Control Systems And Their Components (EE391)

Lec. 5: State Space Representation and its Dynamic Solution

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Lecture Outline

- Dynamic response of SS equations (Transient Solution including both homogenous + forced solutions)
- Diagonalization of the system dynamics matrix **A**
- Going from SS model to TF
- Relationship between poles of TF and eigenvalues of **A**

State Space Equations (Reminder)

For an *n* dimensional system with *p* inputs and *m* outputs

$$\underbrace{\dot{\mathbf{x}}(t)}_{n\times 1} = \underbrace{\mathbf{A}}_{n\times n}\underbrace{\mathbf{x}(t)}_{n\times 1} + \underbrace{\mathbf{B}}_{n\times p}\underbrace{\mathbf{u}(t)}_{p\times 1}$$

$$\underbrace{\mathbf{y}(t)}_{m\times 1} = \underbrace{\mathbf{C}}_{m\times n}\underbrace{\mathbf{x}(t)}_{n\times 1} + \underbrace{\mathbf{D}}_{m\times p}\underbrace{\mathbf{u}(t)}_{p\times 1}$$

- $t \in \mathbb{R}$ denotes time
- $\mathbf{x} \in \mathbb{R}^n$ denotes the state vector
- $\mathbf{u} \in \mathbb{R}^p$ denotes the input vector
- $\mathbf{y} \in \mathbb{R}^m$ denotes the output vector

- $\mathbf{A} \in \mathbb{R}^{n\times n}$ denotes the system dynamic matrix
- $\mathbf{B} \in \mathbb{R}^{n\times p}$ denotes the input matrix
- $\mathbf{C} \in \mathbb{R}^{m\times n}$ denotes the output or sensor matrix
- $\mathbf{D} \in \mathbb{R}^{m\times p}$ denotes the feedthrough matrix

- For LTI systems, the matrices A,B,C and D are all constant, i.e. not f(*t*)
- For time variant systems → A(*t*), B(*t*), C(*t*), D(*t*)

State Space Equations

Problem 2 (from last Lect.): Find the SS formulation for the following system whose i/o relationship is given by the following Diff. Eq.

$$\dddot{y} + 2\ddot{y} + 3\dot{y} + y = 4\ddot{u} - \dot{u} + 5u$$

Answer

$$\frac{Y(s)}{U(s)} = \frac{4s^2 - s + 5}{s^3 + 2s^2 + 3s + 1}$$

$$= \frac{1}{s^3 + 2s^2 + 3s + 1} \cdot \left(4s^2 - s + 5\right) = \frac{Z(s)}{U(s)} \cdot \frac{Y(s)}{Z(s)}$$

$$x_1 = z \qquad x_2 = \dot{z} \qquad x_3 = \ddot{z}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

MATLAB

[A B C D] = tf2ss(num,den)

Note that MATLAB uses a flipped state vector assignment

Solution of SS Equations (Time response)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Homogeneous Solution

When the input is zero and the system is only driven by the initial state variables **x**(0)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

Taking the Laplace transform (assume **A** is constant for LTI system)

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$$

$$\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{X}(s) = \mathbf{x}(0)$$

$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{x}(0)$$

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left\{\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\right\}\mathbf{x}(0)$$

Solution of SS Equations (Time response)

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s}\left(\mathbf{I}-\frac{\mathbf{A}}{s}\right)^{-1}$$

$$\because\left(\mathbf{I}-\frac{\mathbf{A}}{s}\right)\left(\mathbf{I}+\frac{\mathbf{A}}{s}+\frac{\mathbf{A}^2}{s^2}+\frac{\mathbf{A}^3}{s^3}+\ldots\right)=\mathbf{I}+\frac{\mathbf{A}}{s}+\frac{\mathbf{A}^2}{s^2}+\frac{\mathbf{A}^3}{s^3}+\ldots-\frac{\mathbf{A}}{s}-\frac{\mathbf{A}^2}{s^2}-\frac{\mathbf{A}^3}{s^3}=\mathbf{I}$$

$$\therefore\left(\mathbf{I}-\frac{\mathbf{A}}{s}\right)^{-1}=\mathbf{I}+\frac{\mathbf{A}}{s}+\frac{\mathbf{A}^2}{s^2}+\frac{\mathbf{A}^3}{s^3}+\ldots$$

