (EE391)

Lec. 6: SS Dynamic Solution and Realizations

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Dr. Mohamed Hamdy Osman

Lecture Outline

- Diagonalization of the system dynamics matrix **A**
- Going from SS model to TF
- Relationship between poles of TF and eigenvalues of **A**
- Equivalent SS equations
- SS realizations

State Space Equations (Reminder)

For an *n* dimensional system with *p* inputs and *m* outputs

$$\underset{n\times 1}{\dot{\mathbf{x}}(t)} = \underset{n\times n}{\mathbf{A}}\,\underset{n\times 1}{\mathbf{x}(t)} + \underset{n\times p}{\mathbf{B}}\,\underset{p\times 1}{\mathbf{u}(t)}$$

$$\underset{m\times 1}{\mathbf{y}(t)} = \underset{m\times n}{\mathbf{C}}\,\underset{n\times 1}{\mathbf{x}(t)} + \underset{m\times p}{\mathbf{D}}\,\underset{p\times 1}{\mathbf{u}(t)}$$

- $t \in \mathbb{R}$ denotes time
- $\mathbf{x} \in \mathbb{R}^n$ denotes the state vector
- $\mathbf{u} \in \mathbb{R}^p$ denotes the input vector
- $\mathbf{y} \in \mathbb{R}^m$ denotes the output vector

- $\mathbf{A} \in \mathbb{R}^{n\times n}$ denotes the system dynamic matrix
- $\mathbf{B} \in \mathbb{R}^{n\times p}$ denotes the input matrix
- $\mathbf{C} \in \mathbb{R}^{m\times n}$ denotes the output or sensor matrix
- $\mathbf{D} \in \mathbb{R}^{m\times p}$ denotes the feedthrough matrix

- For LTI systems, the matrices A,B,C and D are all constant, i.e. not f(t)
- For time variant systems → A(t), B(t), C(t), D(t)

Solution of SS Equations (Time response)

Interpretation of solution using diagonalization

Back to Homog. Sol. $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$

$$\mathbf{x}(t) = \mathbf{Q}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}\mathbf{Q}^{-1}\mathbf{x}(0)$$

$$= \begin{bmatrix} \vdots & & \vdots \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} \cdots & \mathbf{w}_1^{\mathbf{T}} & \cdots \\ & \vdots & \\ \cdots & \mathbf{w}_n^{\mathbf{T}} & \cdots \end{bmatrix}\mathbf{x}(0)$$

$$= \begin{bmatrix} & & \vdots \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} \mathbf{w}_1^{\mathbf{T}}\mathbf{x}(0) \\ \vdots \\ \mathbf{w}_n^{\mathbf{T}}\mathbf{x}(0) \end{bmatrix}$$

Solution of SS Equations (Time response)

Interpretation of solution using diagonalization

Back to Homog. Sol. $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$

$$\mathbf{x}(t) = \begin{bmatrix} & & \vdots \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1^{\mathbf{T}}\mathbf{x}(0) \\ \vdots \\ \mathbf{w}_n^{\mathbf{T}}\mathbf{x}(0) \end{bmatrix}$$

$$= \begin{bmatrix} \vdots & & \vdots \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t}\mathbf{w}_1^{\mathbf{T}}\mathbf{x}(0) \\ \vdots \\ e^{\lambda_n t}\mathbf{w}_n^{\mathbf{T}}\mathbf{x}(0) \end{bmatrix}$$

$$= \sum_{i=1}^{n} \mathbf{v}_i e^{\lambda_i t}\left(\mathbf{w}_i^{\mathbf{T}}\mathbf{x}(0)\right)$$

Solution of SS Equations (Time response)

Interpretation of solution using diagonalization

Back to Homog. Sol.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

$$\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{v}_i e^{\lambda_i t} \left(\mathbf{w}_i^{\mathbf{T}} \mathbf{x}(0) \right)$$

- Solution is a linear combination of all individual modes ($e^{\lambda_i t}$)
- Eigenvalues λ_i determine the time behavior of each mode
- Eigenvectors $\mathbf{v}_i$ determine how much each mode impacts each of the state variables
- Rows of $\mathbf{Q}^{-1}$, denoted by $\mathbf{w}_i^{\mathbf{T}}$, determine how much each initial state variable contribute to each mode
- Benefit of diagonalization or eigen decomposition is to decouple the modes and write the full time solution as a linear combination of them
- You can also expect that **eigenvalues are related to poles**

