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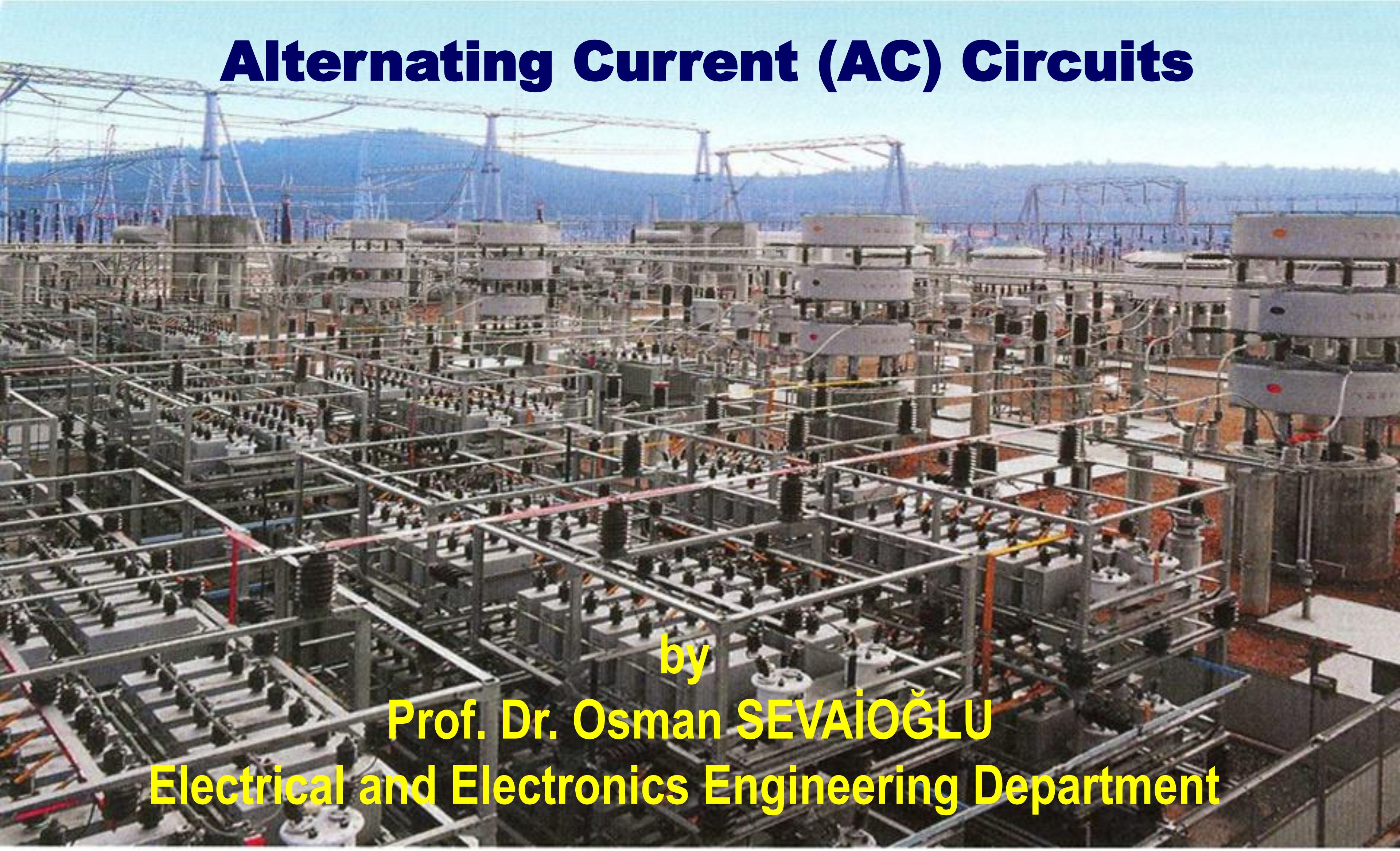
AC Circuits



Alternating Current (AC) Circuits

by

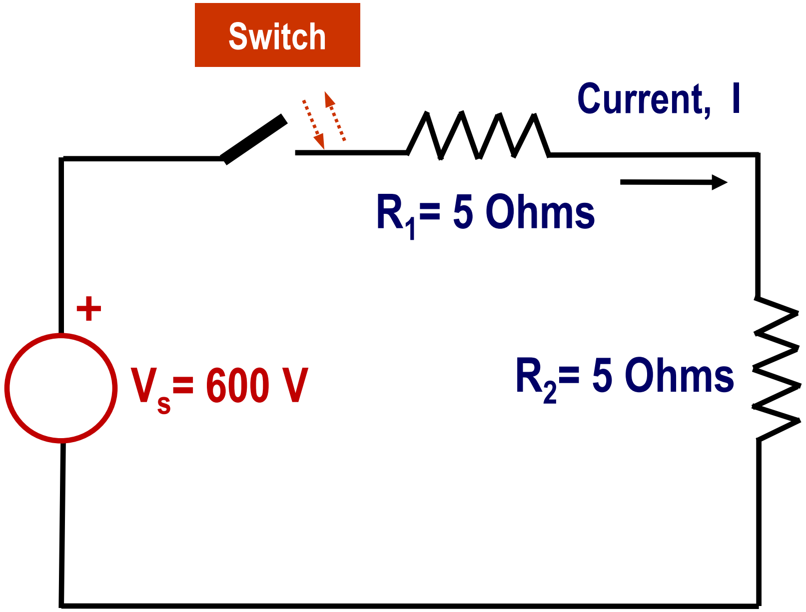
Prof. Dr. Osman SEVAİOĞLU

Electrical and Electronics Engineering Department

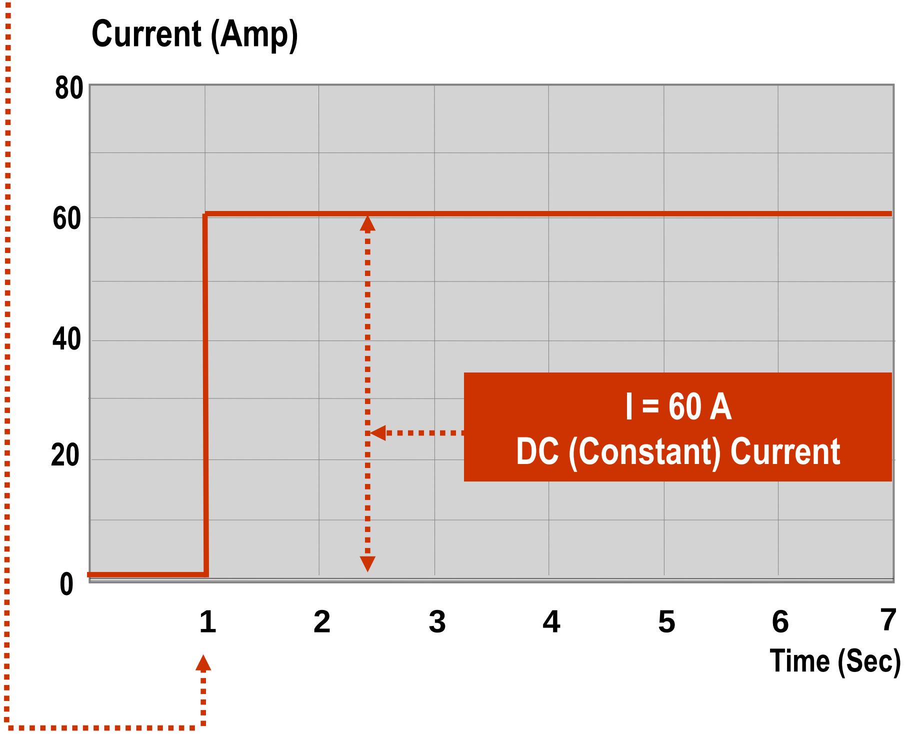
What is Direct Current (DC) ?

Definition

Direct Current (DC) is a current, that does not change in time



Switch is turned "on" at: t = 1 sec



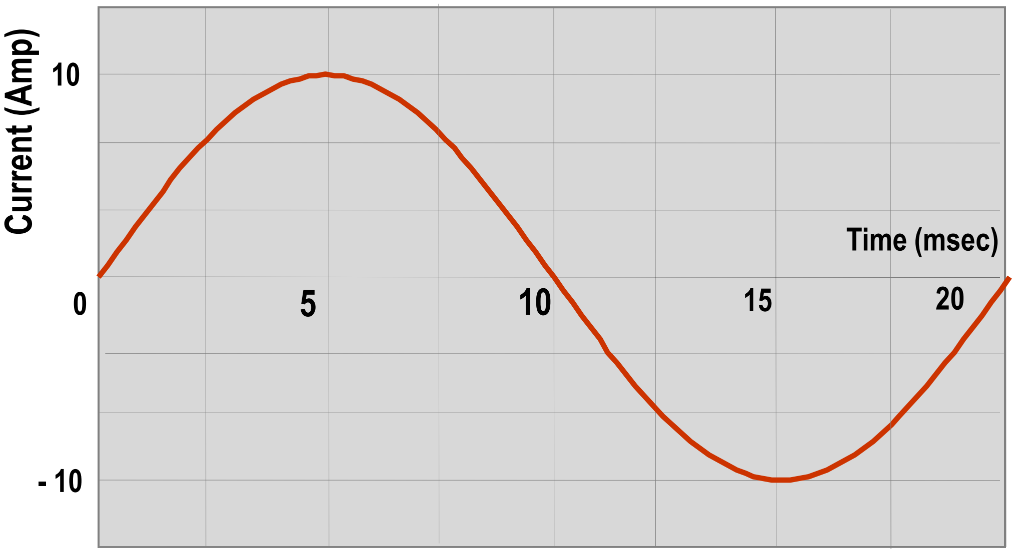
AC Circuits

What is Alternating Current (AC) ?

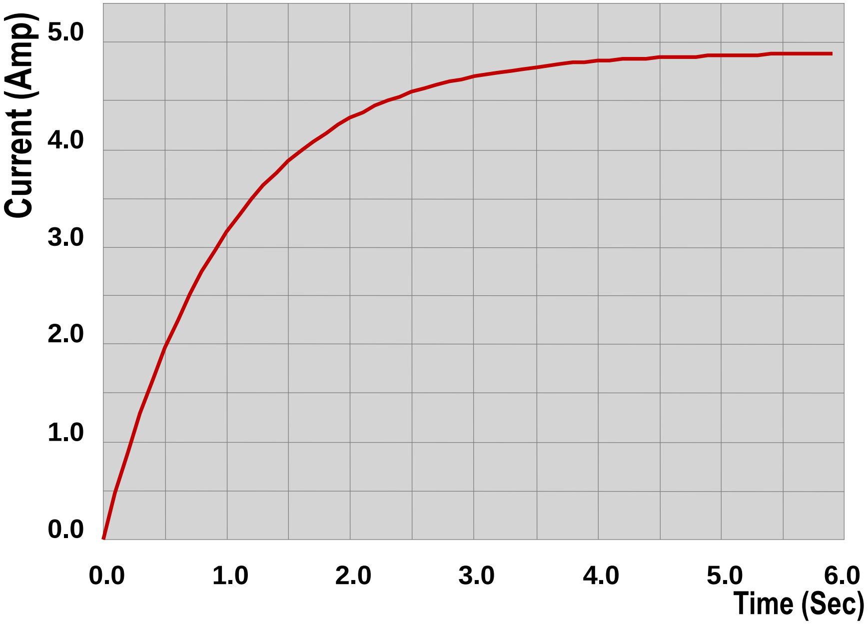
Definition

Alternating Current (AC) is a current, that changes in time

Sinusoidal Alternating Current



Non - Sinusoidal Alternating Current



Parameters of a Sinusoidal Waveform

Definition

Sinusoidal voltage is a voltage with waveform as shown on the RHS

$$V(t) = \hat{V} \sin (wt + \alpha)$$

where

$V(t)$ is the voltage waveform,

$\hat{V}$ is the peak value (amplitude),

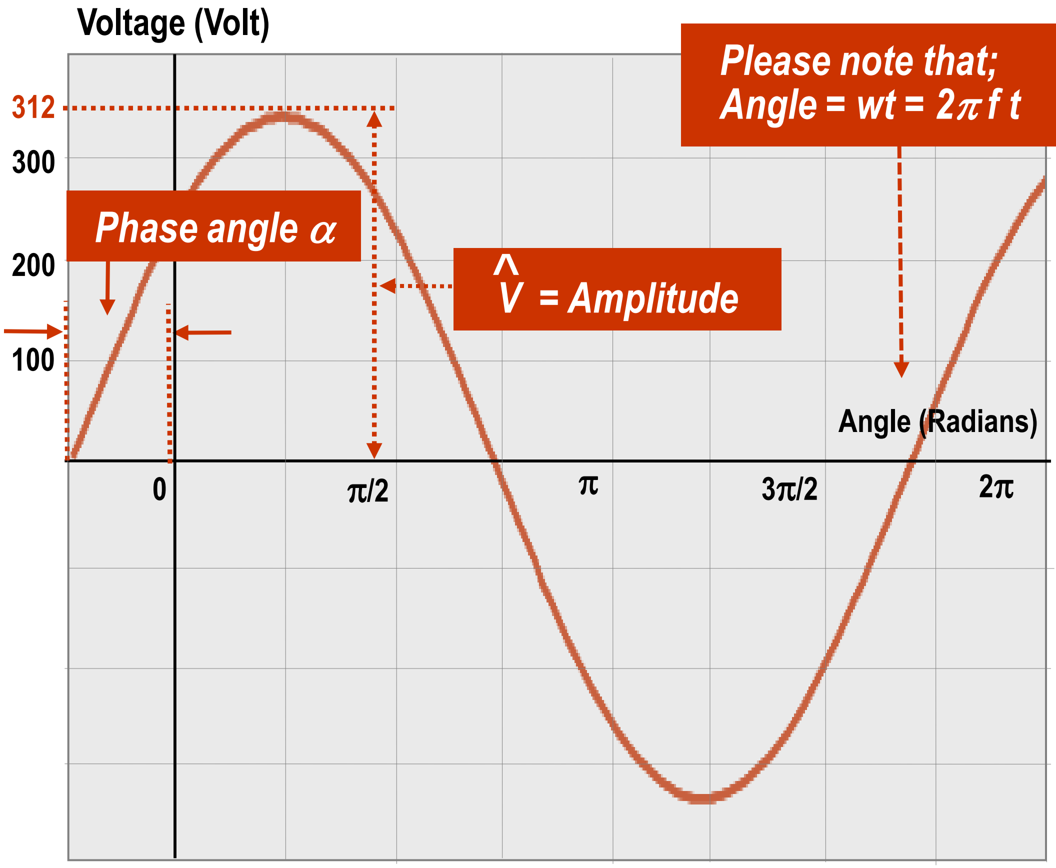
w is the angular frequency,

α is the phase shift, i.e. angle of the voltage at $t = 0$, (phase angle)

$f = 50$ Hz

$w = 2 \pi f = 314$ rad/sec

Sinusoidal Voltage



Please note that;
Angle = $wt = 2\pi f t$

Phase angle α

$\hat{V}$ = Amplitude

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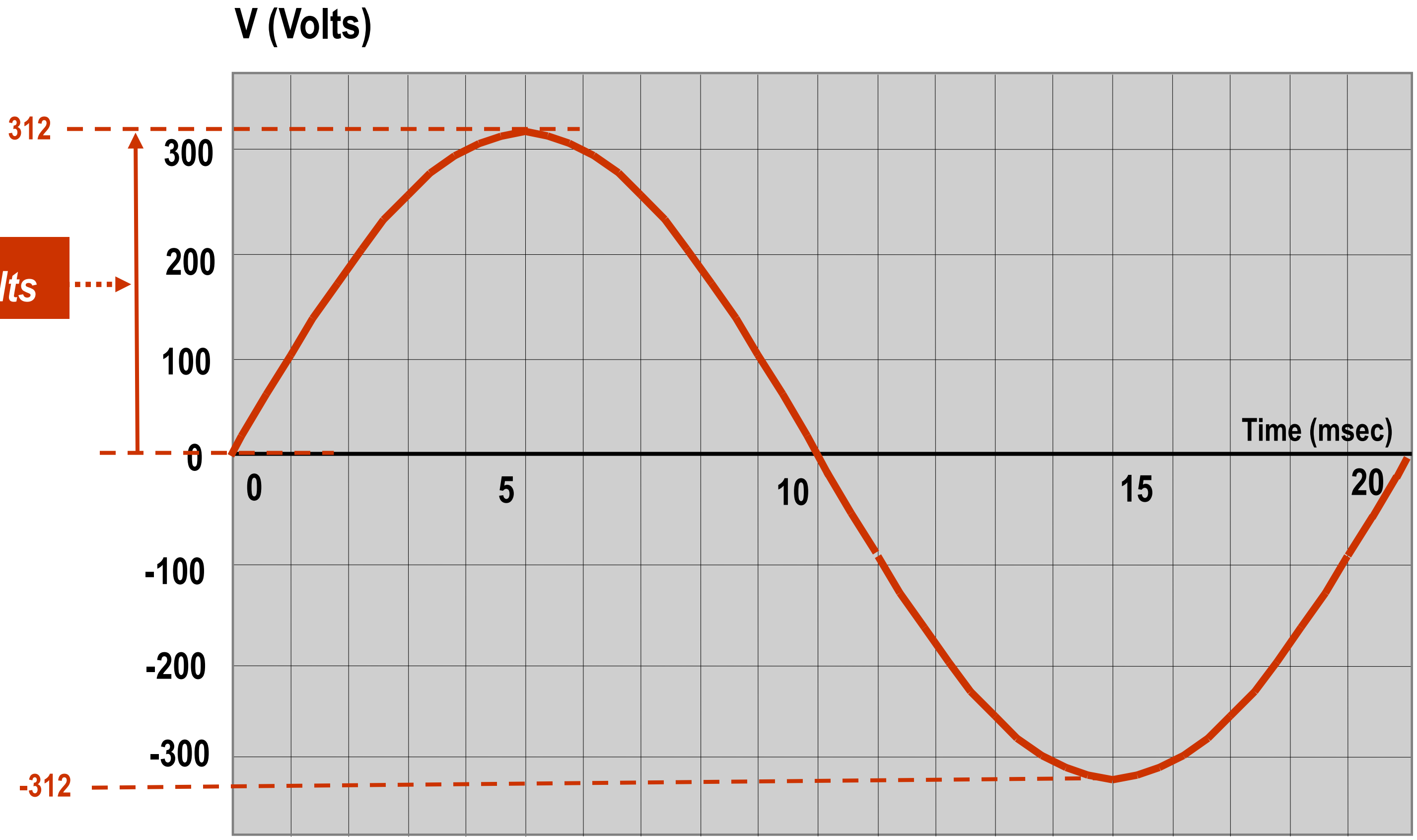
Parameters of a Sinusoidal Waveform

Voltage Waveform



$\hat{V}$ = *Amplitude = 312 Volts*

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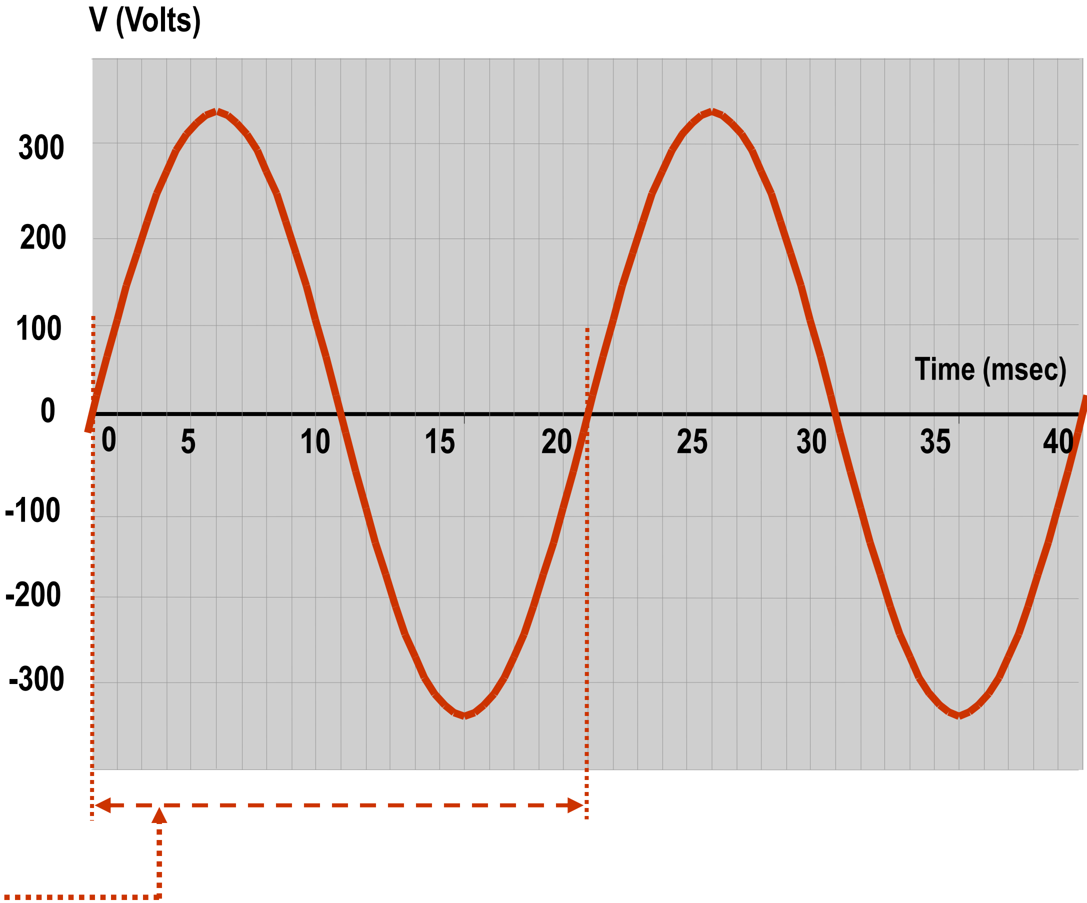
Parameters of a Sinusoidal Waveform