Think Maclaurin Expansion for scalars!!

$$\therefore \mathbf{x}(t)=\mathcal{L}^{-1}\left\{\frac{\mathbf{I}}{s}+\frac{\mathbf{A}}{s^2}+\frac{\mathbf{A}^2}{s^3}+\frac{\mathbf{A}^3}{s^4}+\ldots\right\}\mathbf{x}(0)$$

$$\therefore \mathbf{x}(t)=\left(\mathbf{I}+\mathbf{A}t+\mathbf{A}^2\frac{t^2}{2!}+\mathbf{A}^3\frac{t^3}{3!}+\ldots\right)\mathbf{x}(0)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\}=\frac{1}{n!}t^n$$

Solution of SS Equations (Time response)

$$\therefore \mathbf{x}(t) = \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2!} + \mathbf{A}^3 \frac{t^3}{3!} + \ldots \right) \mathbf{x}(0)$$

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}(0)$$

$$\boxed{\mathbf{x}(t) = \mathbf{\Phi}(t) \mathbf{x}(0)}$$

State Transition Matrix $\boxed{\mathbf{\Phi}(t) = e^{\mathbf{A}t}}$

It gives the updated state variables at time *t* given the initial state variables **x**(0)

Matrix Exponential
in MATLAB
expm(A)

Solution of SS Equations (Time response)

Property of Matrix exponential

$$e^{\mathbf{A}}e^{\mathbf{B}} = e^{(\mathbf{A}+\mathbf{B})} \quad \text{iff} \quad \mathbf{AB}=\mathbf{BA}$$

$$\text{otherwise} \quad e^{\mathbf{A}}e^{\mathbf{B}} \neq e^{(\mathbf{A}+\mathbf{B})}$$

Try to prove it!!

Properties of State Transition Matrix

1) $\mathbf{\Phi}(t_1) = e^{\mathbf{A}t_1}, \quad \mathbf{\Phi}(t_2) = e^{\mathbf{A}t_2}$

$$\mathbf{\Phi}(t_1)\cdot\mathbf{\Phi}(t_2) = e^{\mathbf{A}t_1}\cdot e^{\mathbf{A}t_2} = e^{\mathbf{A}(t_1+t_2)}$$

The second equality holds since $\mathbf{A}t_1$ commutes with $\mathbf{A}t_2$

$$\text{if } t_1 = -t_2 = t \quad \Rightarrow \quad \mathbf{\Phi}(t)\cdot\mathbf{\Phi}(-t) = e^{\mathbf{A}t}\cdot e^{-\mathbf{A}t} = e^{A\cdot 0} = \mathbf{I}$$

$$\therefore \mathbf{\Phi}(-t) \text{ is the inverse of } \mathbf{\Phi}(t) \quad \Rightarrow \quad \boxed{\mathbf{\Phi}^{-1}(t) = \mathbf{\Phi}(-t)}$$

Solution of SS Equations (Time response)

Properties of State Transition Matrix

2) $\because \mathbf{x}(t_0) = \mathbf{\Phi}(t_0)\mathbf{x}(0) = e^{\mathbf{A}t_0}\mathbf{x}(0)$

$\therefore \mathbf{x}(0) = e^{-\mathbf{A}t_0}\mathbf{x}(t_0)$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) = e^{\mathbf{A}t}e^{-\mathbf{A}t_0}\mathbf{x}(t_0) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0)$$

$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) = \mathbf{\Phi}(t-t_0)\mathbf{x}(t_0)}$$

3) $$\boxed{\mathbf{\Phi}(t_2 - t_0) = \mathbf{\Phi}(t_2 - t_1)\cdot\mathbf{\Phi}(t_1 - t_0)}$$

Solution of SS Equations (Time response)