From SS model to Transfer Function

SS model

$$\underset{n\times 1}{\dot{\mathbf{x}}(t)} = \underset{n\times n}{\mathbf{A}}\underset{n\times 1}{\mathbf{x}(t)} + \underset{n\times p}{\mathbf{B}}\underset{p\times 1}{\mathbf{u}(t)}$$

$$\underset{m\times 1}{\mathbf{y}(t)} = \underset{m\times n}{\mathbf{C}}\underset{n\times 1}{\mathbf{x}(t)} + \underset{m\times p}{\mathbf{D}}\underset{p\times 1}{\mathbf{u}(t)}$$

Transfer Function

$$\frac{Y(s)}{U(s)}$$

From SS model to Transfer Function

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Take LT $\quad s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$

$$\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{x}(0) + \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\mathbf{U}(s)$$

since $\quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$

$$\therefore \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

$$\therefore \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \underbrace{\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}}_{\text{Initial state response}}\mathbf{x}(0) + \underbrace{\left[\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]}_{\text{Transfer Function Matrix } m\times p}\mathbf{U}(s)$$

Initial state response

Transfer Function Matrix $m \times p$

From SS model to Transfer Function

$$\mathbf{Y}(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{x}(0) + \left[\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$

To obtain TF, set **x**(0) = 0

$$\mathbf{Y}(s) = \left[\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$

For a SISO system, **Y**(s) and **U**(s) are scalars

$$\frac{Y(s)}{U(s)} = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}$$

MATLAB

tf2ss

ss2tf

From SS model to Transfer Function

Example Find the transfer function of the following state space model

$$\dot{\mathbf{x}} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} -0.5 & 1 \end{bmatrix}\mathbf{x}$$

Solution

$$\frac{Y(s)}{U(s)} = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}$$

From a previous example with the same matrix **A**

$$\left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \begin{bmatrix} \dfrac{s+3}{(s-2)(s+1)} & \dfrac{-5}{(s-2)(s+1)} \\ \dfrac{2}{(s-2)(s+1)} & \dfrac{s-4}{(s-2)(s+1)} \end{bmatrix}$$

From SS model to Transfer Function

Example

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} -0.5 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+3}{(s-2)(s+1)} & \frac{-5}{(s-2)(s+1)} \\ \frac{2}{(s-2)(s+1)} & \frac{s-4}{(s-2)(s+1)} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & 1 \end{bmatrix} \begin{bmatrix} \frac{-2s-1}{(s-2)(s+1)} \\ \frac{s-8}{(s-2)(s+1)} \end{bmatrix}$$

$$= \frac{2s-7.5}{(s-2)(s+1)}$$

<u>We notice that the poles at $s = 2,-1$ are exactly the eigenvalues of **A**</u> we found before

From SS model to Transfer Function

HW problem

Find the transfer function matrix of the following SS model having 3 inputs and 2 outputs

$$\dot{\mathbf{x}} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -2 & 3 & 1 \\ 1 & -5 & 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} -0.5 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{x}$$

Hint

You should still find $[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]$ which is a 2x3 matrix that relates the input and output vectors as follows

$$\mathbf{Y}(s) = \left[\mathbf{C}\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}+\mathbf{D}\right]\mathbf{U}(s)$$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \left[\mathbf{C}\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}+\mathbf{D}\right] \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix}$$

From SS model to Transfer Function

Back to SISO case and general TF

$$\frac{Y(s)}{U(s)} = \mathbf{C}\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}+\mathbf{D}$$

$$\frac{Y(s)}{U(s)} = \mathbf{C}\frac{\operatorname{adj}\left(s\mathbf{I}-\mathbf{A}\right)}{\left|s\mathbf{I}-\mathbf{A}\right|}\mathbf{B}+\mathbf{D}$$