Period and Frequency

Full period = T = 20 msec ⇔ 360 °

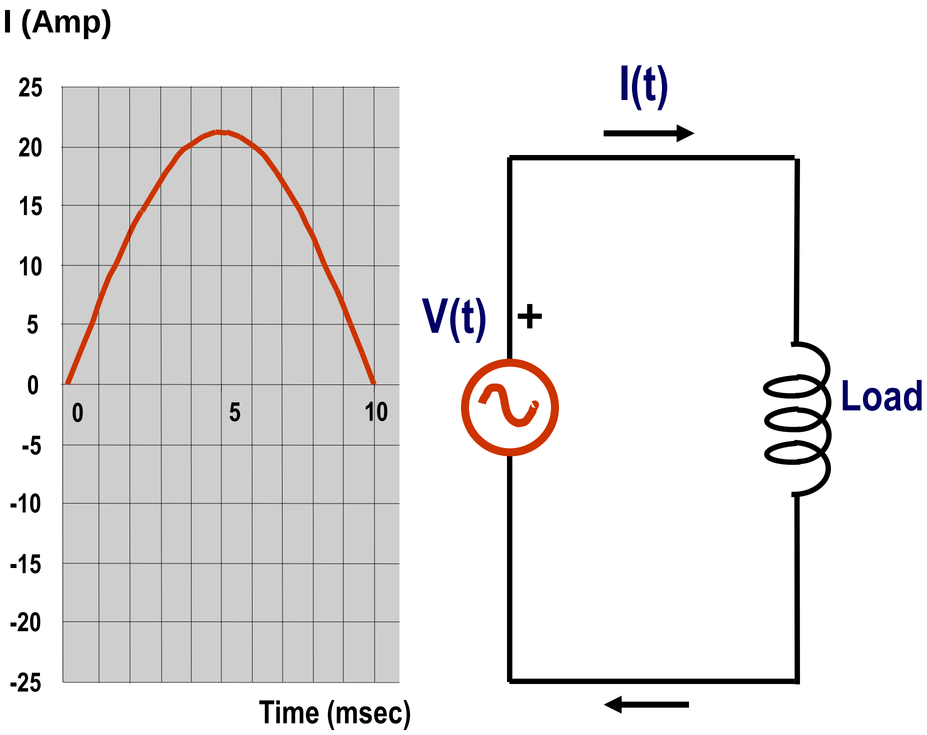
Frequency = $1/T = 1/(20 \times 10^{-3})$
= 50 Hz

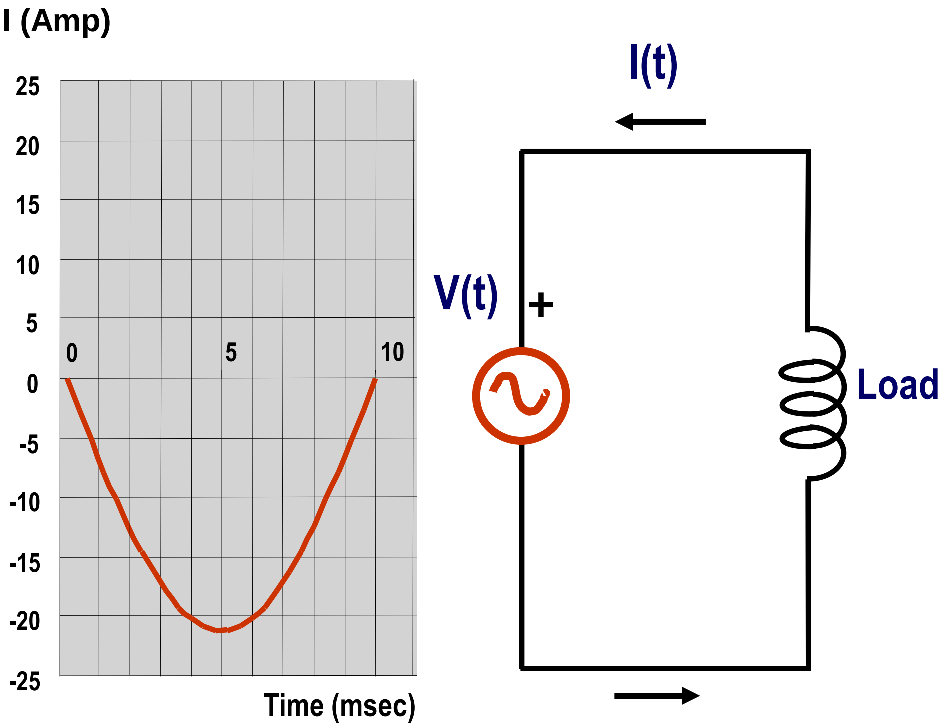


AC Circuits

AC (Alternating Current) Circuit

Positive half cycle	Negative half cycle





AC Circuits

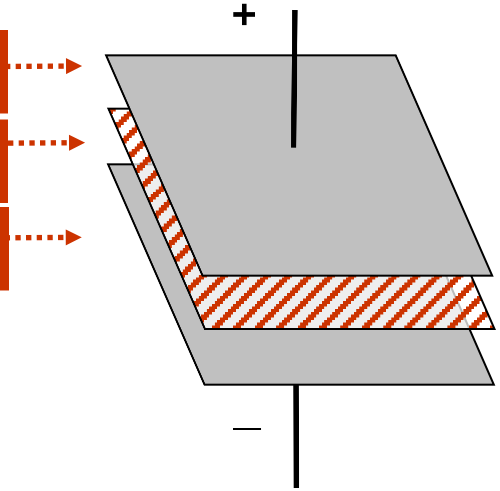
Elements of AC Circuits: Capacitor

Definition

Capacitor is a device that can store electrical charge

The simplest configuration consists of two parallel conducting plates separated by an insulating layer

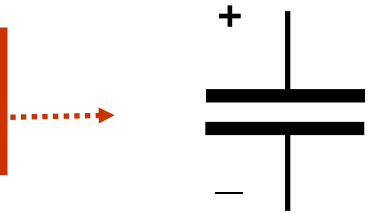
Insulating Layer provides dielectricity (prevents current flow) between positive and negative conductors



Positive conductor

Insulating Layer

Negative conductor



Symbolic representation

Capacitance

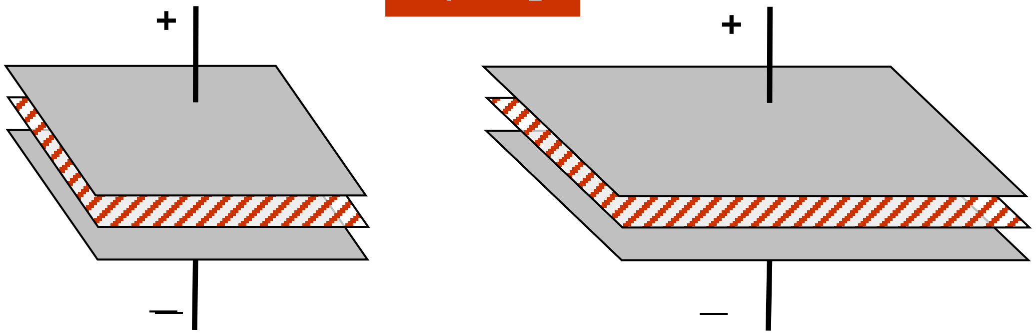
Definition

Capacitors store electrical charge

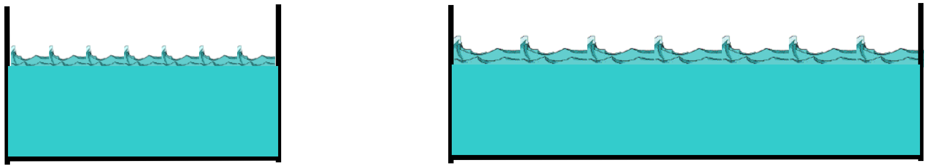
Storage capacity of a capacitor is called **“capacitance”**

Small Capacitance	Large Capacitance
Capacitance = C_1	Capacitance = C_2

$C_1 < C_2$



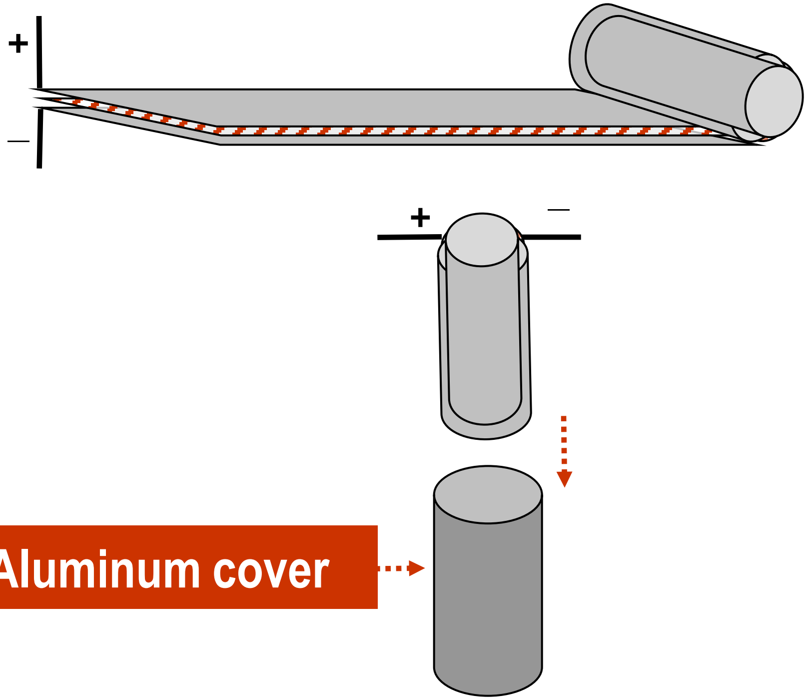
Water (hydroulic) example



Capacitor-Practical Configuration

Geometry

Capacitor plates are packaged in a roll form in order to have smaller size

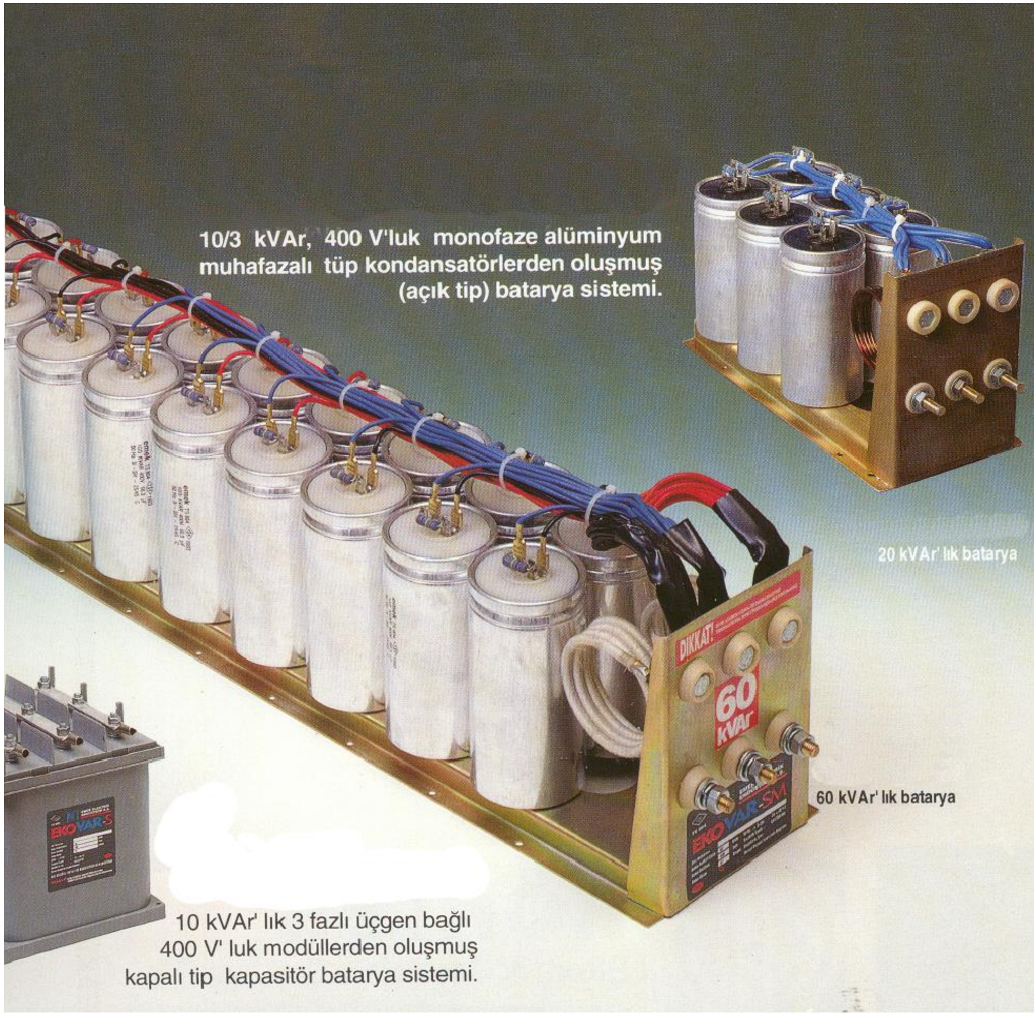


Aluminum cover

Capacitor-Practical Configuration

Geometry

Capacitor cylinders are then connected in parallel in bank form



10/3 kVAr, 400 V'luk monofaze alüminyum muhafazalı tüp kondansatörlerden oluşmuş (açık tip) batarya sistemi.

20 kVAr' lık batarya

60 kVAr' lık batarya

10 kVAr' lık 3 fazlı üçgen bağlı 400 V' luk modüllerden oluşmuş kapalı tip kapasitör batarya sistemi.

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Capacitor-Practical Configuration

Geometry

Capacitor banks

Control relay

AC Circuits

Capacitor-Practical Configuration

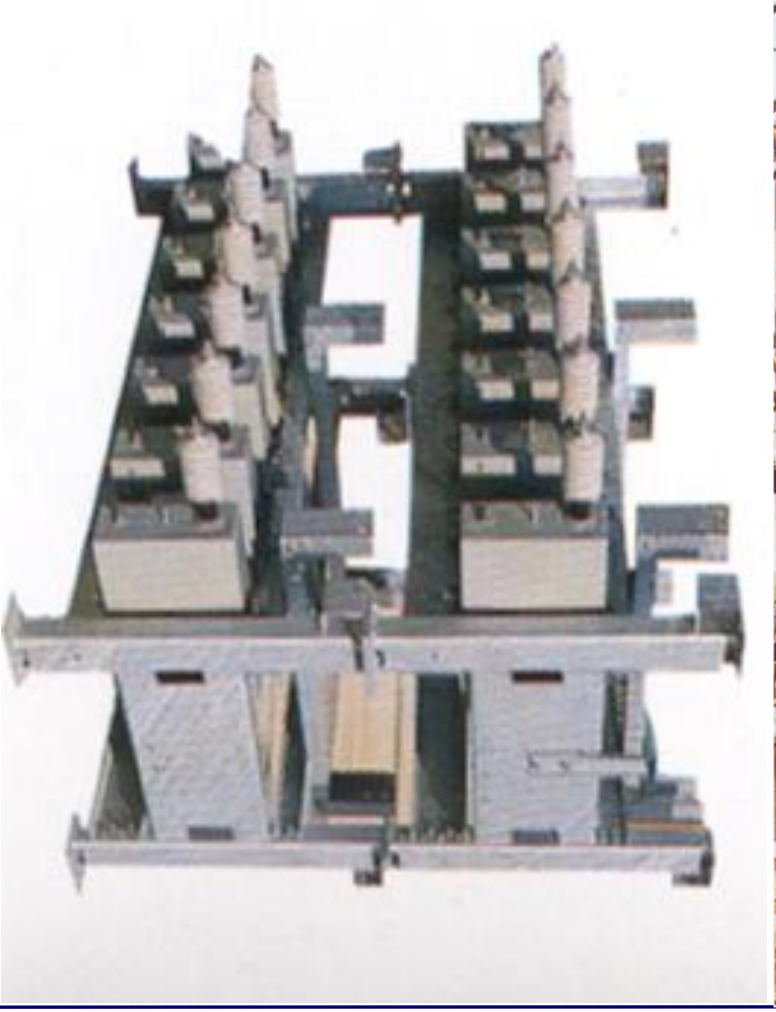
Geometry	*Single Phase*	*Three Phase*
Single and three-phase capacitor banks		