Example

Compute $e^{\mathbf{A}t}$ if $\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left\{\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\right\}$$

$$\left(s\mathbf{I} - \mathbf{A}\right) = \begin{bmatrix} s-4 & 5 \\ -2 & s+3 \end{bmatrix}$$

$$\left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \frac{1}{\left(s-4\right)\left(s+3\right)+10}\begin{bmatrix} s+3 & -5 \\ 2 & s-4 \end{bmatrix}$$

$$= \frac{1}{s^2 - s - 2}\begin{bmatrix} s+3 & -5 \\ 2 & s-4 \end{bmatrix} = \begin{bmatrix} \dfrac{s+3}{(s-2)(s+1)} & \dfrac{-5}{(s-2)(s+1)} \\ \dfrac{2}{(s-2)(s+1)} & \dfrac{s-4}{(s-2)(s+1)} \end{bmatrix}$$

Solution of SS Equations (Time response)

Example

Compute $e^{\mathbf{A}t}$ if $\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left\{\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{s+3}{(s-2)(s+1)} & \frac{-5}{(s-2)(s+1)} \\ \frac{2}{(s-2)(s+1)} & \frac{s-4}{(s-2)(s+1)} \end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix} \frac{5/3}{s-2} - \frac{2/3}{s+1} & \frac{-5/3}{s-2} + \frac{5/3}{s+1} \\ \frac{2/3}{s-2} - \frac{2/3}{s+1} & \frac{-2/3}{s-2} + \frac{5/3}{s+1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{3}e^{2t} - \frac{2}{3}e^{-t} & -\frac{5}{3}e^{2t} + \frac{5}{3}e^{-t} \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} & -\frac{2}{3}e^{2t} + \frac{5}{3}e^{-t} \end{bmatrix}$$

<u>MATLAB</u>

```
syms t
A = [4 -5;2 -3]
expm(A*t)
```

Solution of SS Equations (Time response)

Problem

Compute $e^{\mathbf{A}t}$ if $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Solution of SS Equations (Time response)

Problem

Compute $e^{\mathbf{A}t}$ if $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$

Solution of SS Equations (Time response)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Forced Solution

When the input **u**(t) is non-zero

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Taking the Laplace transform (assume **A** is constant for LTI system)

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{x}(0) + \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left\{\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\right\}\mathbf{x}(0) + \mathcal{L}^{-1}\left\{\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\mathbf{U}(s)\right\}$$

Solution of SS Equations (Time response)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Forced Solution

When the input **u**(t) is non-zero

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}\mathbf{x}(0) + \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)\right\}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{x}(t) = \underbrace{e^{\mathbf{A}t}\mathbf{x}(0)}_{\text{Homogenous solution}} + \underbrace{\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau}_{\text{Forced solution}}$$

$$\boxed{\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau}$$

Solution of SS Equations (Time response)

Example

Solve the following SS equations, i.e. find **x**(t), if $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the input u(t) is a unit step function

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t)$$

Solution

From previous example with the same **A**, we found the state transition matrix

$$e^{\mathbf{A}t} = \begin{bmatrix} \frac{5}{3}e^{2t} - \frac{2}{3}e^{-t} & -\frac{5}{3}e^{2t} + \frac{5}{3}e^{-t} \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} & -\frac{2}{3}e^{2t} + \frac{5}{3}e^{-t} \end{bmatrix}$$

Substitute in

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Solution of SS Equations (Time response)

$$\mathbf{x}(t)=\begin{bmatrix}\frac{5}{3}e^{2t}-\frac{2}{3}e^{-t} & -\frac{5}{3}e^{2t}+\frac{5}{3}e^{-t}\\ \frac{2}{3}e^{2t}-\frac{2}{3}e^{-t} & -\frac{2}{3}e^{2t}+\frac{5}{3}e^{-t}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$+\int_0^t\begin{bmatrix}\frac{5}{3}e^{2(t-\tau)}-\frac{2}{3}e^{-(t-\tau)} & -\frac{5}{3}e^{2(t-\tau)}+\frac{5}{3}e^{-(t-\tau)}\\ \frac{2}{3}e^{2(t-\tau)}-\frac{2}{3}e^{-(t-\tau)} & -\frac{2}{3}e^{2(t-\tau)}+\frac{5}{3}e^{-(t-\tau)}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\cdot 1\,d\tau$$

$$\mathbf{x}(t)=\begin{bmatrix}\frac{5}{3}e^{2t}-\frac{2}{3}e^{-t}\\ \frac{2}{3}e^{2t}-\frac{2}{3}e^{-t}\end{bmatrix}+\int_0^t\begin{bmatrix}\frac{5}{3}e^{2(t-\tau)}-\frac{2}{3}e^{-(t-\tau)}\\ \frac{2}{3}e^{2(t-\tau)}-\frac{2}{3}e^{-(t-\tau)}\end{bmatrix}d\tau$$