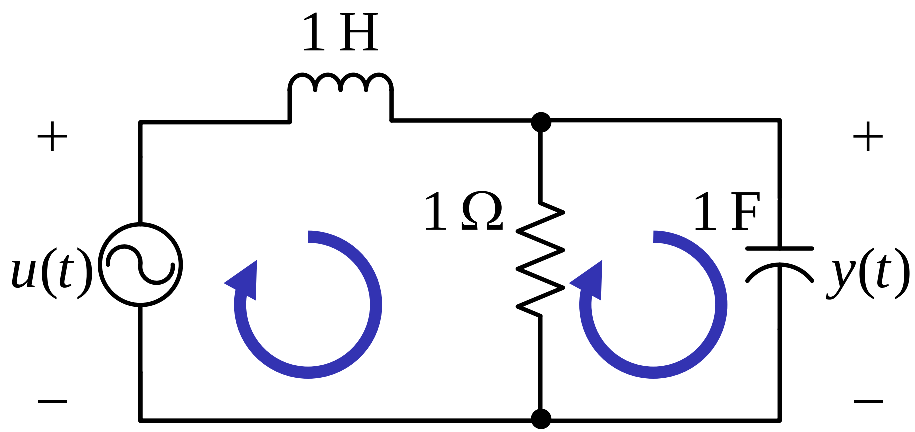
$$= \frac{\mathbf{C}\operatorname{adj}\left(s\mathbf{I}-\mathbf{A}\right)\mathbf{B}+\mathbf{D}\left|s\mathbf{I}-\mathbf{A}\right|}{\left|s\mathbf{I}-\mathbf{A}\right|}$$

- Clearly the poles of the TF are the values of s that makes $|s\mathbf{I}-\mathbf{A}| = 0$ which are the also the eigenvalues of **A**
- We can easily predict that what was said on poles can be exactly said on eigenvalues (e.g. the condition of BIBO stability is Re{eigenvalues} < 0

Equivalent SS Equations

Example

Find two SS representations for the this circuit. Use the underneath two assignments of state variables



State variables (1st)

- x_1 : inductor current i_L
- x_2 : capacitor voltage v_c

State variables (2nd)

- $\tilde{x}_1$: current of left loop
- $\tilde{x}_2$: current of right loop

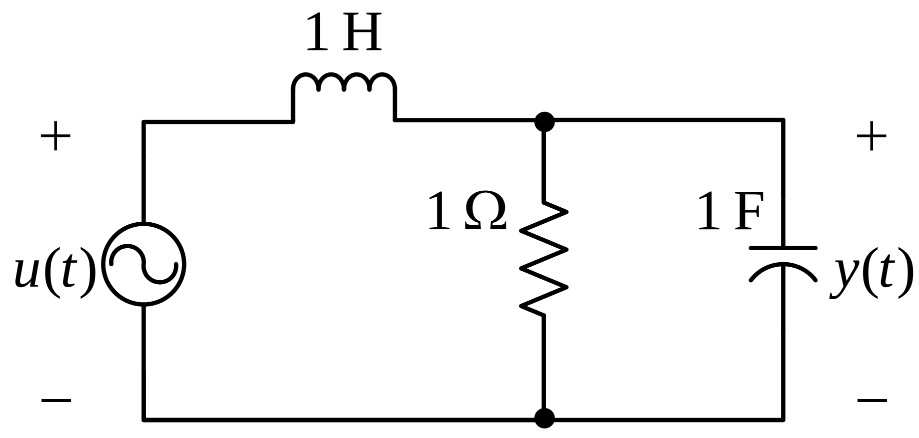
Find the relation (transformation **P**) between the two state vectors in the 1st and 2nd realizations

$$\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x}$$

Equivalent SS Equations

State variables (1st)

- x_1 : inductor current i_L
- x_2 : capacitor voltage v_c



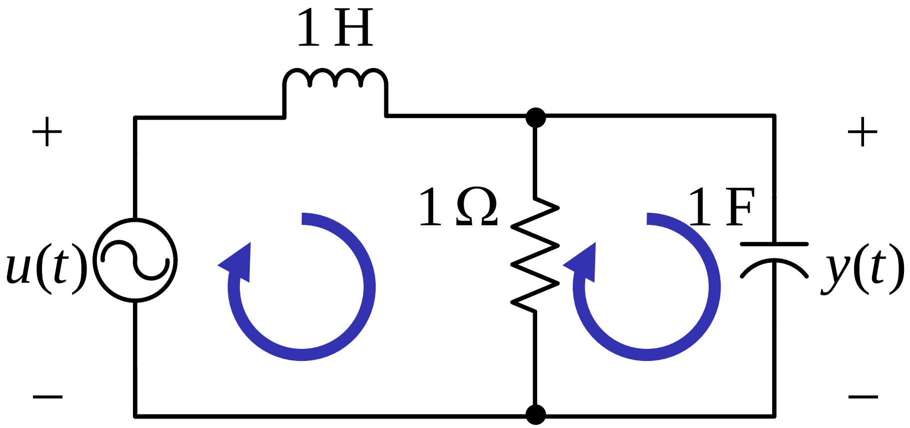
$$u = \dot{x}_1 + x_2 \Rightarrow \dot{x}_1 = -x_2 + u$$

$$x_1 = x_2 + \dot{x}_2 \Rightarrow \dot{x}_2 = x_1 - x_2$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u$$