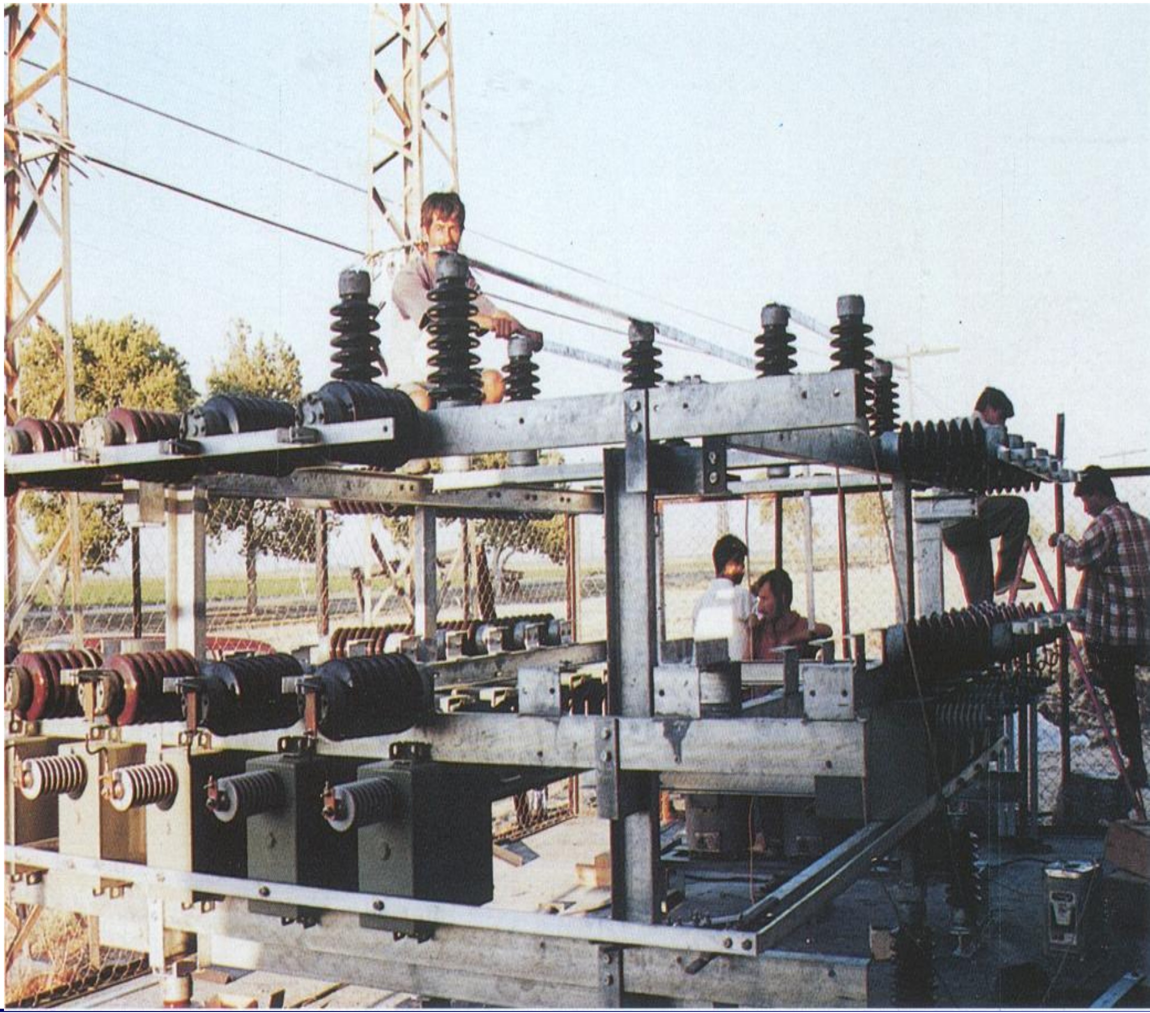
AC Circuits

MV (Medium Voltage) Shunt Capacitor Banks

AC Circuits

MV (Medium Voltage) Shunt Capacitor Banks





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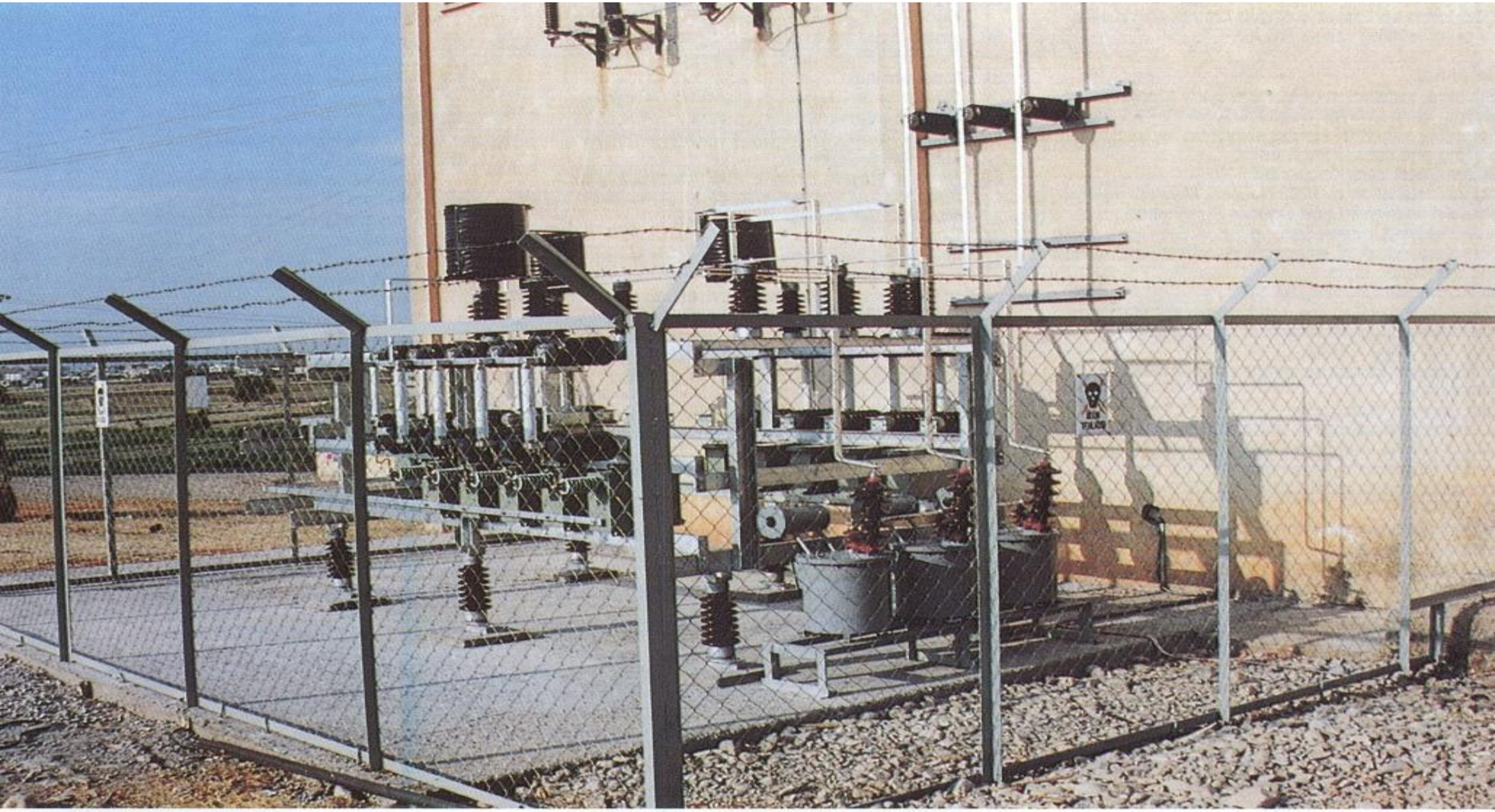
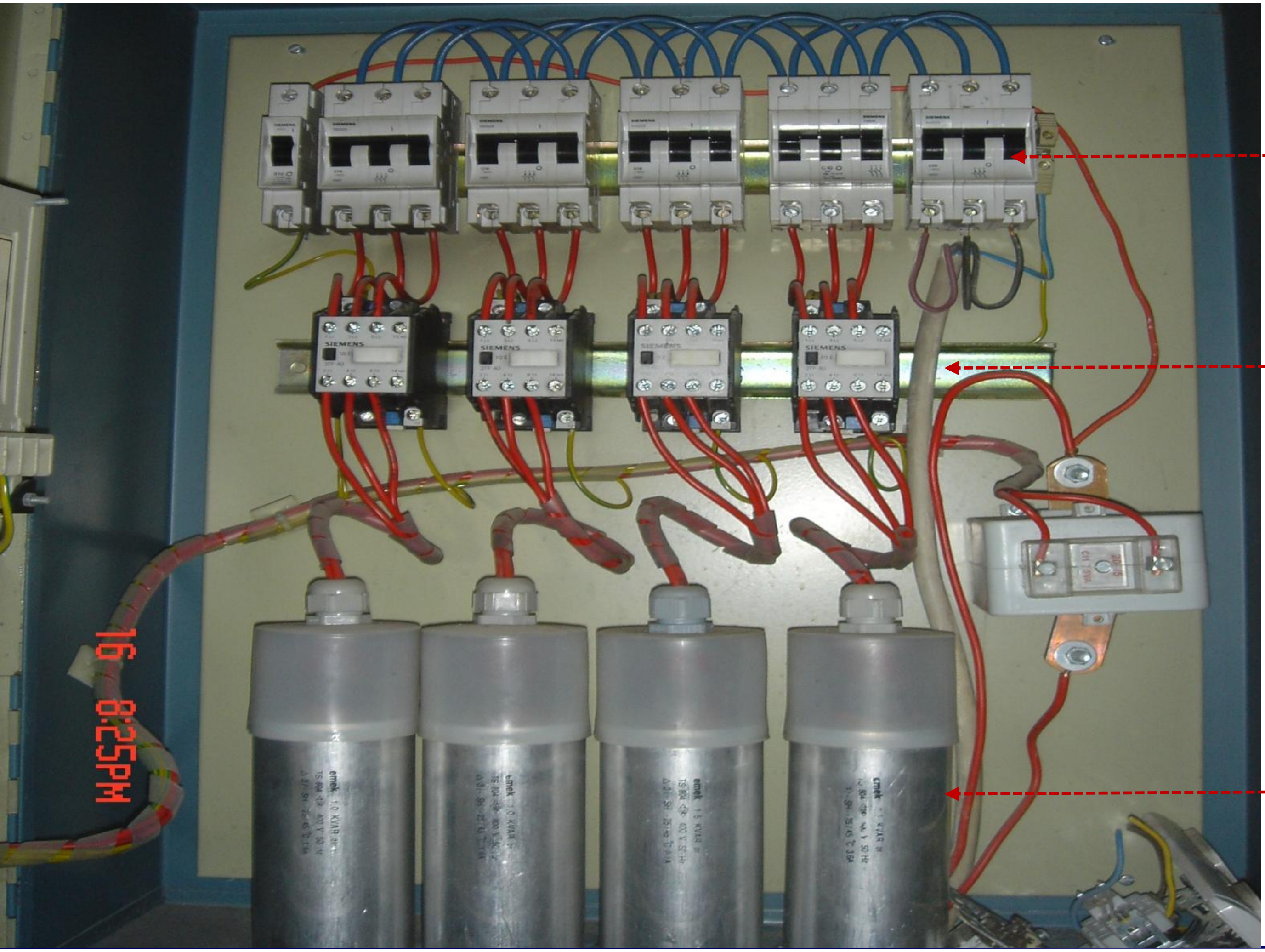
MV (Medium Voltage) Shunt Capacitor Banks



LV (Low Voltage) Capacitor Banks



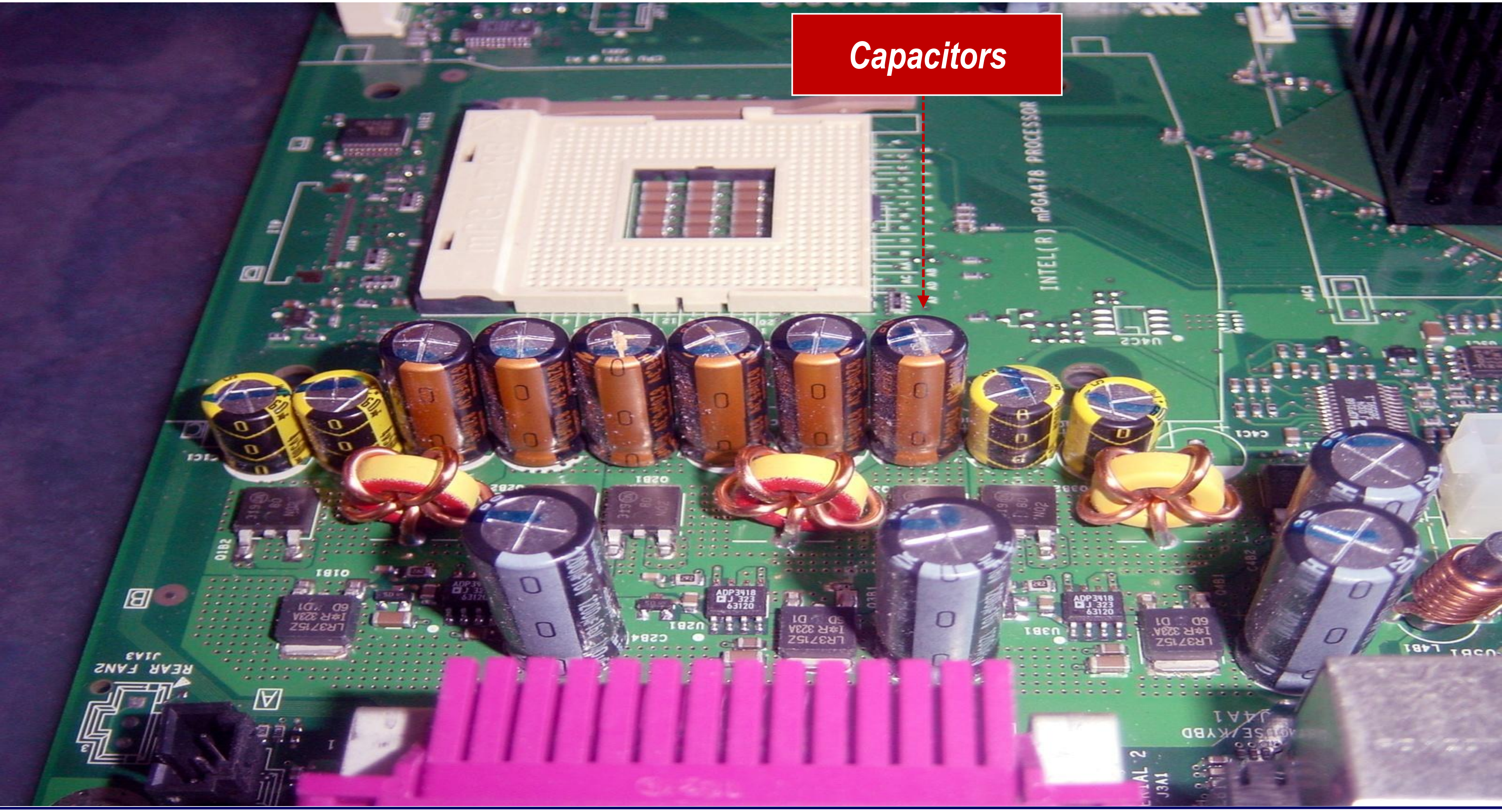
Protective Breakers

Contactors

Capacitor Bank

AC Circuits

Electronic Capacitors in a Motherboard



Capacitors

AC Circuits

Basic Relation

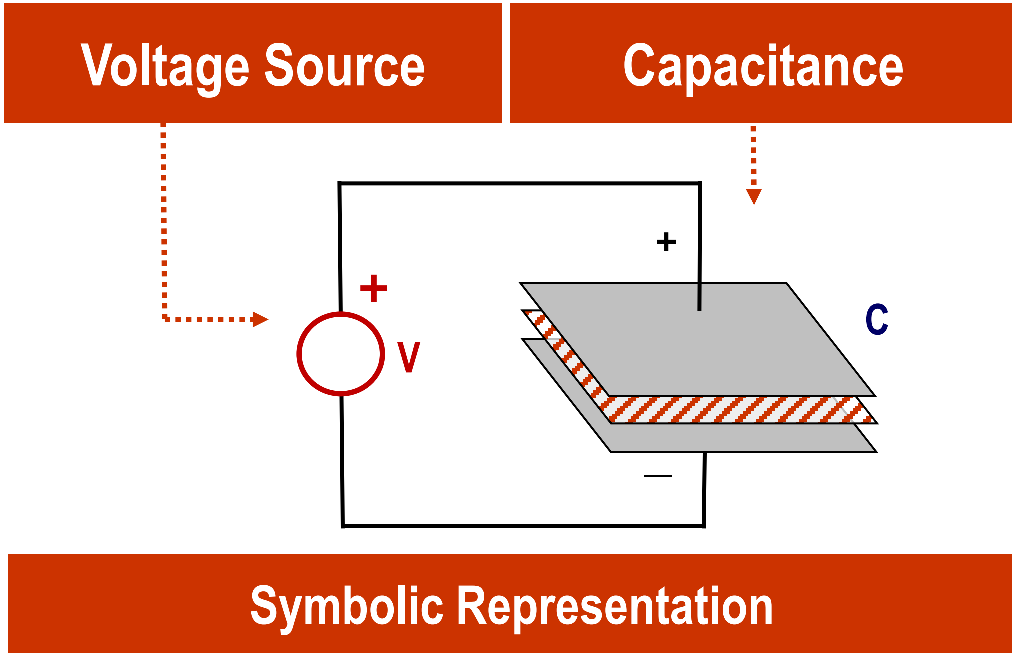
Basic Principle

- Charge stored in a capacitor is proportional to the capacitance C,
- Charge stored in a capacitor is proportional to the voltage V applied

or

Q = C V

where, Q is charge stored (Coulombs),
V is voltage (Volts),
C is capacitance (Farads)



AC Circuits

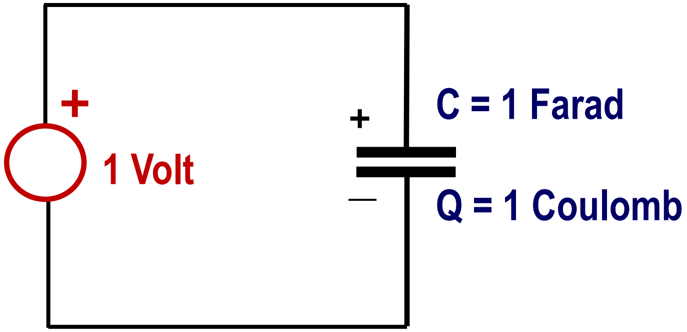
Definition of Farad

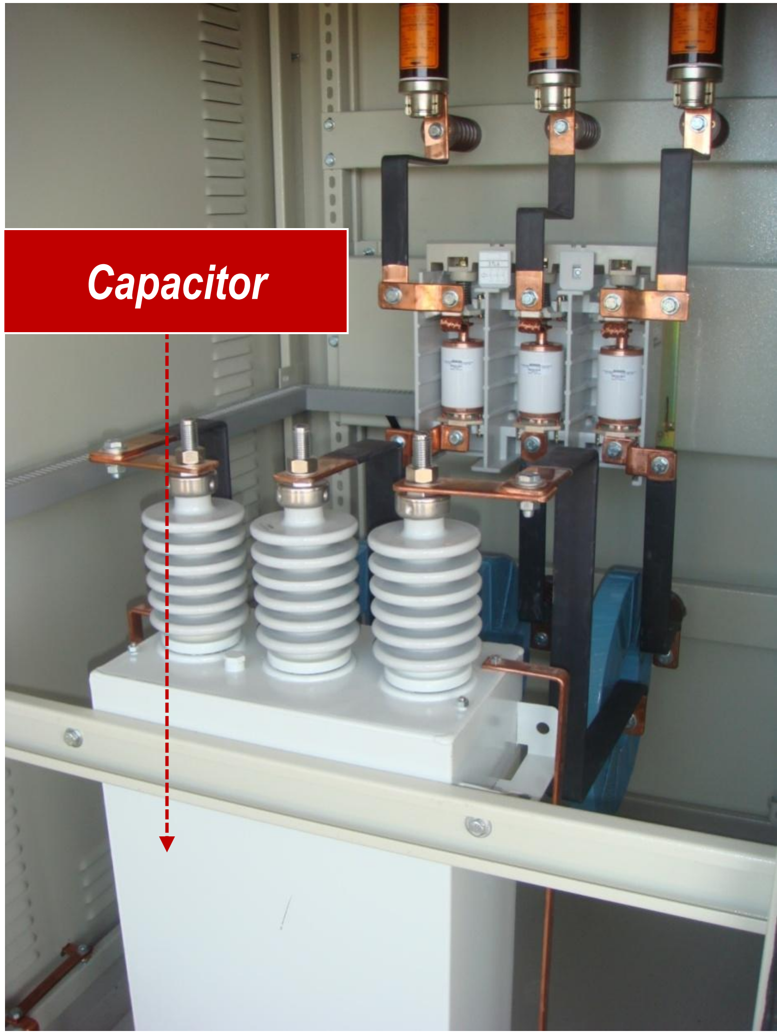
Definition

1 Farad is the capacitance that creates 1 Volt voltage difference between the terminals of the plates when charged by 1 Coulomb of electrical charge

$$Q = C\,V$$

where, Q = 1 Coulomb,
V = 1 Volt,
C = 1 Farad





AC Circuits

Current in a Capacitance

Definition

The relation;

$$Q = C\ V$$

may be written in time domain as;

$$Q(t) = C\ V(t)$$

or differentiating both sides with respect to time

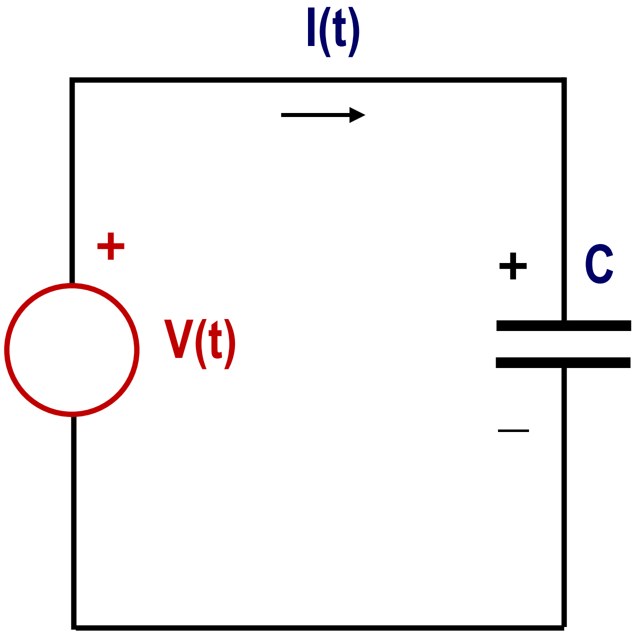
$$dQ(t)/dt = C\ dV(t)/dt$$

remembering that;

$$dQ(t)/dt = I(t)$$

It can be written that;