Solution of SS Equations (Time response)

$$\mathbf{x}(t) = \begin{bmatrix} \frac{5}{3}e^{2t} - \frac{2}{3}e^{-t} \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} \end{bmatrix} + \begin{bmatrix} \frac{5}{6}\left(e^{2t} - 1\right) - \frac{2}{3}\left(1 - e^{-t}\right) \\ \frac{1}{3}\left(e^{2t} - 1\right) - \frac{2}{3}\left(1 - e^{-t}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2}e^{2t} - \frac{3}{2} \\ e^{2t} - 1 \end{bmatrix}$$

Solution of SS Equations (Time response)

Problem (Method 2)

Repeat computing $e^{\mathbf{A}t}$ if $\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ in an easier way (use eigen decomposition to diagonalize **A** first)

Solution

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2!} + \ldots$$

$$= \mathbf{Q}\mathbf{Q}^{-1} + \mathbf{Q}\Lambda\mathbf{Q}^{-1}t + \mathbf{Q}\Lambda\mathbf{Q}^{-1}\mathbf{Q}\Lambda\mathbf{Q}^{-1}\frac{t^2}{2!} + \ldots.$$

$$= \mathbf{Q}\mathbf{Q}^{-1} + \mathbf{Q}\Lambda t\mathbf{Q}^{-1} + \mathbf{Q}\Lambda^2\frac{t^2}{2!}\mathbf{Q}^{-1} + \ldots.$$

$$= \mathbf{Q}e^{\Lambda t}\mathbf{Q}^{-1}$$

$$= \mathbf{Q}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\mathbf{Q}^{-1}$$

Q: Eigen vector matrix
Λ: diagonal matrix with eigen vaues on main diagonal

Solution of SS Equations (Time response)

Problem (Method 2)

Repeat computing $e^{\mathbf{A}t}$ if $\mathbf{A}=\begin{bmatrix}4 & -5\\ 2 & -3\end{bmatrix}$ in an easier way (use eigen decomposition to diagonalize **A** first)

Solution

$$e^{\mathbf{A}t}=\mathbf{Q}\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}\mathbf{Q}^{-1}$$

$$=\begin{bmatrix}1 & 1\\ 2/5 & 1\end{bmatrix}\begin{bmatrix}e^{2t} & 0\\ 0 & e^{-t}\end{bmatrix}\begin{bmatrix}1 & 1\\ 2/5 & 1\end{bmatrix}^{-1}$$

$$=\begin{bmatrix}1 & 1\\ 2/5 & 1\end{bmatrix}\begin{bmatrix}e^{2t} & 0\\ 0 & e^{-t}\end{bmatrix}\begin{bmatrix}5/3 & -5/3\\ -2/3 & 5/3\end{bmatrix}=\begin{bmatrix}\frac{5}{3}e^{2t}-\frac{2}{3}e^{-t} & -\frac{5}{3}e^{2t}+\frac{5}{3}e^{-t}\\ \frac{2}{3}e^{2t}-\frac{2}{3}e^{-t} & -\frac{2}{3}e^{2t}+\frac{5}{3}e^{-t}\end{bmatrix}$$

Solution of SS Equations (Time response)

Interpretation of solution using diagonalization

Back to Homog. Sol. $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$

$$\mathbf{x}(t) = \mathbf{Q}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}\mathbf{Q}^{-1}\mathbf{x}(0)$$

$$= \begin{bmatrix} \vdots & & \vdots \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} \cdots & \mathbf{w}_1^{\mathbf{T}} & \cdots \\ & \vdots & \\ \cdots & \mathbf{w}_n^{\mathbf{T}} & \cdots \end{bmatrix}\mathbf{x}(0)$$

$$= \begin{bmatrix} & & \vdots \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} \mathbf{w}_1^{\mathbf{T}}\mathbf{x}(0) \\ \vdots \\ \mathbf{w}_n^{\mathbf{T}}\mathbf{x}(0) \end{bmatrix}$$