Equivalent SS Equations



State variables (2nd)

- $\tilde{x}_1$: current of left loop
- $\tilde{x}_2$: current of right loop

$$u = \dot{\tilde{x}}_1 + (\tilde{x}_1 - \tilde{x}_2) \Rightarrow \dot{\tilde{x}}_1 = -\tilde{x}_1 + \tilde{x}_2 + u$$

$$\tilde{x}_2 = 1 \cdot \frac{d}{dt}\{1 \cdot (\tilde{x}_1 - \tilde{x}_2)\} \Rightarrow \dot{\tilde{x}}_1 - \dot{\tilde{x}}_2 = \tilde{x}_2 \Rightarrow \dot{\tilde{x}}_2 = -\tilde{x}_1 + u$$

$$\therefore \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + 0u$$

The two State vectors are equivalent and can be related by the transformation **P**

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x}$$

Equivalent SS Equations

- Consider an *n*-dimensional state space equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- Let **P** be an $n \times n$ real nonsingular matrix, and let $\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x}$. Then, the SS equations

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \tilde{\mathbf{C}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{D}}\mathbf{u}(t)$$

where

$$\tilde{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}, \quad \tilde{\mathbf{B}} = \mathbf{P}\mathbf{B}, \quad \tilde{\mathbf{C}} = \mathbf{C}\mathbf{P}^{-1}, \quad \tilde{\mathbf{D}} = \mathbf{D}$$

is said to be algebraically equivalent with the original state space equations

- $\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x}$ is called an equivalence (or similarity) transformation

Equivalent SS Equations

start from $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$

$\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x} \quad \Rightarrow$ substitute with $\mathbf{x} = \mathbf{P}^{-1}\tilde{\mathbf{x}}$

$$\mathbf{P}^{-1}\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\mathbf{P}^{-1}\tilde{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\dot{\tilde{\mathbf{x}}}(t) = \underbrace{\mathbf{P}\mathbf{A}\mathbf{P}^{-1}}_{\tilde{\mathbf{A}}}\tilde{\mathbf{x}}(t) + \underbrace{\mathbf{P}\mathbf{B}}_{\tilde{\mathbf{B}}}\mathbf{u}(t)$$

start from $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$

$\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x} \quad \Rightarrow$ substitute with $\mathbf{x} = \mathbf{P}^{-1}\tilde{\mathbf{x}}$

$$\mathbf{y}(t) = \underbrace{\mathbf{C}\mathbf{P}^{-1}}_{\tilde{\mathbf{C}}}\tilde{\mathbf{x}}(t) + \underbrace{\mathbf{D}}_{\tilde{\mathbf{D}}}\mathbf{u}(t)$$

Equivalent SS Equations

Problem

Prove that the similarity transformation $\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x}$ does not change the eigenvalues of **A,** i.e. prove that eigenvalues of A are the same as $\tilde{\mathbf{A}}$

Problem

Prove that the similarity transformation $\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x}$ does not change the Transfer function matrix

$$\tilde{\mathbf{T}} = \tilde{\mathbf{C}}\left(s\mathbf{I} - \tilde{\mathbf{A}}\right)^{-1}\tilde{\mathbf{B}} + \tilde{\mathbf{D}}$$

$$= \mathbf{C}\mathbf{P}^{-1}\left(s\mathbf{I} - \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\right)^{-1}\mathbf{P}\mathbf{B} + \mathbf{D}$$

$$= \mathbf{C}\mathbf{P}^{-1}\left(s\mathbf{P}\mathbf{P}^{-1} - \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\right)^{-1}\mathbf{P}\mathbf{B} + \mathbf{D}$$

$$= \mathbf{C}\mathbf{P}^{-1}\left[\mathbf{P}\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{P}^{-1}\right]^{-1}\mathbf{P}\mathbf{B} + \mathbf{D}$$

$$= \mathbf{C}\mathbf{P}^{-1}\mathbf{P}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{P}^{-1}\mathbf{P}\mathbf{B} + \mathbf{D}$$

$$= \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D} = \mathbf{T}$$

SS Realizations

- The problem concerning how to describe a system in state space equations, provided that the transfer function of a system, $G(s)$, is available, is called **Realization Problem**.

$$\boldsymbol{G(s)} \Longrightarrow \mathbf{A, B, C, D}$$

- There are infinite realizations for the same TF
- We are only interested in minimal realizations (least number of state variables n)
- There are standard realizations that we will study