$$I(t) = C\ dV(t)\ /\ dt$$



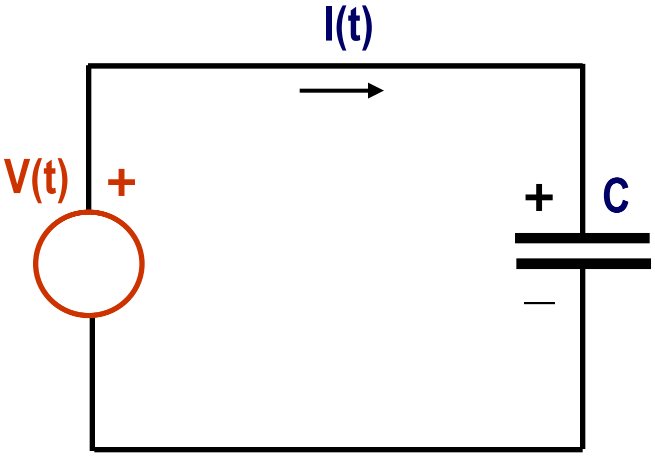
AC Circuits

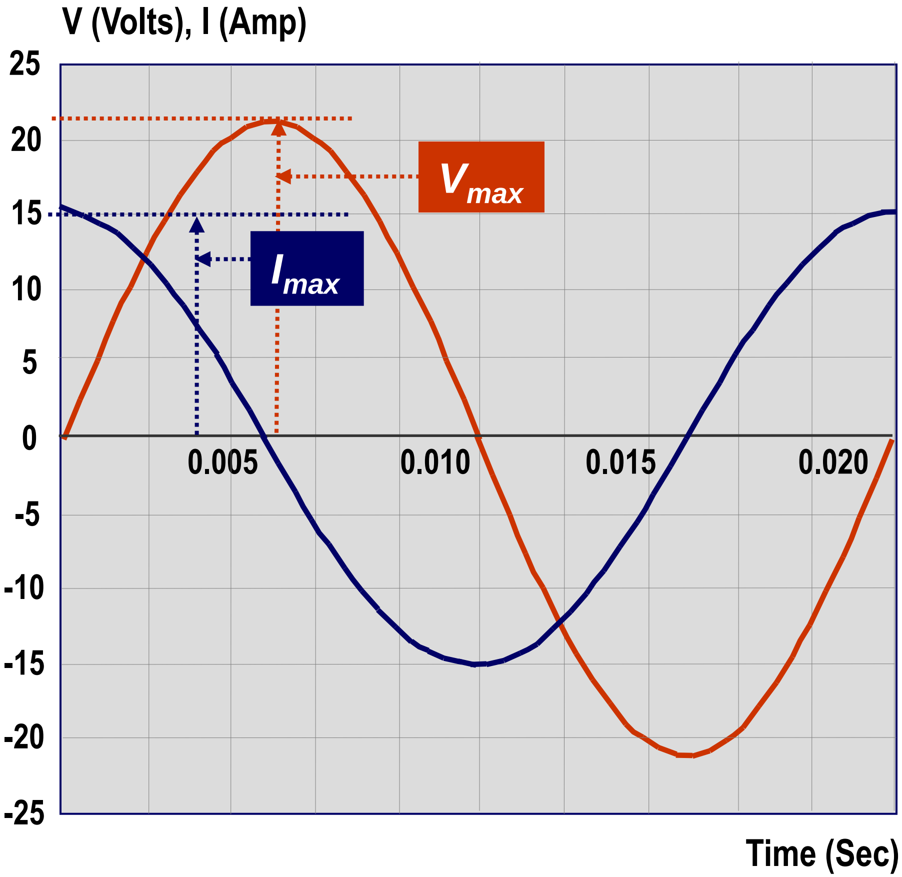
Current in a Capacitance

Phase Shift between Current and Voltage Waveforms

$$I(t) = C\, d\,V(t) / dt$$
$$= C\, d/dt\ \ V_{max} \sin wt$$
$$= C\, V_{max}\, w \cos wt$$
$$= I_{max} \cos wt$$

where, $I_{max} = C\, V_{max}\, w$





AC Circuits

Current in a Capacitance

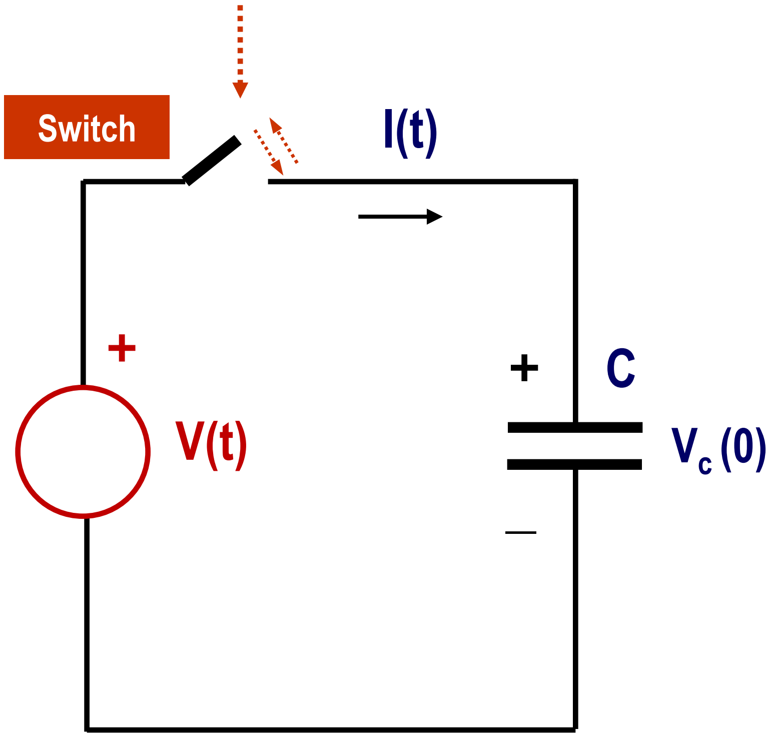
Definition

The above equation may be integrated with respect to time, yielding the following voltage - current relation for a Capacitor

$$V(t) = (1/C) \int I(t)dt + V(0)$$

where *V(0)* is the initial voltage across the capacitor, representing the initial voltage due to the initial charge stored in the capacitor

Switch is turned "on" at: t = 0 sec



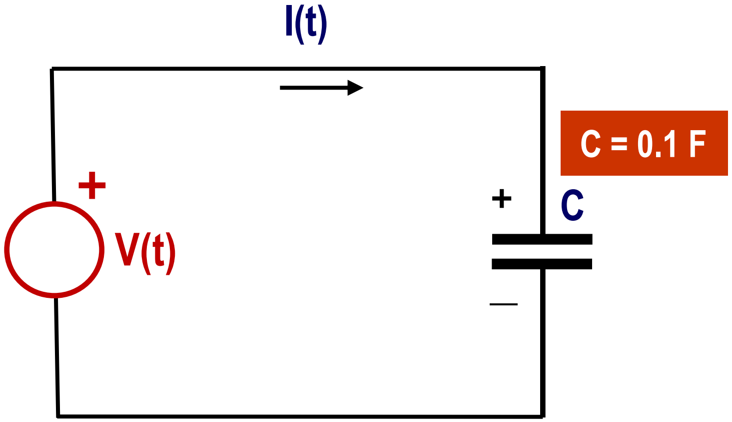
AC Circuits

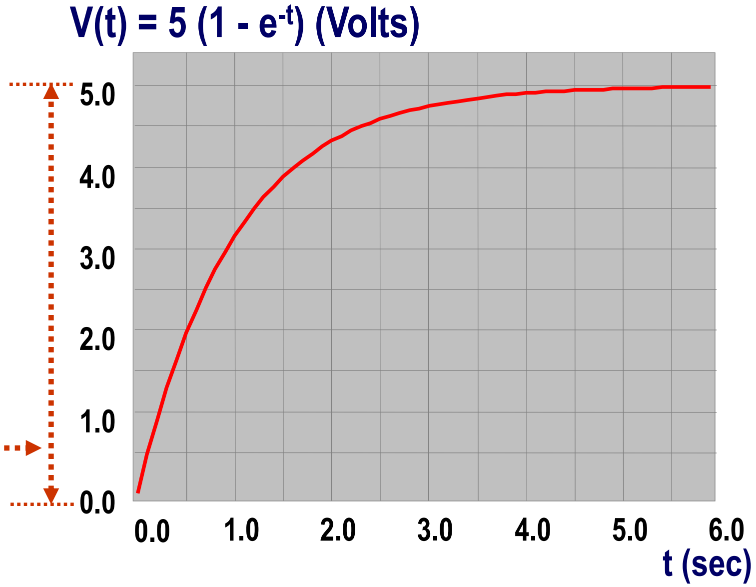
Example - 1

Problem

Determine the time waveform of the current flowing in the circuit shown on the RHS by assuming that the capacitor is charged by the exponential voltage V(t) shown in the figure

$V(t) = 5\ (1 – e^{-t})$ Volts





V_{max} = Maximum voltage that can be reached = 5 Volts

$Q_{max} = C \times V_{max}$ = Maximum charge that can be stored

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AC Circuits

Example - 1

Solution

$I(t) = C\, dV(t)/dt$

$V(t) = 5\,(1 - e^{-t})$ Volts

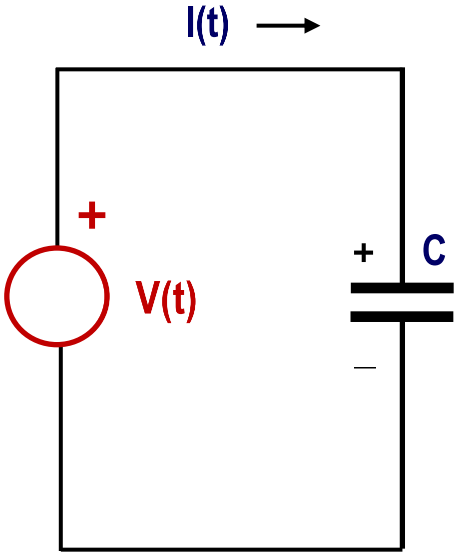
Hence,

$I(t) = C\, d\, V(t) / dt$

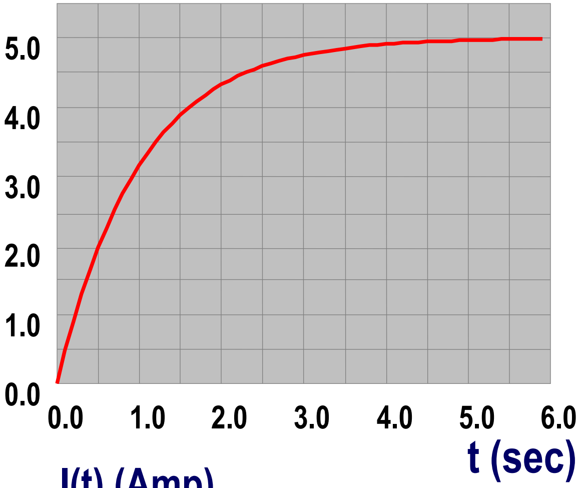
$= C\, d/dt\; 5\,(1 - e^{-t})$

$= 0.1 \times 5\, e^{-t}$

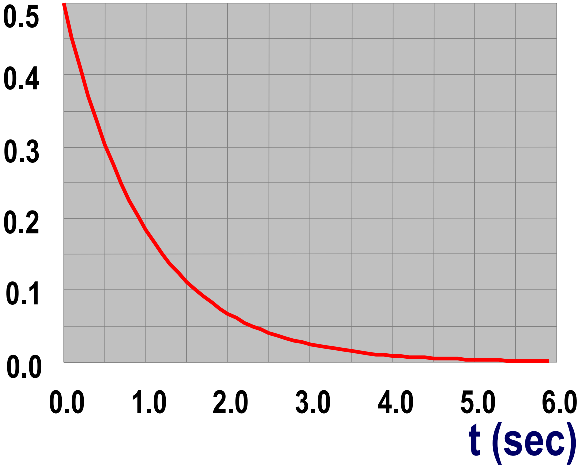
$= 0.5 \times e^{-t}$ Ampers



$V(t) = 5\,(1 - e^{-t})$ (Volts)



I(t) (Amp)



Example - 1

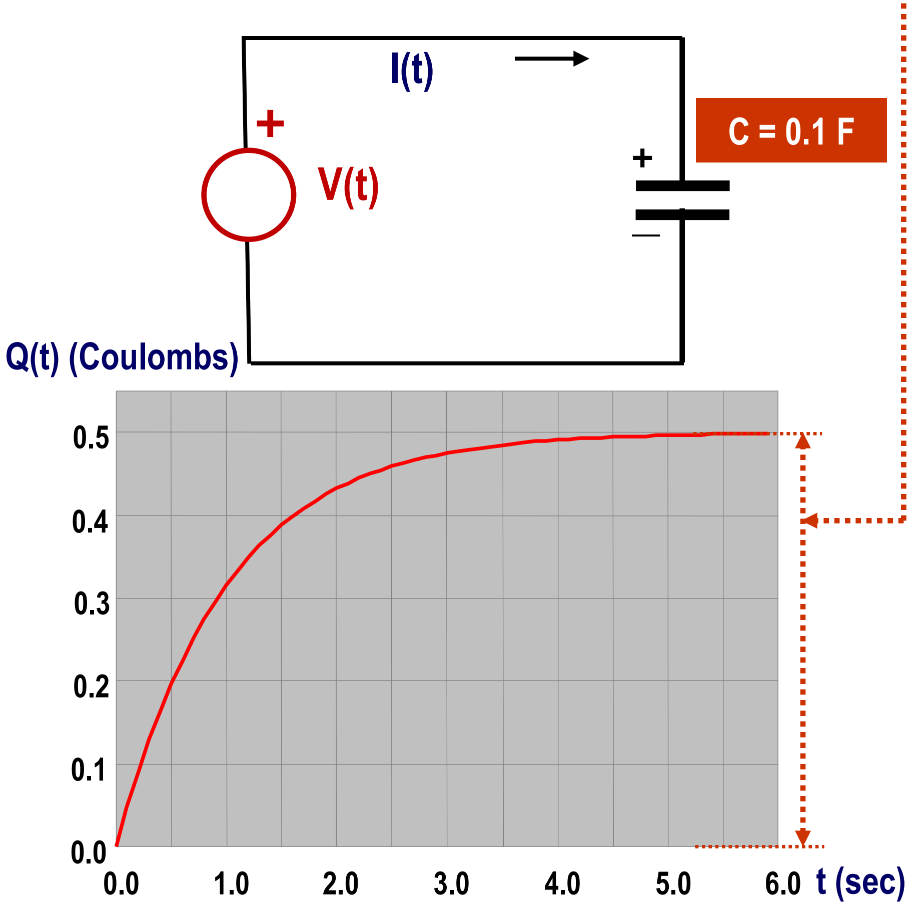
Charge Stored in a Capacitor

Now, determine the time waveform of the charge stored in the capacitor

Charge stored in the capacitor starts from zero and gradually increases to its final value

$$Q(t) = C \times V(t)$$
$$= 0.1 \times 5\,(1 - e^{-t})$$
$$= 0.5 \times (1 - e^{-t}) \text{ Coulombs}$$

$Q_{max} = C \times V_{max}$ = Maximum charge that can be stored



Example - 2

Problem

Current source shown in the circuit shown on the RHS provides 10 A constant current within the time interval;

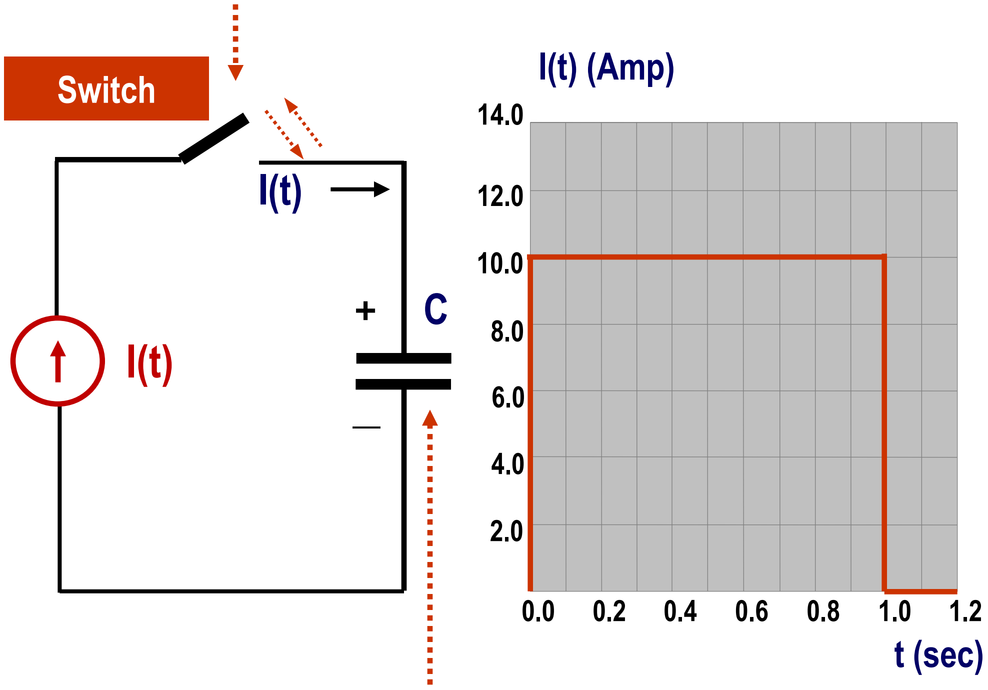
t ε [0, 1 sec]

Capacitor is initially charged to 2 Volts voltage

Determine the voltage across the capacitor within the time interval;

t ε [0, 1 sec]

Switch is turned “on” at: t = 0 sec,
“off” at: t = 1.0 sec.



$C = 1\ F \qquad V_c(0) = 2\ Volts$

AC Circuits

Example - 2

Solution

Voltage across the capacitance can be expressed as

$$V(t) = (1/C) \int I(t)dt + V(0)$$

where,

$$V(0) = 2 \text{ Volts}$$

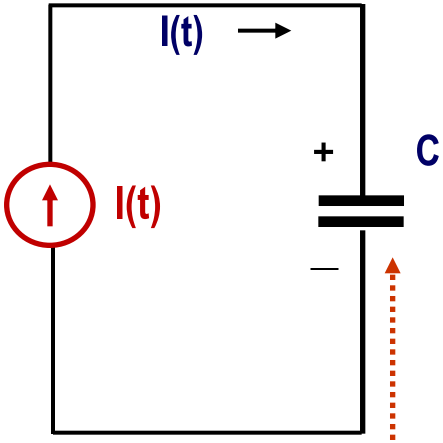
is the initial voltage across the capacitor

Hence;

$$V(t) = (1/C) \int I(t)\, dt + 2$$

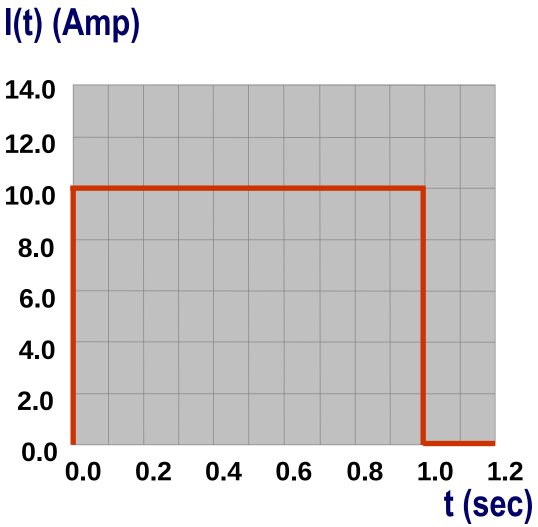
$$= 1 \times 10 \int dt + 2$$

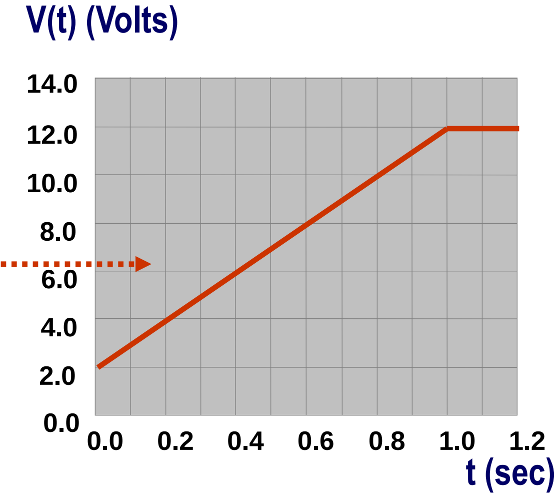
$$= 10\, t + 2 \quad \text{Volts}$$



C = 1 F
$V_c(0)$ = 2 Volts

$V(t) = 10\, t + 2$





Solution of R-C Circuits

Problem

Solve the “RC circuit” shown on the RHS for current waveform I(t) flowing in the circuit when the switch is turned “on” at t = 0