Solution of SS Equations (Time response)

Interpretation of solution using diagonalization

Back to Homog. Sol. $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$

$$\mathbf{x}(t) = \begin{bmatrix} & & \vdots \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1^{\mathbf{T}}\mathbf{x}(0) \\ \vdots \\ \mathbf{w}_n^{\mathbf{T}}\mathbf{x}(0) \end{bmatrix}$$

$$= \begin{bmatrix} \vdots & & \vdots \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t}\mathbf{w}_1^{\mathbf{T}}\mathbf{x}(0) \\ \vdots \\ e^{\lambda_n t}\mathbf{w}_n^{\mathbf{T}}\mathbf{x}(0) \end{bmatrix}$$

$$= \sum_{i=1}^{n} \mathbf{v}_i e^{\lambda_i t}\left(\mathbf{w}_i^{\mathbf{T}}\mathbf{x}(0)\right)$$

Solution of SS Equations (Time response)

Interpretation of solution using diagonalization

Back to Homog. Sol. $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$

$$\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{v}_i e^{\lambda_i t} \left(\mathbf{w}_i^{\mathbf{T}} \mathbf{x}(0) \right)$$

- Solution is a linear combination of all individual modes ($e^{\lambda_i t}$)
- Eigenvalues λ_i determine the time behavior of each mode
- Eigenvectors $\mathbf{v}_i$ determine how much each mode impacts each of the state variables
- Rows of $\mathbf{Q}^{-1}$, denoted by $\mathbf{w}_i^{\mathbf{T}}$, determine how much each initial state variable contribute to each mode
- Benefit of diagonalization or eigen decomposition is to decouple the modes and write the full time solution as a linear combination of them
- You can also expect that **eigenvalues are related to poles**

From SS model to Transfer Function

SS model

$$\underset{n\times 1}{\dot{\mathbf{x}}(t)} = \underset{n\times n}{\mathbf{A}}\underset{n\times 1}{\mathbf{x}(t)} + \underset{n\times p}{\mathbf{B}}\underset{p\times 1}{\mathbf{u}(t)}$$

$$\underset{m\times 1}{\mathbf{y}(t)} = \underset{m\times n}{\mathbf{C}}\underset{n\times 1}{\mathbf{x}(t)} + \underset{m\times p}{\mathbf{D}}\underset{p\times 1}{\mathbf{u}(t)}$$

Transfer Function

$$\frac{Y(s)}{U(s)}$$

From SS model to Transfer Function

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Take LT $\quad s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$

$$\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{x}(0) + \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\mathbf{U}(s)$$

since $\quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$

$$\therefore \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

$$\therefore \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \underbrace{\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}}_{\text{Initial state response}}\mathbf{x}(0) + \underbrace{\left[\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]}_{\text{Transfer Function Matrix } m\times p}\mathbf{U}(s)$$

Initial state response

Transfer Function Matrix *m*×*p*

From SS model to Transfer Function

$$\mathbf{Y}(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{x}(0) + \left[\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$

To obtain TF, set **x**(0) = 0

$$\mathbf{Y}(s) = \left[\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$

For a SISO system, **Y**(s) and **U**(s) are scalars

$$\frac{Y(s)}{U(s)} = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}$$

From SS model to Transfer Function

Example

Find the transfer function of the following state space model

$$\dot{\mathbf{x}} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} -0.5 & 1 \end{bmatrix}\mathbf{x}$$

Solution

$$\frac{Y(s)}{U(s)} = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}$$

From a previous example with the same matrix **A**

$$\left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \begin{bmatrix} \dfrac{s+3}{(s-2)(s+1)} & \dfrac{-5}{(s-2)(s+1)} \\ \dfrac{2}{(s-2)(s+1)} & \dfrac{s-4}{(s-2)(s+1)} \end{bmatrix}$$