  - Controller Canonical Form (CCF)
  - Observer Canonical Form (OCF)
  - Modal (Diagonal) Canonical Form (DCF)

SS Realizations

Controller Canonical Form (CCF)

Assume order of numerator is less than that of denominator by 1

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\ldots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\ldots+a_1 s+a_0}$$

$$=(b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\ldots+b_1 s+b_0)\times\frac{1}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\ldots+a_1 s+a_0}$$

$$=\frac{Y(s)}{Z(s)}\times\frac{Z(s)}{U(s)}$$

<u>For the first part</u>

$$\frac{Z(s)}{U(s)}=\frac{1}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\ldots+a_1 s+a_0}$$

$$z^{(n)}(t)+a_{n-1}z^{(n-1)}(t)+\ldots.+a_1\dot{z}(t)+a_0 z(t)=u(t)$$

SS Realizations

Controller Canonical Form (CCF)

For the first part

$$\frac{Z(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0}$$

$$z^{(n)}(t) + a_{n-1}z^{(n-1)}(t) + \ldots. + a_1\dot{z}(t) + a_0 z(t) = u(t)$$

$$\text{let} \quad x_1 = z, \quad x_2 = \dot{z}, \quad x_3 = \ddot{z}, \ \ldots, \ x_n = z^{(n-1)}$$

$$\therefore \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_4, \ldots,$$

$$\dot{x}_n = z^{(n)} = -a_{n-1}z^{(n-1)} - \ldots - a_1\dot{z} - a_0 z + u$$

$$= -a_{n-1}x_n - \ldots - a_1 x_2 - a_0 x_1 + u$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & \cdots & -a_{n-3} & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

SS Realizations

Controller Canonical Form (CCF)

For the second part

$$\frac{Y(s)}{Z(s)} = (b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_1 s + b_0)$$

$$y(t) = b_{n-1}z^{(n-1)}(t) + b_{n-2}z^{(n-2)}(t) + \ldots. + b_1\dot{z}(t) + b_0 z(t)$$

$$= b_{n-1}x_n + b_{n-2}x_{n-1} + \ldots. + b_1 x_2 + b_0 x_1$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + 0 \cdot u(t)$$

SS Realizations

Controller Canonical Form (CCF)

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\ldots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\ldots+a_1 s+a_0}$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \vdots\\ \dot{x}_n(t)\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0 & \ldots & 0\\ 0 & 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ -a_0 & -a_1 & \cdots & -a_{n-3} & -a_{n-2} & -a_{n-1}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ \vdots\\ x_n(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ \vdots\\ 1\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}+0\cdot u(t)$$

SS Realizations

Diagonal Canonical Form (DCF)

$$G(s) = \frac{Y(s)}{U(s)} = \left\{ \sum_{i=1}^{n} \frac{r_i}{s - \lambda_i} + r_0 \right\} U(s)$$

Use Partial Fraction Expansion for TF

- In case **all poles are distinct**, we define:

$$X_1(s) = \frac{1}{s-\lambda_1} U(s) \quad \longrightarrow \quad \dot{x}_1(t) = \lambda_1 x_1(t) + u(t)$$

$$X_2(s) = \frac{1}{s-\lambda_2} U(s) \quad \longrightarrow \quad \dot{x}_2(t) = \lambda_2 x_2(t) + u(t)$$

$$\vdots$$

$$X_n(s) = \frac{1}{s-\lambda_n} U(s) \quad \longrightarrow \quad \dot{x}_n(t) = \lambda_n x_n(t) + u(t)$$

$$Y(s) = r_1 X_1(s) + r_2 X_2(s) + \cdots + r_n X_n(s) + r_0 U(s) \quad \longrightarrow \quad y(t) = r_1 x_1(t) + r_2 x_2(t) + \cdots + r_n x_n(t) + r_0 u(t)$$

SS Realizations

Diagonal Canonical Form (DCF)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + r_0 u(t)$$

- Matrix **A** for DCF is diagonal, i.e. all state variables are decoupled
- DCF can be obtained by finding the eigenvalues λ's and eigenvector matrix **Q**
- Having distinct eigenvalues is exactly the same as having distinct poles (non-repeated poles) in which case **A** is diagonalizable

SS Realizations

Observer Canonical Form (OCF)