Solution

Writing down KVL for the circuit

$$V(t) = R\,I(t) + V_C(t)$$
$$= R\,I(t) + (1/C) \int I(t)dt + V_c(0)$$

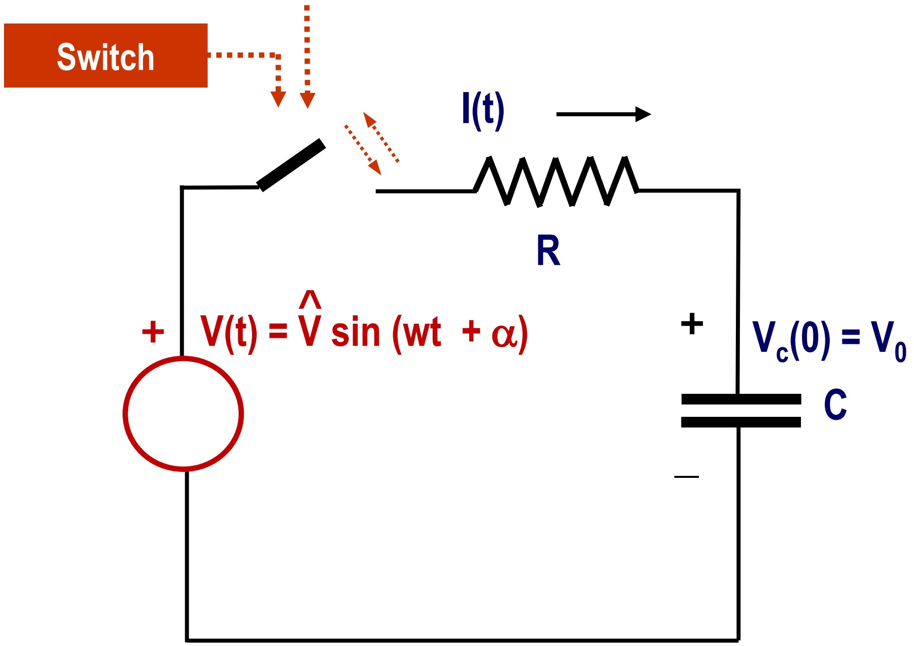
Differentiating both sides wrt time once;

$$d/dt\, V(t) = R\, d/dt\, I(t) + (1/C)\, I(t)$$

or dividing both sides by R

$$d/dt\, I(t) + (1/RC)\, I(t) = (1/R)\, d/dt\, V(t)$$

Switch is turned “on” at: t = 0 sec



$d/dt\, Vc(0) = 0$

A first order ordinary differential equation

AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

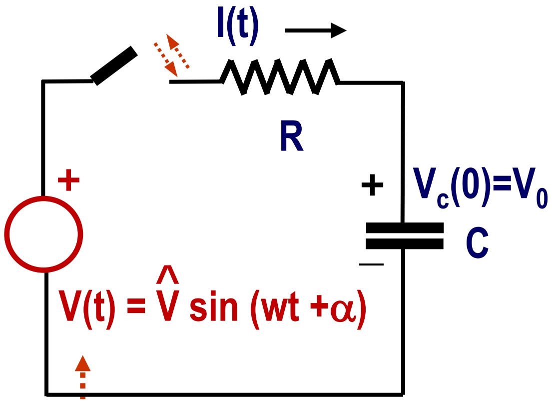
Solution

Solve the resulting first order ordinary differential equation (ODE)

$$\frac{d}{dt} I(t) + (1/RC)\, I(t) = (1/R) \frac{d}{dt} V(t)$$

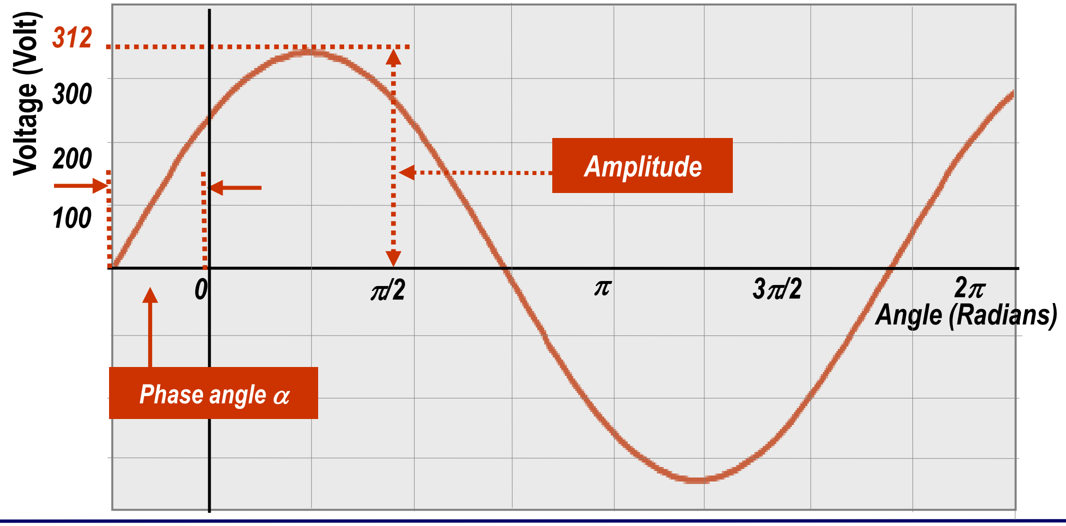
$$\frac{d}{dt} I(t) + (1/RC)\, I(t) = (1/R) \frac{d}{dt} \left(\hat{V} \sin (wt + \alpha) \right)$$

$$\frac{d}{dt} I(t) + (1/RC)\, I(t) = (\hat{V}/R)\, w \cos (wt + \alpha)$$



$$V(t) = \hat{V} \sin (wt + \alpha)$$

$$\frac{d}{dt} V(t) = \hat{V}\, w \cos(wt + \alpha)$$



Amplitude

Phase angle α

AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

Solution

Solve the resulting first order ordinary differential equation (ODE)

$$dI(t) / dt + (1/RC)\, I(t) = (\hat{V}/R)\, w \cos (wt + \alpha)$$

Define an integration factor $\mu(t) = e^{t/RC}$

Multiply both sides of the above ODE by this factor;

$$\mu(t)\, dI(t)/dt + I(t)\, (1/RC)\, \mu(t) = \mu(t)\, (\hat{V}/R)\, w \cos (wt + \alpha)$$

$$\mu(t)\, dI(t)/dt + I(t)\, d/dt\, \mu(t) = (\hat{V}/R)\, \mu(t)\, w \cos (wt + \alpha)$$

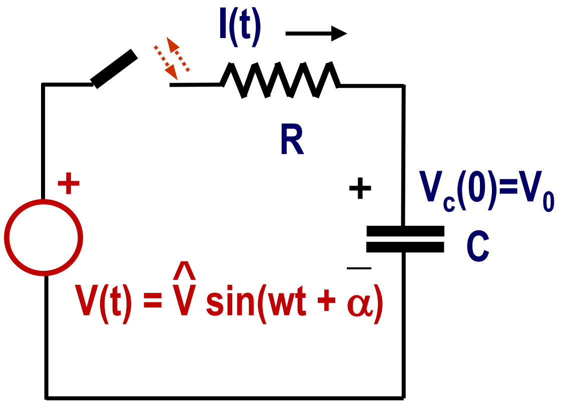
$$d/dt\, [\mu(t)\, I(t)] = (\hat{V}/R)\, \mu(t)\, w \cos (wt + \alpha)$$

$$\cancel{\int d/dt}\, [\mu(t)\, I(t)]\, \cancel{dt} = (\hat{V}/R) \int \mu(t)\, w \cos (wt + \alpha)\, dt + I(0)$$

$$\mu(t)\, I(t) = (\hat{V}/R)\, w \int \mu(t) \cos (wt + \alpha)\, dt + I(0)$$

$$I(t) = \hat{I}\, \mu(t)^{-1}\, w \int \mu(t) \cos(wt + \alpha)\, dt + \mu(t)^{-1} I(0)$$

Switch is turned "on" at:
t = 0 sec



$$d/dt\, \mu(t) = (1/RC)\, \mu(t)$$

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Solution of the resulting First Order Ordinary Differential Equation

Solution (Continued)

Subsituting the integration factor $\mu(t) = e^{t/RC}$ into the above solution;

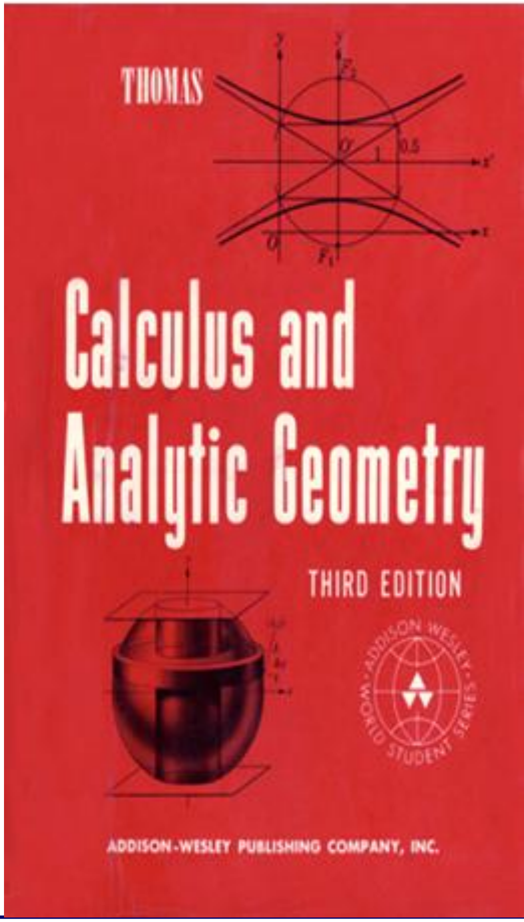
$$I(t) = \hat{I}\, e^{-t/RC}\, w \int e^{t/RC} \cos(wt + \alpha)\, dt + e^{-t/RC}\, I(0)$$

$$= \hat{I}\, e^{-t/RC}\, w \int e^{t/RC} (\cos wt \cos\alpha - \sin wt \sin\alpha)\, dt + e^{-t/RC}\, I(0)$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

<u>Taken from the Reference:</u> Calculus and Analytic Geometry, Thomas, Addison Wesley, Third Ed. 1965

Reference


THOMAS

Calculus and Analytic Geometry

THIRD EDITION

ADDISON-WESLEY PUBLISHING COMPANY, INC.

AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

Solution (Continued)

Now, proceeding;

$$I(t) = \hat{I}\, e^{-t/RC}\, w \int e^{t/RC} (\cos wt \cos\alpha - \sin wt \sin\alpha)\, dt + e^{-t/RC} I(0)$$

$$= \hat{I}\, e^{-t/RC}\, w \left[\cos\alpha \underbrace{\int e^{t/RC} \cos wt\, dt} - \sin\alpha \underbrace{\int e^{t/RC} \sin wt\, dt} \right] + e^{-t/RC} I(0)$$

$$\int e^{ax} \cos bx\ dx = e^{ax} \frac{b \sin bx + a \cos bx}{a^2 + b^2}$$

$$\int e^{ax} \sin bx\ dx = e^{ax} \frac{a \sin bx - b \cos bx}{a^2 + b^2}$$

Taken from the Reference:Calculus and Analytic Geometry, Thomas, Addison Wesley, Third Ed. 1965, pp. 369

AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

Solution (Continued)

Now, proceeding;

$$I(t) = \hat{I}\, e^{-t/RC}\, w\, [\, \cos\alpha \underbrace{\int e^{t/RC} \cos wt\, dt} - \sin\alpha \underbrace{\int e^{t/RC} \sin wt\, dt}\,] + e^{-t/RC}\, I(0)$$

$$\int e^{t/RC} \cos wt\, dt = e^{t/RC} \frac{w \sin wt + (1/RC) \cos wt}{(1/RC)^2 + w^2}$$

$$\int e^{t/RC} \sin wt\, dt = e^{t/RC} \frac{(1/RC) \sin wt - w \cos wt}{(1/RC)^2 + w^2}$$

AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

Solution (Continued)

Now, proceeding;

$$I(t) = \hat{I}\, e^{-t/RC}\, w\, [\cos\alpha \int e^{t/RC} \cos wt\, dt - \sin\alpha \int e^{t/RC} \sin wt\, dt\,] + e^{-t/RC}\, I(0)$$

$$I(t) = \hat{I}\, e^{-t/RC}\, w\, [\cos\alpha\, e^{t/RC}\, \frac{w \sin wt + (1/RC) \cos wt}{(1/RC)^2 + w^2} - \sin\alpha\;\; e^{t/RC}\, \frac{(1/RC) \sin wt - w \cos wt}{(1/RC)^2 + w^2}\,] + e^{-t/RC}\, I(0)$$

$$I(t) = \hat{I}\, \cancel{e^{-t/RC}}\, w\, [\cos\alpha\, \cancel{e^{t/RC}}\, \frac{w \sin wt + (1/RC) \cos wt}{(1/RC)^2 + w^2} - \sin\alpha\, \cancel{e^{t/RC}}\, \frac{(1/RC) \sin wt - w \cos wt}{(1/RC)^2 + w^2}\,] + e^{-t/RC}\, I(0)$$

$$I(t) = \frac{\hat{I}\, w}{(1/RC)^2 + w^2}\, [\cos\alpha\; [w \sin wt + (1/RC) \cos wt\,] - \sin\alpha\; [(1/RC) \sin wt - w \cos wt\,]\,] + e^{-t/RC}\, I(0)$$

$$I(t) = \frac{\hat{I}\, w}{(1/RC)^2 + w^2}\, [\,(w\cos\alpha - (1/RC) \sin\alpha\,) \sin wt + ((1/RC) \cos\alpha + w \sin\alpha\,)\; \cos wt\,] + e^{-t/RC}\, I(0)$$

AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

Solution (Continued)

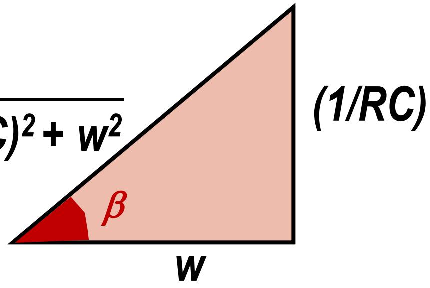
$$I(t) = \frac{\hat{I}\, w}{(1/RC)^2 + w^2} \left[(w\cos\alpha \; - (1/RC)\sin\alpha)\sin wt + ((1/RC)\cos\alpha + w\sin\alpha)\; \cos wt \; \right] + e^{-t/RC} I(0)$$

$$I(t) = \frac{\hat{I}\, w}{\sqrt{(1/RC)^2 + w^2}} \left[\frac{w\cos\alpha \; - \; (1/RC)\sin\alpha}{\sqrt{(1/RC)^2 + w^2}} \sin wt + \frac{(1/RC)\cos\alpha + w\sin\alpha}{\sqrt{(1/RC)^2 + w^2}} \cos wt \right] + e^{-t/RC} I(0)$$

$$w / \sqrt{(1/RC)^2 + w^2} = \cos\beta$$

$$(1/RC) / \sqrt{(1/RC)^2 + w^2} = \sin\beta$$

$$w^2 + (1/RC)^2 = r^2 \dashrightarrow r = \sqrt{(1/RC)^2 + w^2}$$



AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

Solution (Continued)

$$I(t) = \frac{\hat{I}\, w}{\sqrt{(1/RC)^2 + w^2}} \left[\frac{w\cos\alpha - (1/RC)\sin\alpha}{\sqrt{(1/RC)^2 + w^2}} \sin wt + \frac{(1/RC)\cos\alpha + w\sin\alpha}{\sqrt{(1/RC)^2 + w^2}} \cos wt \right] + e^{-t/RC}\, I(0)$$

$$I(t) = \frac{\hat{I}\, w}{\sqrt{(1/RC)^2 + w^2}} \left[\underbrace{(\cos\beta\cos\alpha - \sin\beta\sin\alpha)}_{\cos\beta\cos\alpha\ -\ \sin\beta\ \sin\alpha = \cos(\alpha + \beta)} \sin wt + \underbrace{(\sin\beta\cos\alpha + \cos\beta\sin\alpha)}_{\sin\beta\ \cos\alpha\ +\ \cos\beta\ \sin\alpha\ =\ \sin(\alpha + \beta)} \cos wt \right] + e^{-t/RC}\, I(0)$$

AC Circuits

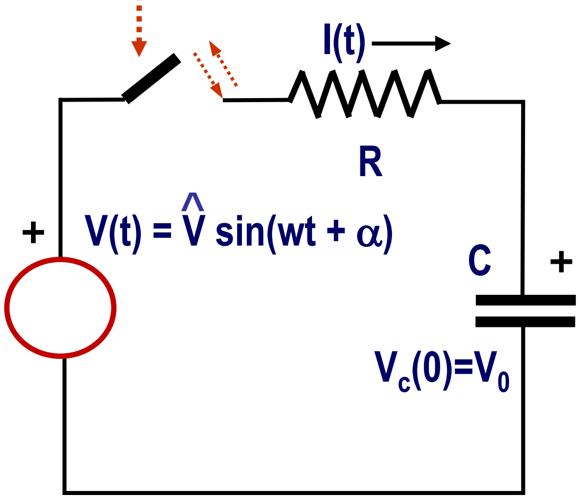
Solution of the resulting First Order Ordinary Differential Equation

Solution (Continued)

$$I(t) = \frac{\hat{I}\, w}{\sqrt{(1/RC)^2 + w^2}} \underbrace{\left[\ \cos(\alpha+\beta)\ \sin wt\ + \sin(\alpha+\beta)\ \cos wt\ \right]}_{\sin(\,wt+\alpha+\beta\,)} + e^{-t/RC}\, I(0)$$

Switch is turned “on” at: t = 0 sec

$$I(t) = \frac{\hat{I}\, w}{\sqrt{(1/RC)^2 + w^2}} \sin(\,wt+\alpha+\beta\,) + e^{-t/RC}\, I(0)$$



AC Circuits

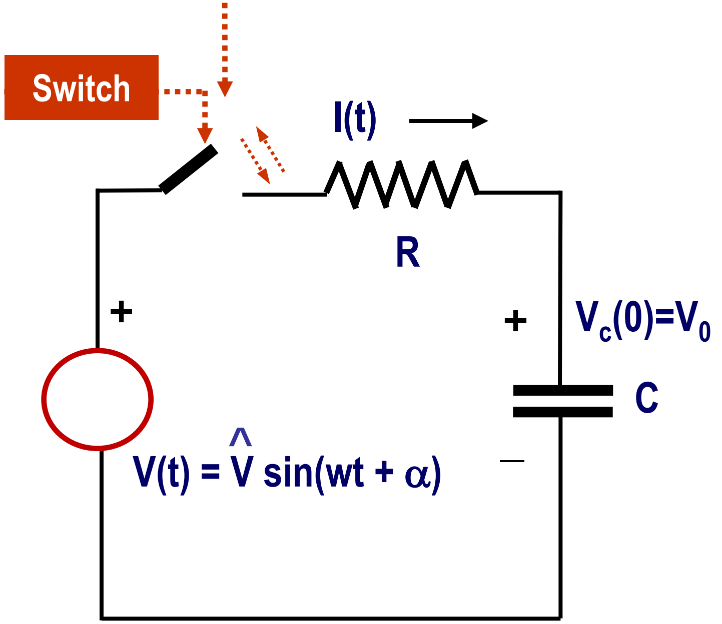
Solution of the resulting First Order Ordinary Differential Equation

Solution (Continued)

Subsituting the above term into the solution;

$$I(t) = \underbrace{\frac{\hat{I}\, w}{\sqrt{(1/RC)^2 + w^2}} \sin(wt + \alpha + \beta)}_{I_{ss}(t) = \text{Steady-State Term}} + \underbrace{e^{-t/RC}\, I(0)}_{I_{tr}(t) = \text{Transient Term}}$$

Switch is turned “on” at: t = 0 sec



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AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

Calculation of Initial Value of I(t)

Initially, the system is an AC circuit operating in steady state, with a resistance R and capacitance C drived by an AC source

$$V(t) = \hat{V} \sin(wt + \alpha)$$

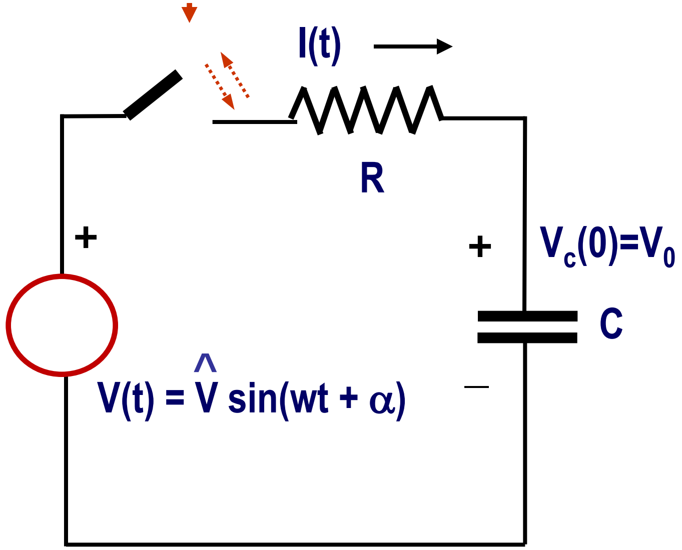
Hence the current *I(t)* depends on the source voltage *V(t)* and *Z = R + j X*

$$I\angle\theta \quad = V\angle\alpha \quad / (R + jX)$$

$$I\angle\theta \quad = V\angle\alpha \quad / Z \angle -\mathrm{Tan}^{-1}(X/R)$$

$$I\angle\theta \quad = \hat{V} \; / Z \quad \angle \alpha + \mathrm{Tan}^{-1}(X/R)$$