From SS model to Transfer Function

Example

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} -0.5 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{s+3}{(s-2)(s+1)} & \dfrac{-5}{(s-2)(s+1)} \\ \dfrac{2}{(s-2)(s+1)} & \dfrac{s-4}{(s-2)(s+1)} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{-2s-1}{(s-2)(s+1)} \\ \dfrac{s-8}{(s-2)(s+1)} \end{bmatrix}$$

$$= \frac{2s-7.5}{(s-2)(s+1)}$$

We notice that the poles at $s = 2,-1$ are exactly the eigenvalues of **A** we found before

From SS model to Transfer Function

HW problem

Find the transfer function matrix of the following SS model having 3 inputs and 2 outputs

$$\dot{\mathbf{x}} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -2 & 3 & 1 \\ 1 & -5 & 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} -0.5 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{x}$$

Hint

You should still find $[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]$ which is a 2x3 matrix that relates the input and output vectors as follows

$$\mathbf{Y}(s) = \left[\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \left[\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right] \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix}$$

From SS model to Transfer Function

Back to SISO case and general TF

$$\frac{Y(s)}{U(s)} = \mathbf{C}\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}+\mathbf{D}$$

$$\frac{Y(s)}{U(s)} = \mathbf{C}\frac{\operatorname{adj}\left(s\mathbf{I}-\mathbf{A}\right)}{\left|s\mathbf{I}-\mathbf{A}\right|}\mathbf{B}+\mathbf{D}$$

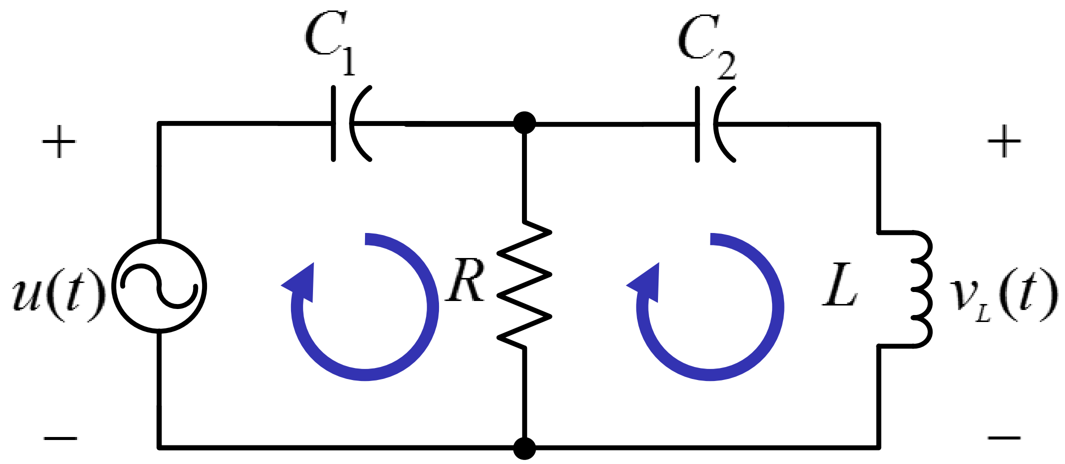
$$= \frac{\mathbf{C}\operatorname{adj}\left(s\mathbf{I}-\mathbf{A}\right)\mathbf{B}+\mathbf{D}\left|s\mathbf{I}-\mathbf{A}\right|}{\left|s\mathbf{I}-\mathbf{A}\right|}$$

- Clearly the poles of the TF are the values of s that makes $|s\mathbf{I}-\mathbf{A}| = 0$ which are the also the eigenvalues of **A**
- We can easily predict that what was said on poles can be exactly said on eigenvalues (e.g. the condition of BIBO stability is Re{eigenvalues} < 0

From SS model to Transfer Function

HW problem

Find two SS representations for the this circuit. Use the underneath two assignments of state variables



<u>State variables (1st)</u>

- $\tilde{x}_1$: current of left loop
- $\tilde{x}_2$: current of right loop

<u>State variables (2nd)</u>

- x_1 : inductor current i_L
- x_2 : capacitor voltage v_c

Find the relation (transformation **P**) between the two state vectors in the 1st and 2nd realizations

$$\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x}$$