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}$$

$$s^nY(s)+a_{n-1}s^{n-1}Y(s)+\cdots+a_1sY(s)+a_0Y(s)=$$
$$b_{n-1}s^{n-1}U(s)+b_{n-2}s^{n-2}U(s)+\cdots+b_1sU(s)+b_0U(s)$$

$$Y(s)+a_{n-1}\frac{Y(s)}{s}+\cdots+a_1\frac{Y(s)}{s^{n-1}}+a_0\frac{Y(s)}{s^n}=$$
$$b_{n-1}\frac{U(s)}{s}+b_{n-2}\frac{U(s)}{s^2}+\cdots+b_1\frac{U(s)}{s^{n-1}}+b_0\frac{U(s)}{s^n}$$

$$Y(s)=\frac{1}{s}\left\{\Big(b_{n-1}U(s)-a_{n-1}Y(s)\Big)+\frac{1}{s}\left\{\Big(b_{n-2}U(s)-a_{n-2}Y(s)\Big)+\frac{1}{s}(\cdots)+\right.\right.$$
$$\left.\left.\underbrace{\frac{1}{s}\left\{b_0U(s)-a_0Y(s)\right\}}_{X_1(s)}\right\}\cdots\right\}$$

SS Realizations

Observer Canonical Form (OCF)

$$X_1(s) = \frac{1}{s}\left\{b_0 U(s) - a_0 Y(s)\right\} \longrightarrow \dot{x}_1(t) = b_0 u(t) - a_0 y(t)$$

$$X_2(s) = \frac{1}{s}\left\{\left(b_1 U(s) - a_1 Y(s)\right) + X_1(s)\right\} \longrightarrow \dot{x}_2(t) = b_1 u(t) - a_1 y(t) + x_1(t)$$

$$\vdots \qquad\qquad \vdots$$

$$X_n(s) = \frac{1}{s}\left\{\left(b_{n-1} U(s) - a_{n-1} Y(s)\right) + X_{n-1}(s)\right\} \longrightarrow \dot{x}_n(t) = b_{n-1} u(t) - a_{n-1} y(t) + x_{n-1}(t)$$

$$Y(s) = X_n(s) \longrightarrow y(t) = x_n(t)$$

SS Realizations

Observer Canonical Form (OCF)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & -a_0 \\ 1 & 0 & \cdots & -a_1 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + 0u(t)$$

SS Realizations

Example

Find the SS realization of the following TF in CCF, DCF, OCF

$$G(s) = \frac{4s^3 + 25s^2 + 45s + 34}{2s^3 + 12s^2 + 20s + 16}$$

And prove that all 3 realizations are equivalent

Solution

$$G(s) = \frac{4s^3 + 25s^2 + 45s + 34}{2s^3 + 12s^2 + 20s + 16} = 2 + \frac{s^2 + 5s + 2}{2s^3 + 12s^2 + 20s + 16} = 2 + \frac{\frac{1}{2}s^2 + 2\frac{1}{2}s + 1}{s^3 + 6s^2 + 10s + 8}$$

SS Realizations

Solution

$$G(s)=\frac{4s^3+25s^2+45s+34}{2s^3+12s^2+20s+16}=2+\frac{s^2+5s+2}{2s^3+12s^2+20s+16}=2+\frac{\frac{1}{2}s^2+2\frac{1}{2}s+1}{s^3+6s^2+10s+8}$$

CCF

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -8 & -10 & -6\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}1 & 2.5 & 0.5\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+2\cdot u(t)$$

SS Realizations

Solution

$$G(s)=\frac{4s^3+25s^2+45s+34}{2s^3+12s^2+20s+16}=2+\frac{s^2+5s+2}{2s^3+12s^2+20s+16}=2+\frac{\frac{1}{2}s^2+2\frac{1}{2}s+1}{s^3+6s^2+10s+8}$$

OCF

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}0 & 0 & -8\\ 1 & 0 & -10\\ 0 & 1 & -6\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+\begin{bmatrix}1\\ 2.5\\ 0.5\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+2\cdot u(t)$$

SS Realizations

Solution

$$G(s)=\frac{4s^3+25s^2+45s+34}{2s^3+12s^2+20s+16}=2+\frac{s^2+5s+2}{2s^3+12s^2+20s+16}=2+\frac{\frac{1}{2}s^2+2\frac{1}{2}s+1}{s^3+6s^2+10s+8}$$

DCF

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}-1+j & 0 & 0\\ 0 & -1-j & 0\\ 0 & 0 & -4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}0.3+j0.15 & 0.3-j0.15 & -0.1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+2\cdot u(t)$$

Use MATLAB to find partial fraction expansion

[r,p,k] = residue([4 25 45 34],[2 12 20 16])