Switch is turned “on” at: t = 0 sec



Please note that capacitive reactance has negative angle

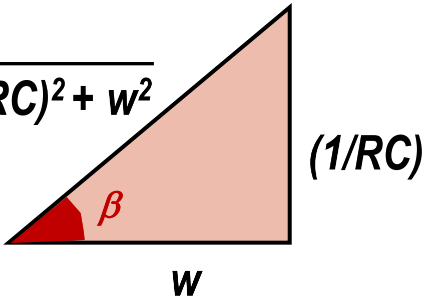
AC Circuits

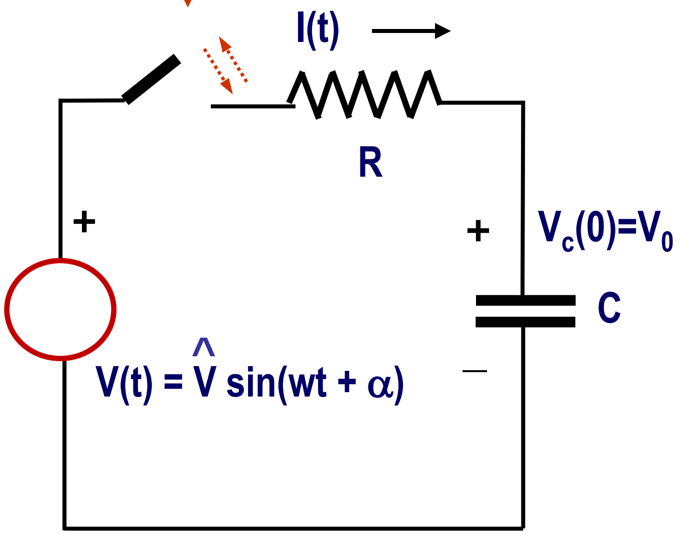
Solution of the resulting First Order Ordinary Differential Equation

Calculation of Initial Value of I(t)

Switch is turned "on" at: t = 0 sec

$w^2 + (1/RC)^2 = r^2$ ⇢ $r = \sqrt{(1/RC)^2 + w^2}$





$X = 1 / (wC) \rightarrow \quad X / R = 1 / (wRC)$

$Tan^{-1} (X / R) = Tan^{-1} (1 / (wRC) = Tan^{-1}((1/RC) / w) = \beta$

$I (t) = \hat{V} / Z \; \sin (wt + \alpha + Tan^{-1} (X / R)))$

$I (0) = \hat{V} / Z \; \sin (\alpha + Tan^{-1} (X / R))$

$I (0) = \hat{V} / Z \; \sin (\alpha + \beta)$

AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

Solution (Continued)

Subsituting the above term into the solution, the final form of the solution waveform becomes;

$$I(t) = \underbrace{\frac{\hat{I}\, w}{\sqrt{(1/RC)^2 + w^2}} \sin(\,wt + \alpha + \beta\,)}_{I_{ss}(t) = \text{Steady-State Term}} + \underbrace{e^{-t/RC}\ \hat{V}\,/\,Z\ \sin(\alpha + \beta)}_{I_{tr}(t) = \text{Transient Term}}$$

AC Circuits

R-C Circuits: Example

Example

Now assume that the parameters of the circuit on the RHS are as follows;

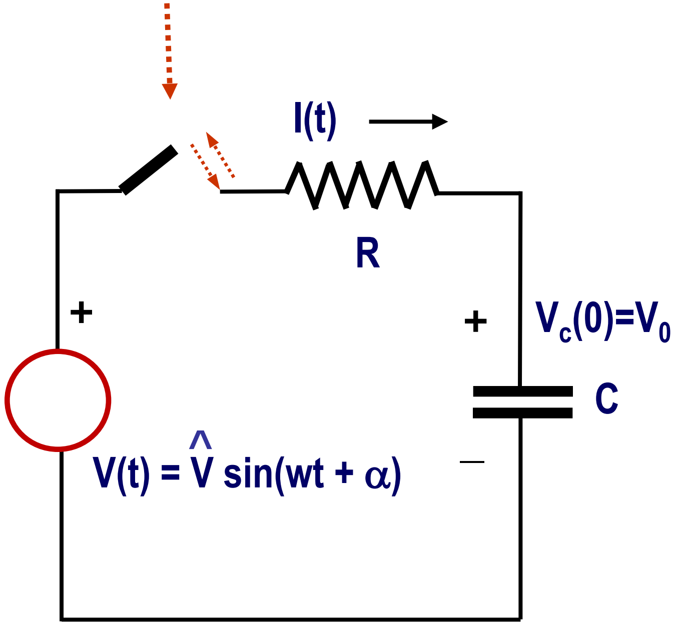
$V(t) = \hat{V} \sin wt = 312 \sin wt$ Volts

$R = 10$ Ohms

$C = 10\ \mu$ Farads

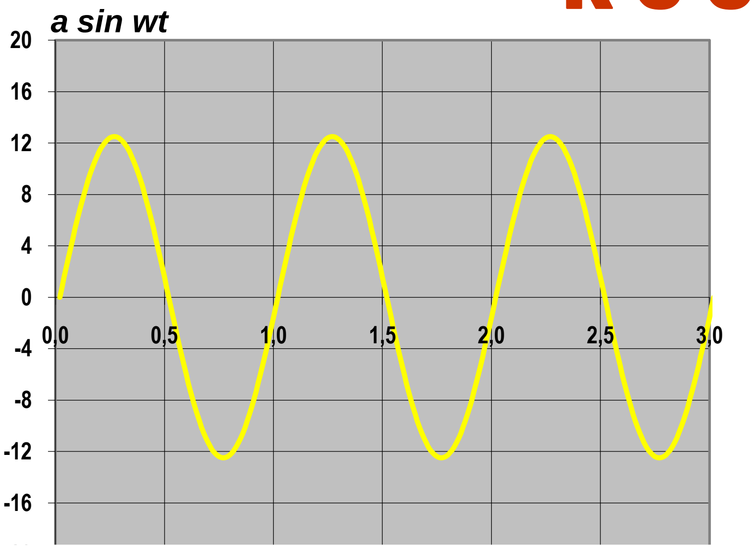
$$I(t) = \underbrace{\frac{\hat{I}\, w}{\sqrt{(1/RC)^2 + w^2}} \sin(wt + \alpha + \beta)}_{I_{ss}(t) = \text{Steady-State Term}} + \underbrace{e^{-t/RC}\, I(0)}_{I_{tr}(t) = \text{Transient Term}}$$

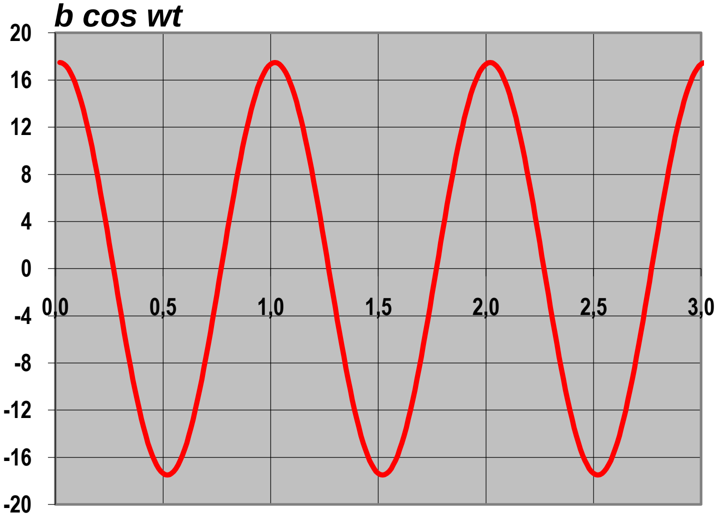
Switch is turned “on” at: t = 0 sec



AC Circuits

R-C Circuits: Example


a sin wt


b cos wt

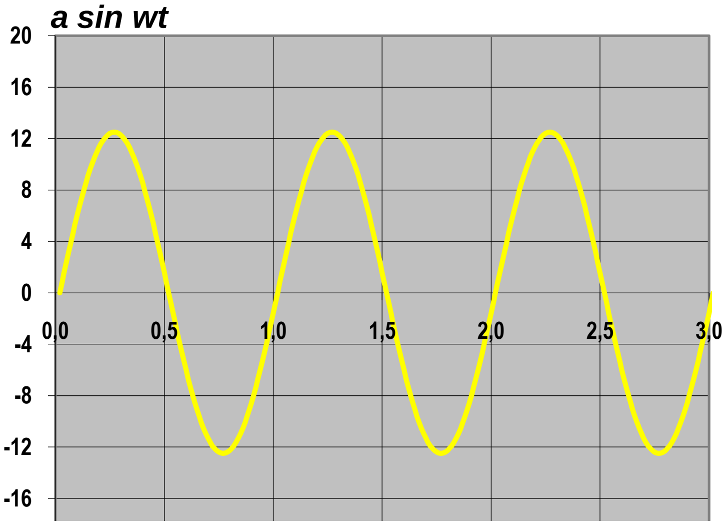
Steady-State Term

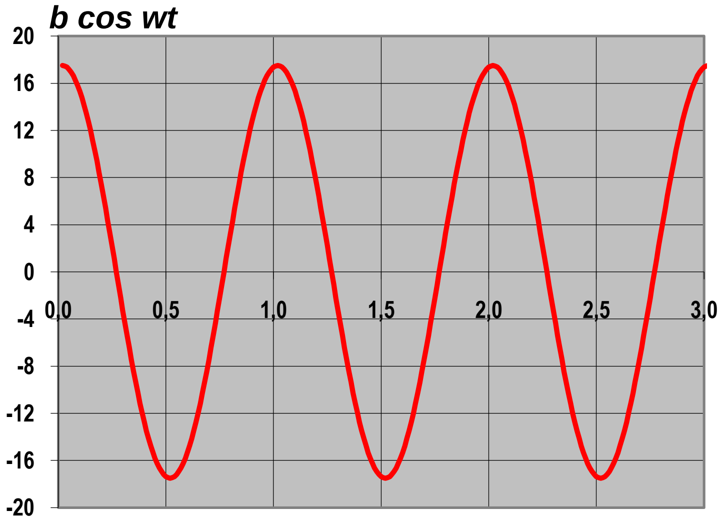
$$I(t) = \frac{\hat{I}\,w}{\sqrt{(1/RC)^2 + w^2}}\left[\underbrace{\cos(\alpha+\beta)}_{a}\sin wt + \underbrace{\sin(\alpha+\beta)}_{b}\cos wt\right]$$

$$a \sin wt + b \cos wt$$

$$= \sin(wt + \alpha + \beta)$$

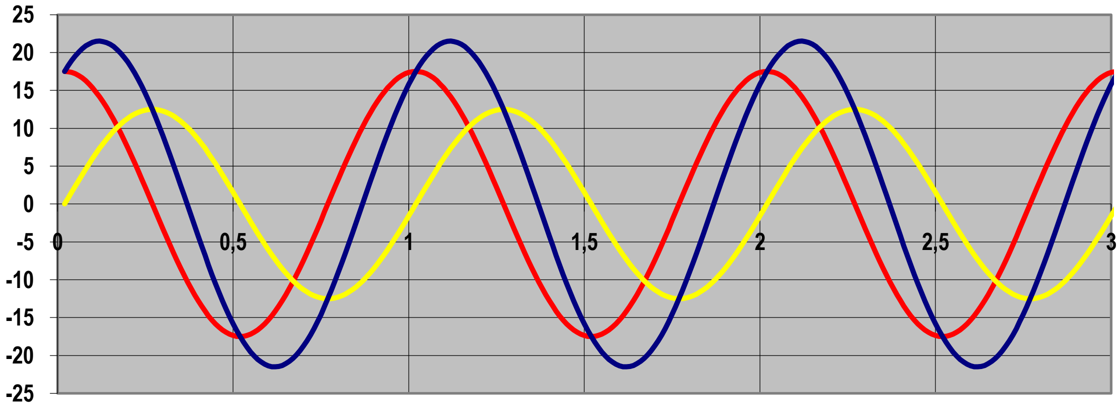
R-C Circuits: Example





Steady-State Term

$$I(t) = \frac{\hat{I}\, w}{\sqrt{(1/RC)^2 + w^2}} \sin(wt + \beta + \alpha)$$



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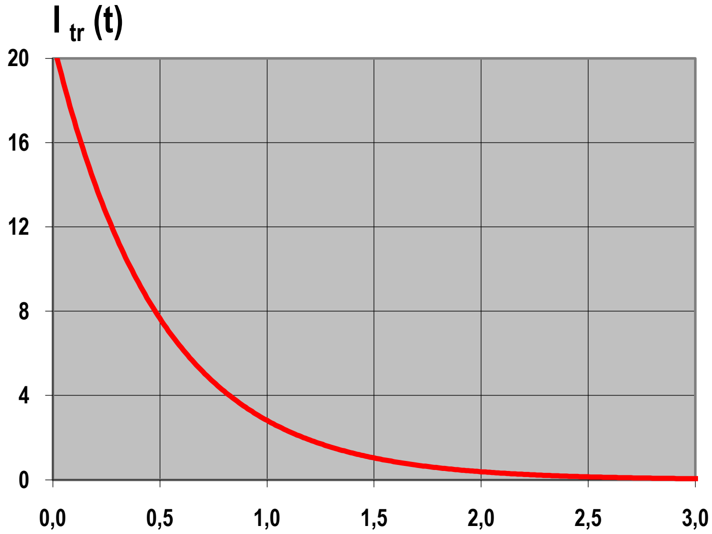
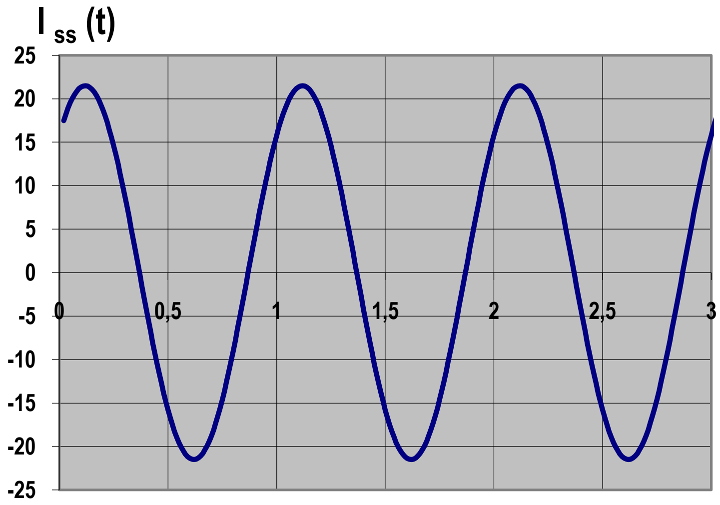
AC Circuits

R-C Circuits: Example

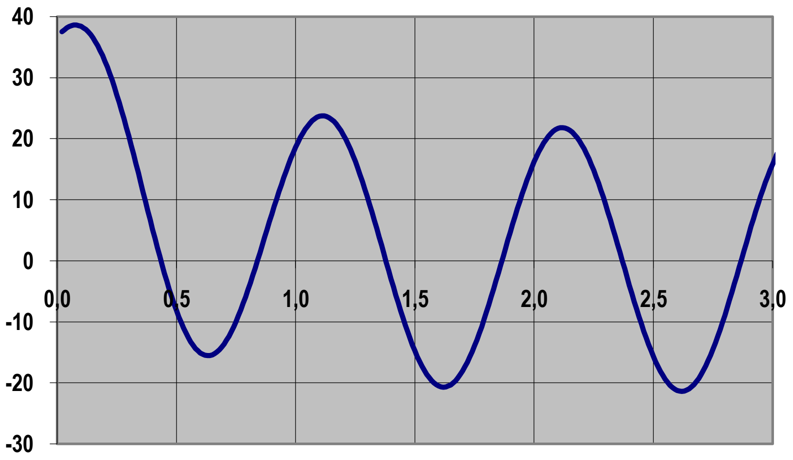


Overall Solution

$$I(t) = \underbrace{\frac{\hat{I}\,w}{\sqrt{(1/RC)^2 + w^2}} \sin(\,wt + \beta + \alpha\,)}_{I_{ss}(t) = \text{Steady-State Term}} + \underbrace{e^{-t/RC}\, I(0)}_{I_{tr}(t) = \text{Transient Term}}$$



$I_{ss}(t) + I_{tr}(t)$

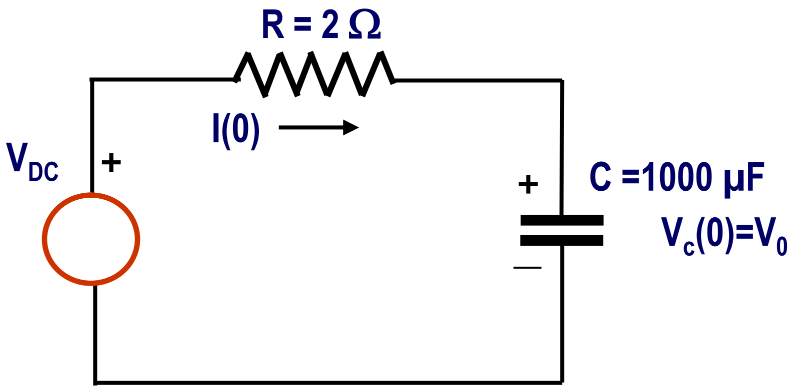


RC Circuits with DC Voltage Source - Two Simple Rules

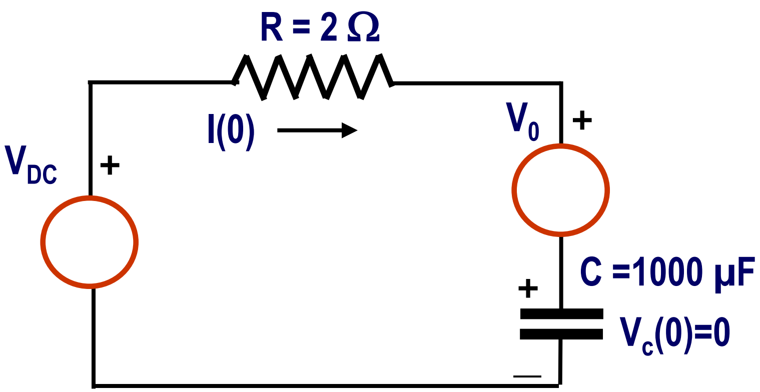
Rule - 1

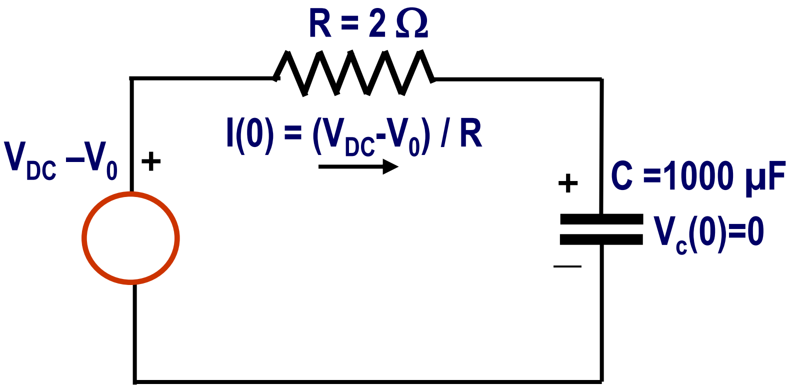
A filled capacitor acts initially as a DC voltage source due to the stored charge

$$V(t) = (1 / C) \int_{-\infty}^{0} I(t)dt = V(0) = V_0$$



The initial voltage of this capacitor may then be represented as a DC voltage source in series with an uncharged capacitor





RC Circuits with DC Voltage Source - Two Simple Rules

The current waveform will then be;

$$I(t) = I(\infty) + [\, I(0) - I(\infty)\,]\, e^{-t/\tau}$$

Where τ is called the time constant of the circuit defined as;

τ = RC = The time required a for capacitor to reach 63 % of its full charge

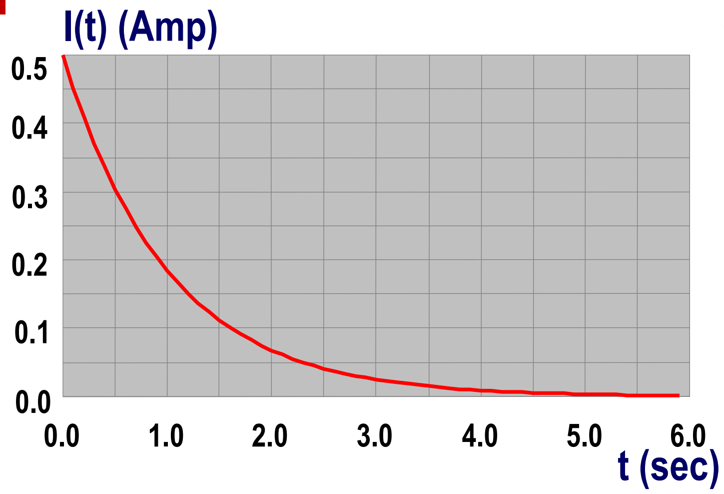
$= 2\,\Omega \times 1000\,\mu F = 2 \times 1000 \times 10^{-6} = 0.002$ sec

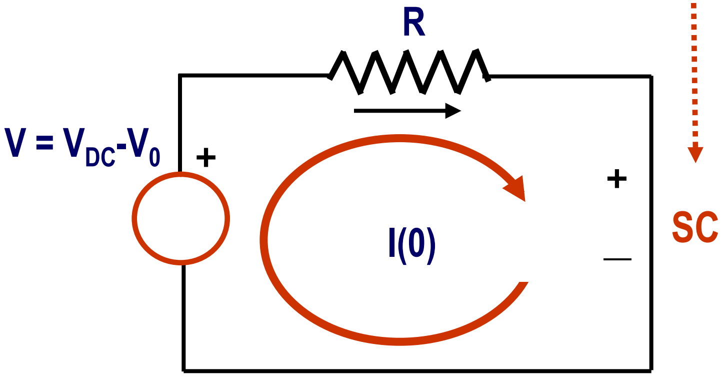
Then the initial value of current will be;

$$I(0) = (\,V_{DC} - V_0\,) / R$$

Please note that an uncharged capacitor acts effectively as short circuit, i.e. $V_0 = 0$

$I(0) = (\,V_{DC} - V_0\,) / R$





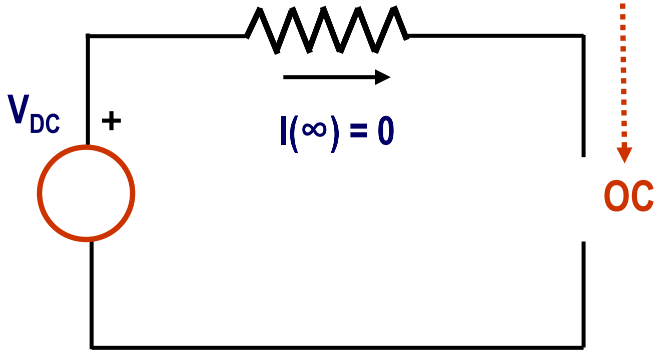
AC Circuits

RC Circuits with DC Voltage Source - Two Simple Rules

Rule - 2

A fully charged capacitor acts finally as open circuit to DC current

$I(t) = C\ d/dt\ V(t) = C\ d/dt\ (constant) = 0\ \ (OC)$



Then the final value of current will be;

$I(\infty) = 0$

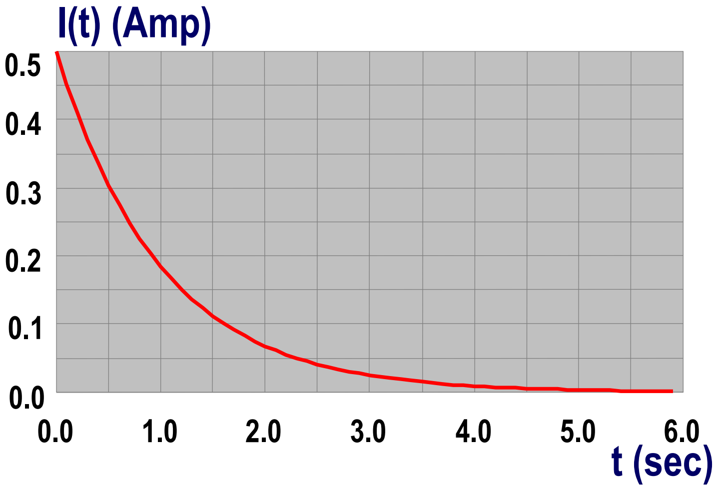
The current waveform will then be;

$$I(t) = I(\infty) + [\ I(0) - I(\infty)\ ]\ e^{-t/\tau}$$

Where τ is called the time constant of the circuit defined as;

τ = RC = The time required a for capacitor to reach 63 % of its full charge
$= 2\ \Omega \times 1000\ \mu F = 2 \times 1000 \times 10^{-6} = 2$ msec

$I(\infty) = 0$



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AC Circuits

RC Circuits with DC Voltage Source - Two Simple Rules

Solution

Substituting the above expressions into the current expression

The initial value of current will be;

$$I(0) = (V_{DC} – V_0) / R$$

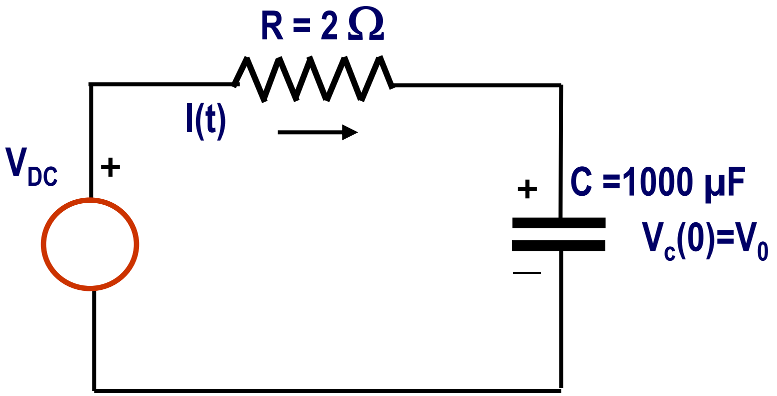
The final value of current will be;

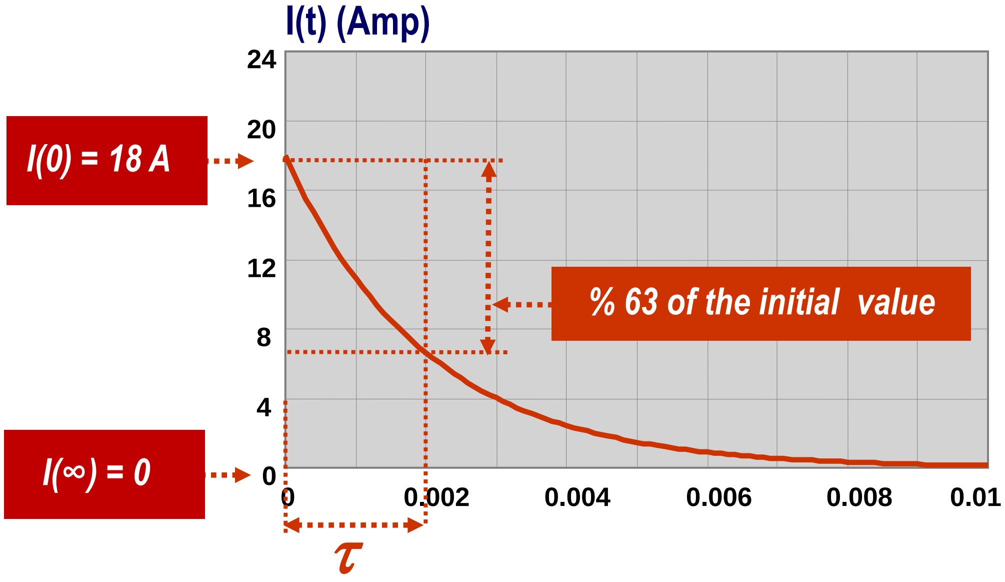
$$I(\infty) = 0$$

The current waveform will then be;

$$I(t) = I(\infty) + [I(0) - I(\infty)] e^{-t/\tau}$$

$$I(t) = [(V_{DC} – V_0) / R] e^{-t/\tau}$$





I(0) = 18 A

% 63 of the initial value

I(∞) = 0

AC Circuits

Meaning of the Time Constant τ

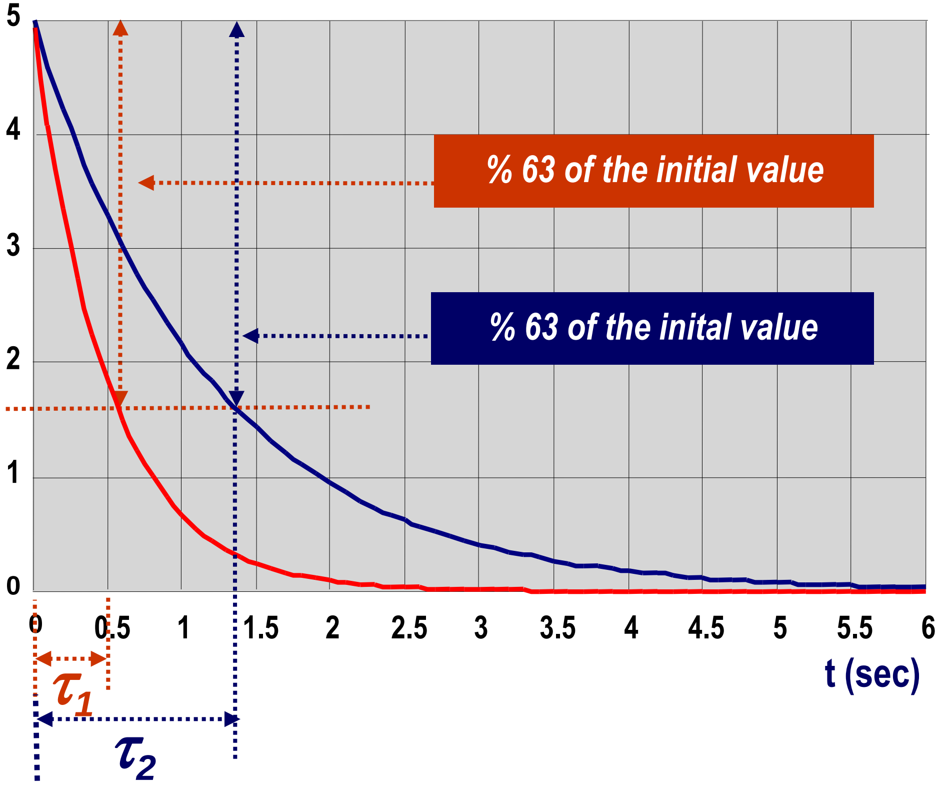
Definition

Time constant τ of a electric circuit is the duration for the current to get reduced by 63 % of its initial value

Time constant τ of an RC circuit is simply expressed as:

$$\tau = RC$$

The Effect of τ on decay



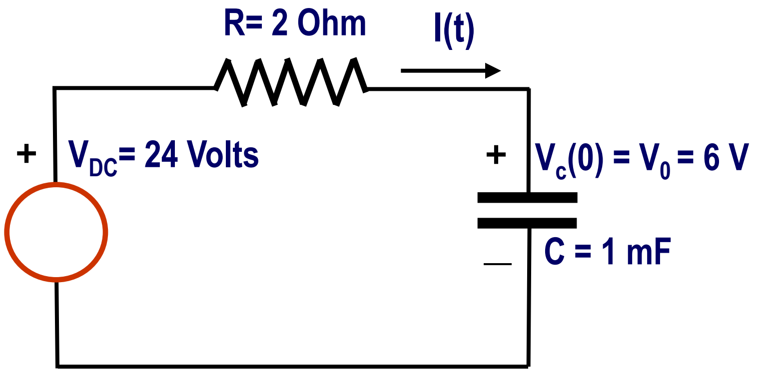
$\tau_2 > \tau_1$

AC Circuits

Example

Problem

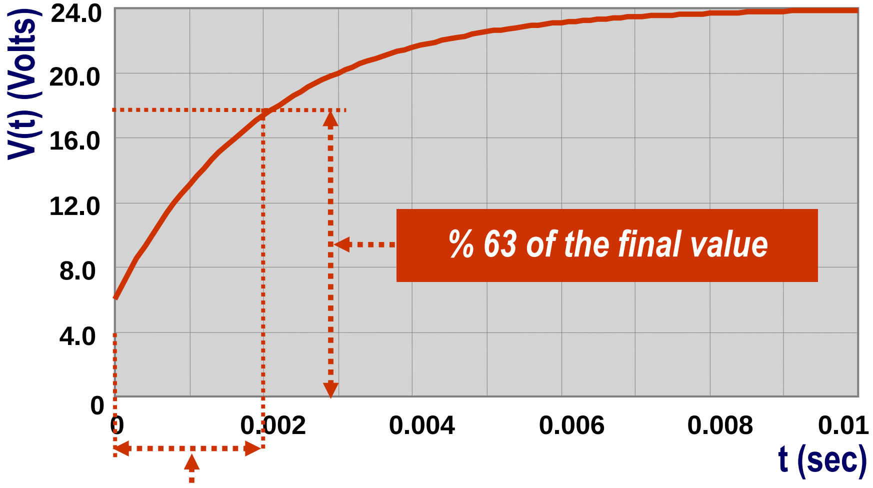
Find the voltage waveform across the 1 mF capacitor shown on the RHS, when it has an initial voltage of 6 Volts and charged by a 24 Volts DC voltage source through a wire with 2 Ohm resistance



Capacitor will behave as a DC source at the beginning and as OC at the end, hence;

$$V(0) = V_0 = 6 \text{ Volts}$$

$$V(\infty) = V_S = 24 \text{ Volts}$$



% 63 of the final value

The voltage waveform will then be;

$$V(t) = V(\infty) + [V(0) - V(\infty)]\, e^{-t/\tau}$$

$$V(t) = 24 + (6 - 24)\, e^{-t/0.002} = 24 - 18\, e^{-t/0.002} \text{ Volts}$$

τ = RC = Time Constant: The time required for a capacitor to reach 63 % of full charge = 2 x 1000 μF = 2 x 0.001 = 0.002 sec

AC Circuits

Energy Stored in a Capacitor

Instantaneous Energy Stored in a Capacitor

Assuming that the instantaneous voltage across the capacitor is $V_c(t)$

$$P(t) = V_C(t)\, I(t)$$

$$W_C(t) = \int P(t)\, dt$$

$$= \int V_C(t)\, I(t)\, dt$$

$$= \int V_C(t)\, C\, dV_C(t) / dt\ \ dt$$

$$= C \int V_C(t)\, dV_C(t)$$

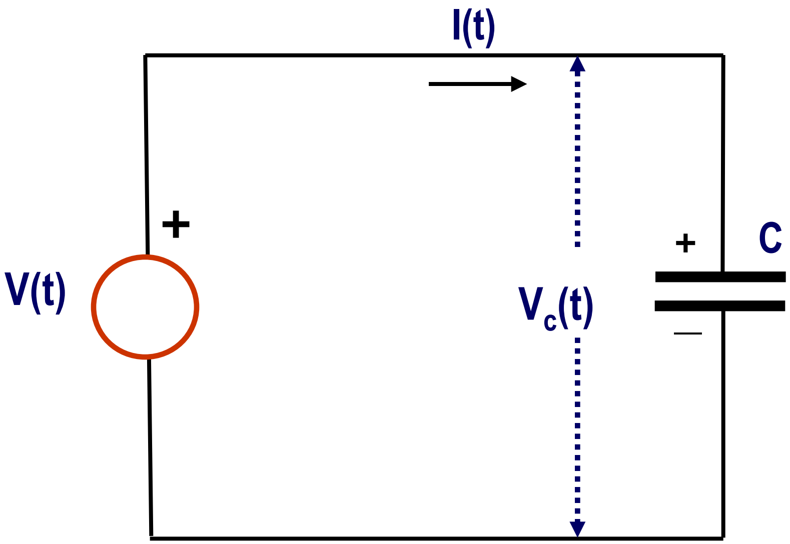
or

$$W_C(t) = (1/2)\, C\, V_C^2(t)$$

Example

Calculate the stored energy in a 10 µF capacitor fully charged with a 12 Volts DC voltage

$$W_c = (1/2)\ 10 \times 10^{-6} \times 12^2 = 720 \times 10^{-6} \text{ Joule}$$



AC Circuits

Example

Problem

Find the instantaneous energy in the capacitor for the voltage shown in the figure

$V(t) = \hat{V} \sin wt$

$W_C(t) = (1/2)\, C\, V_C^2(t)$

$W_c(t) = (1/2)\, 10 \times 10^{-6}\, \hat{V}^2 \sin^2 wt$

$= 0.000005 \times 312^2 \sin^2 wt$

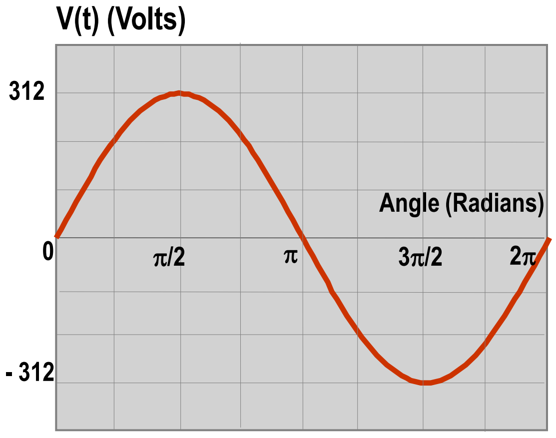
$= 0.4867 \sin^2 wt$

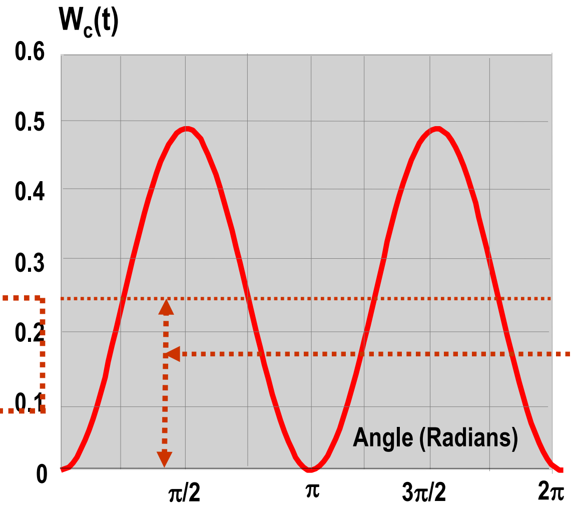
$= 0.4867\, (\, ½ – ½ \cos 2wt\,)$

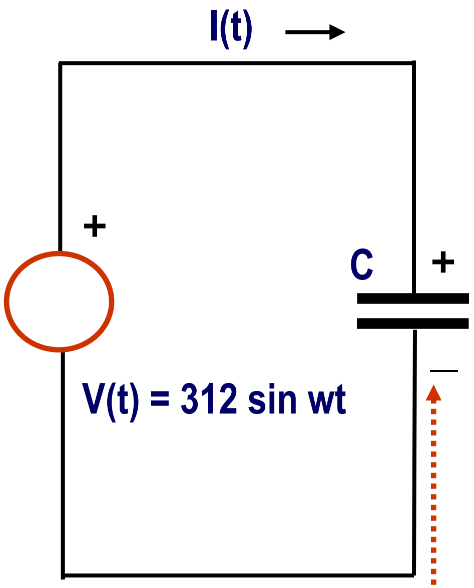
$\sin^2 wt = 1 – \cos^2 wt = 1 – (\, 1 + \cos 2wt\,) / 2$

$= ½ – ½ \cos 2wt$

$V(t) = \hat{V} \sin wt$







$V(t) = 312 \sin wt$

$C = 10\, \mu F$

Mean of $W_c(t) > 0$

AC Circuits

Series Connected Capacitances

Series connected capacitances

$$V_1(t) = (1/C_1) \int I(t)dt$$

$$V_2(t) = (1/C_2) \int I(t)dt$$

$$+ \text{-----------} = + \text{------------------------------------}$$

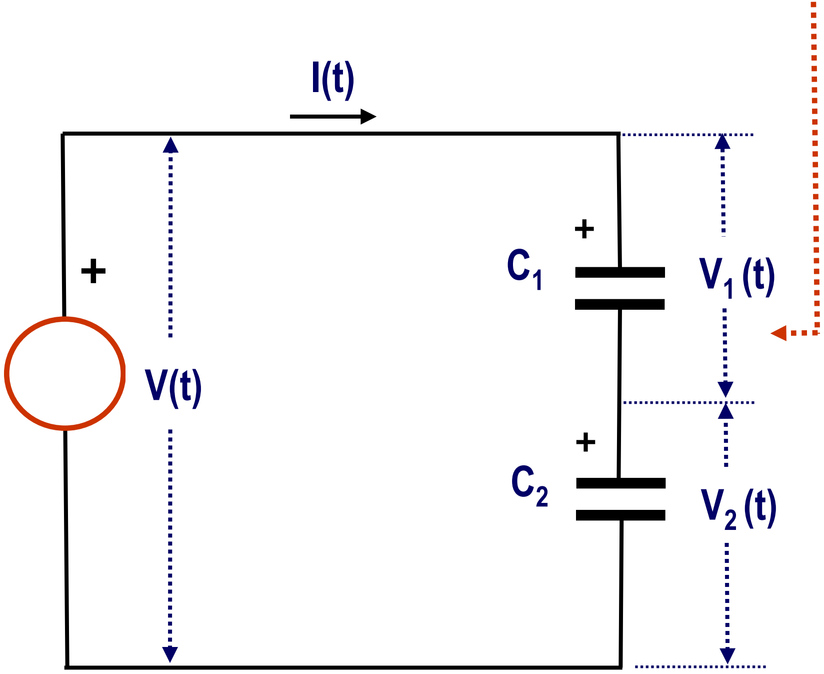
$$V(t) = [\,(1/C_1) + (1/C_2)\,] \int I(t)dt$$

$$= (1/C_{tot}) \int I(t)dt$$

Hence,

$$C_{tot} = \frac{1}{(1/C_1) + (1/C_2)}$$

Series connected capacitances are combined in the same way as for shunt connected resistances



AC Circuits

Series and Shunt Connected Capacitances

Shunt connected capacitances

$$I_1(t) = C_1 \, d\,V(t) / dt$$

$$I_2(t) = C_2 \, d\,V(t) / dt$$

$$+ \text{----------} = + \text{---------------------------}$$

$$I\,(t) = (C_1 + C_2)\, d\,V(t) / dt$$

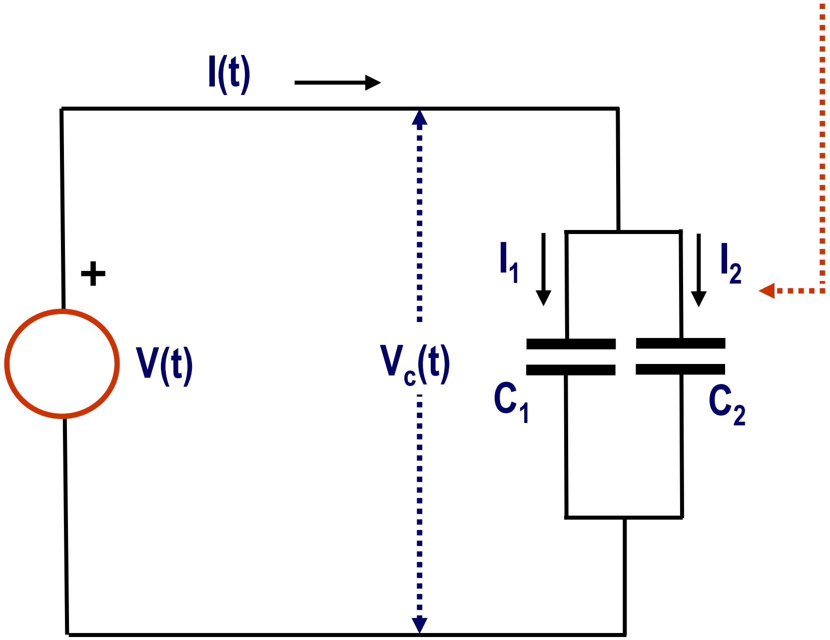
$$= C_{tot} \quad d\,V(t) / dt$$

Where,

$$C_{\,tot} = C_1 + C_2$$

is the total capacitance

Shunt connected capacitances are simply added



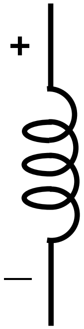
Inductance

Definition

Inductance is a winding or coil of wire around a core

Core may be either insulator or a ferromagnetic material

Symbolic representation

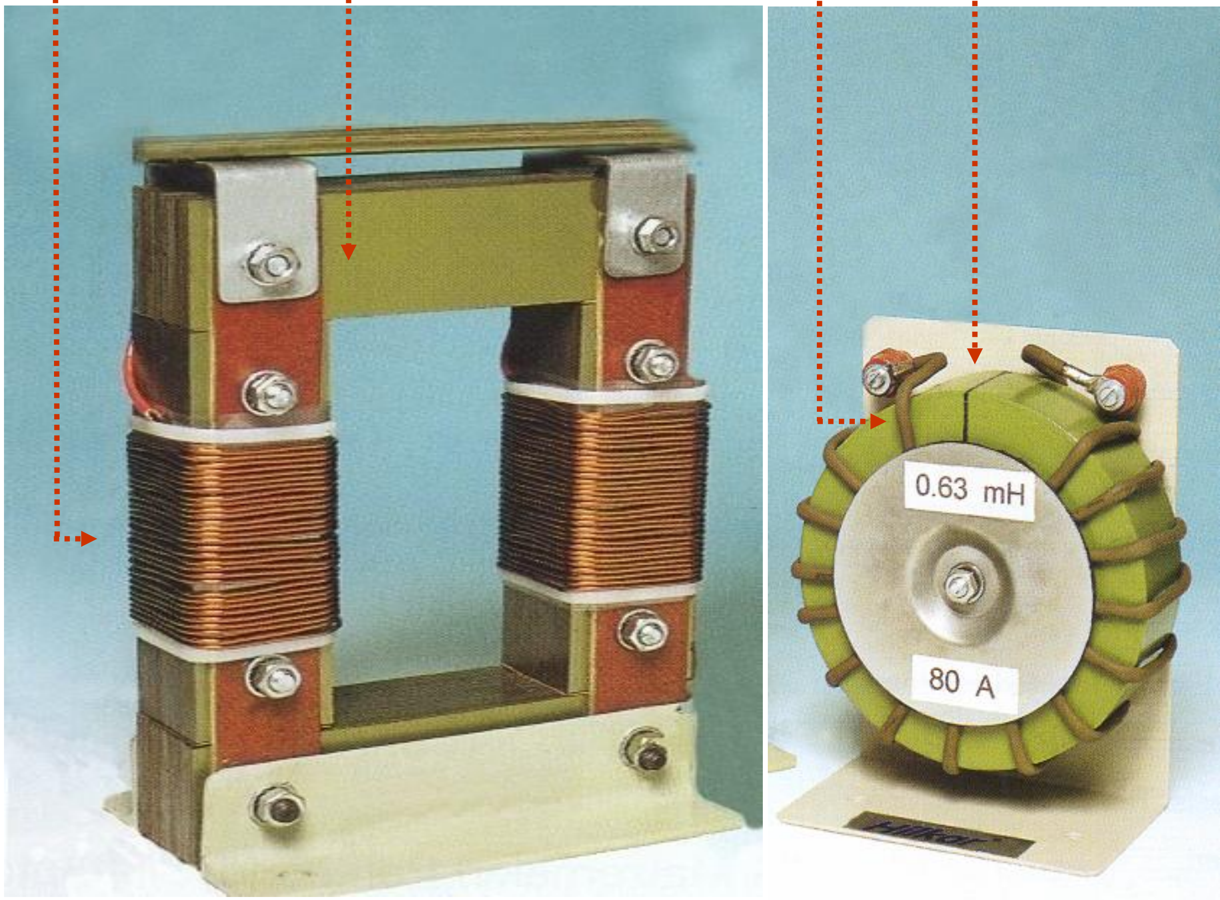


Coil

Core

Toroidal Coil

Toroidal Core



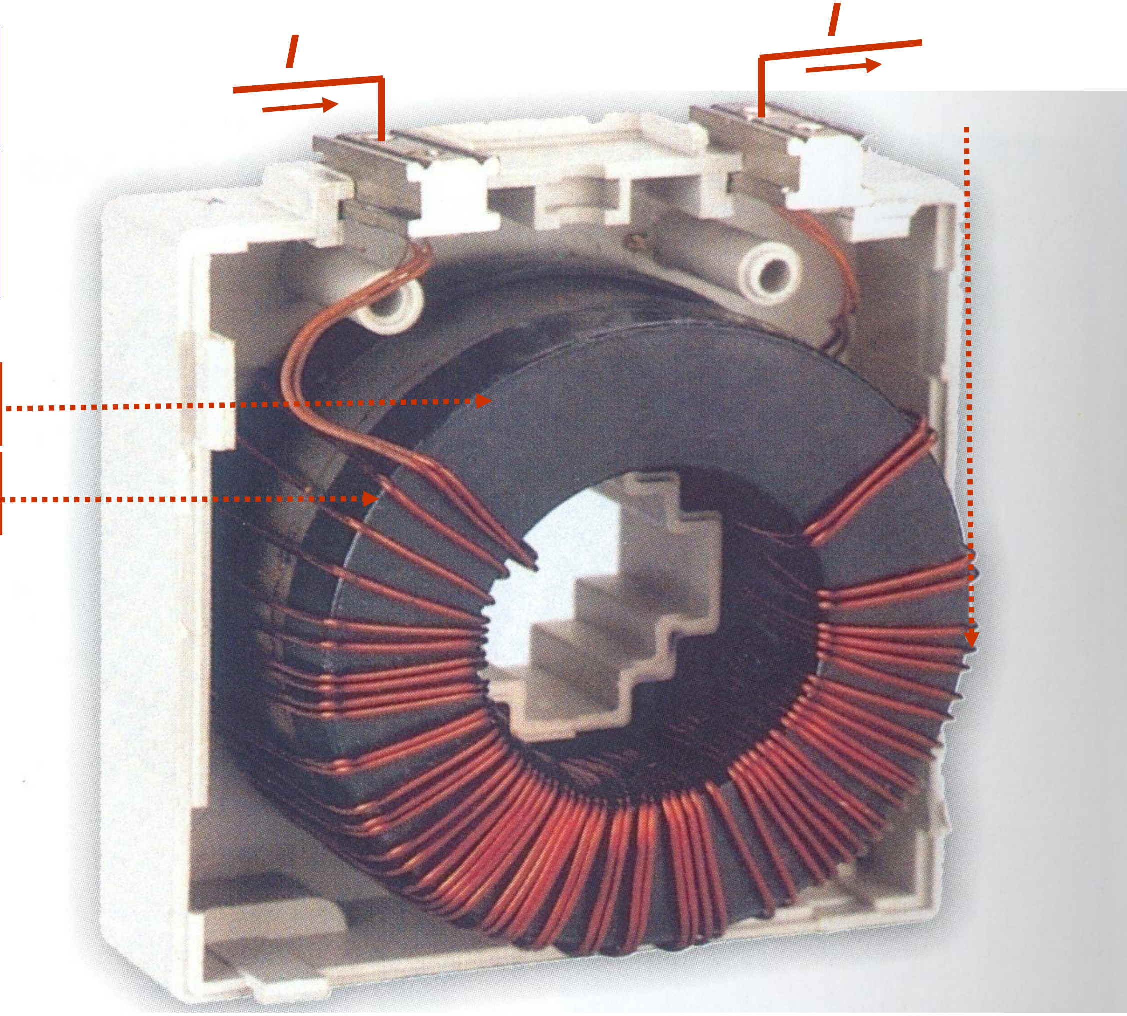
Ferrite Core Toroidal Inductor

Definition

Ferriet core inductor has a toroidal ferrit core inside

Ferrite core

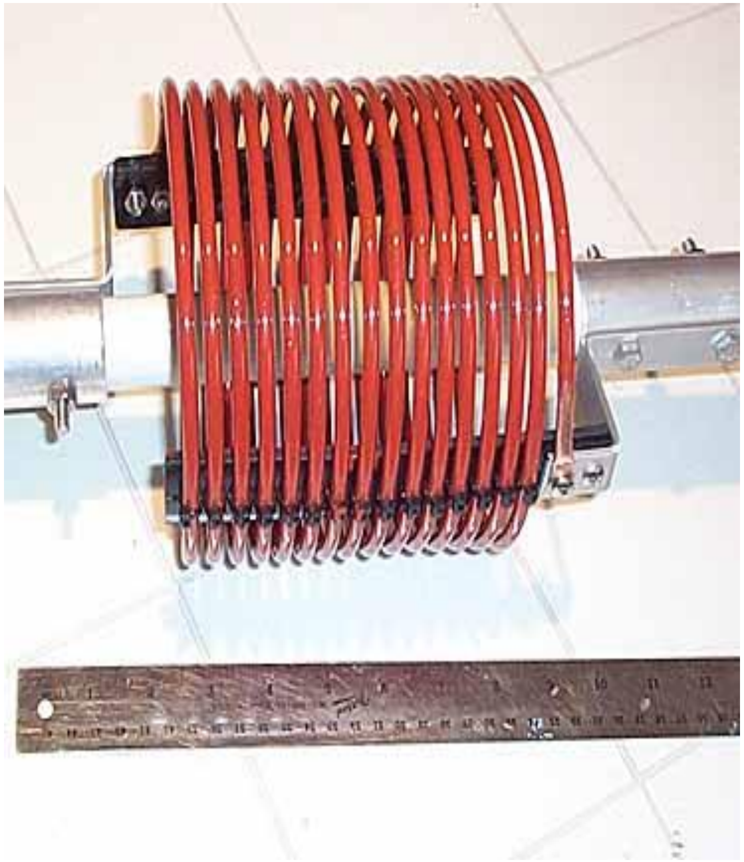
Toroidal coil



Air Core Inductor

Configuration

Air core inductor has no core inside



AC Circuits

Basic Relation

Definition

Voltage across an inductor is proportional to the rate of change of current

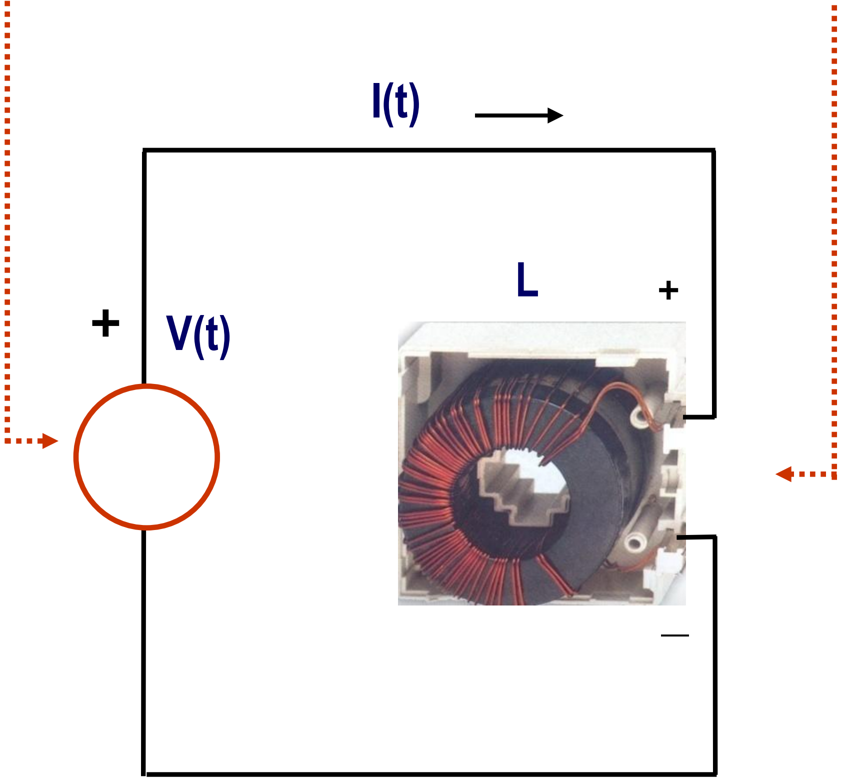
$$V(t) = L\, d\, I(t) / dt$$

where, V(t) is the voltage across the inductance,
I(t) is the current flowing through,
L is the inductance (Henry)

1 Henry is the value of inductance defined as

1 Henry = 1 Volt x 1 second / 1 Amp

Voltage Source V(t) **Inductance L**



AC Circuits

Current in an Inductance

Definition

The equation;

$$V(t) = L\, d\, I(t) / dt$$

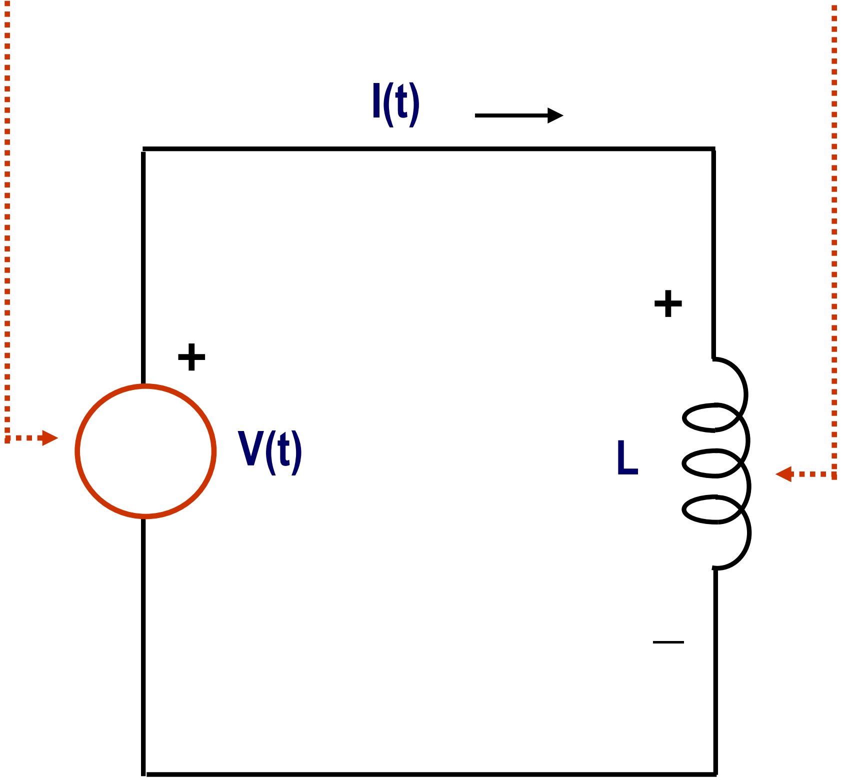
can be written in inverse form as

$$I(t) = (1/L) \int V(t)dt + I(0)$$

where I(0) is the current initially flowing in the inductor

Voltage Source V(t)

Inductance L

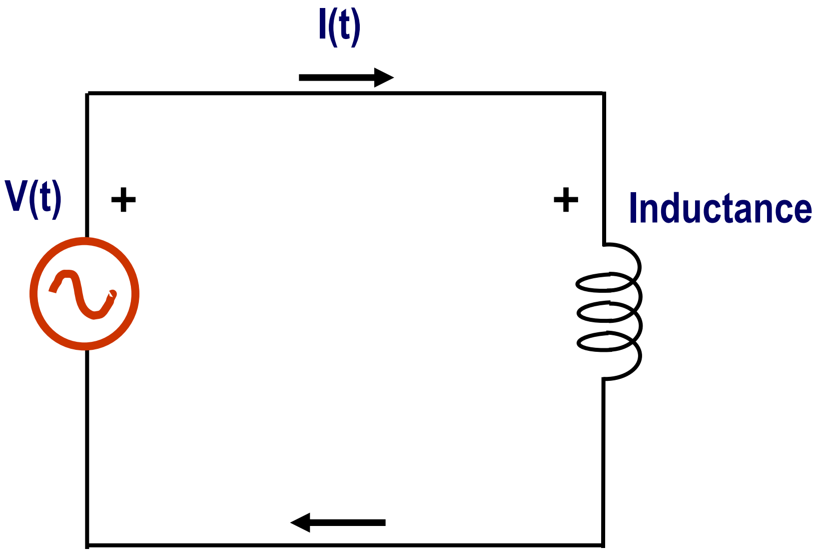


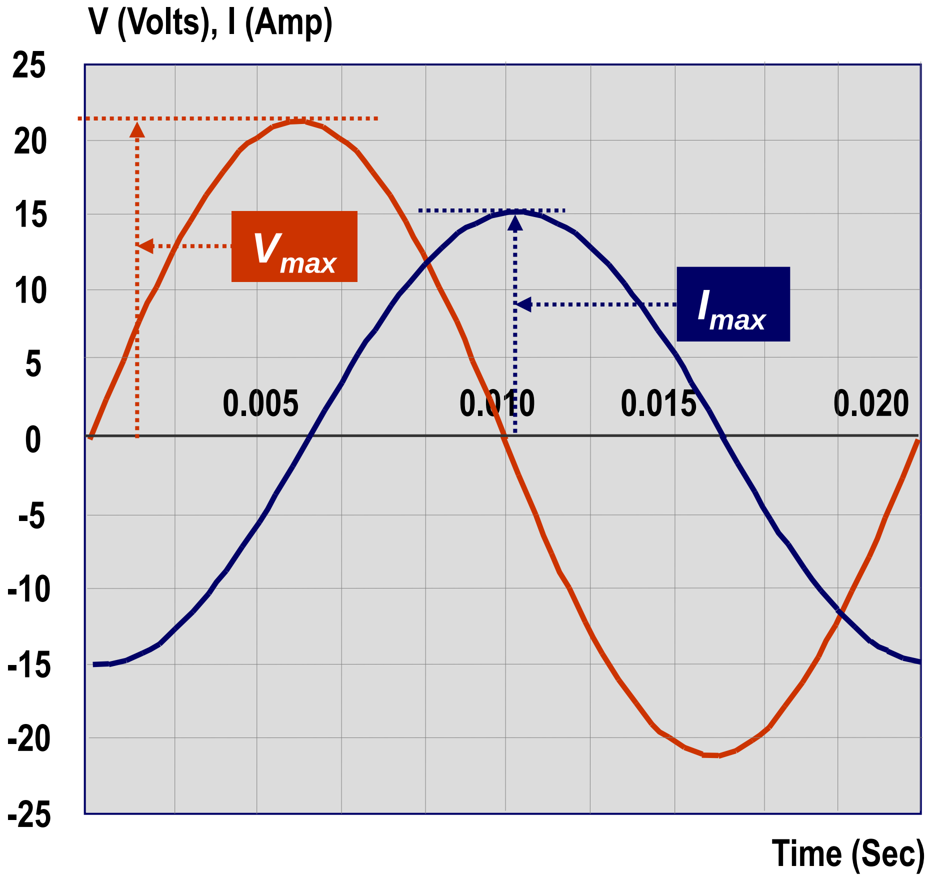
AC Circuits

Current in an Inductance

Phase Shift between Current and Voltage Waveforms

$$
\begin{aligned}
I(t) &= (1/L) \int V(t)\, dt \\
&= (1/L) \int V_{max} \sin wt\, dt \\
&= - (V_{max} / wL) \cos wt \\
&= - I_{max} \cos wt
\end{aligned}
$$





AC Circuits

Series and Shunt Connected Inductors

Series connected inductors are added

$$V_1(t) = L_1 \, d\,I(t) / dt$$

$$V_2(t) = L_2 \, d\,I(t) / dt$$

+ ---------- + ------------------------

$$V(t) = (L_1 + L_2) \, d\,I(t) / dt$$

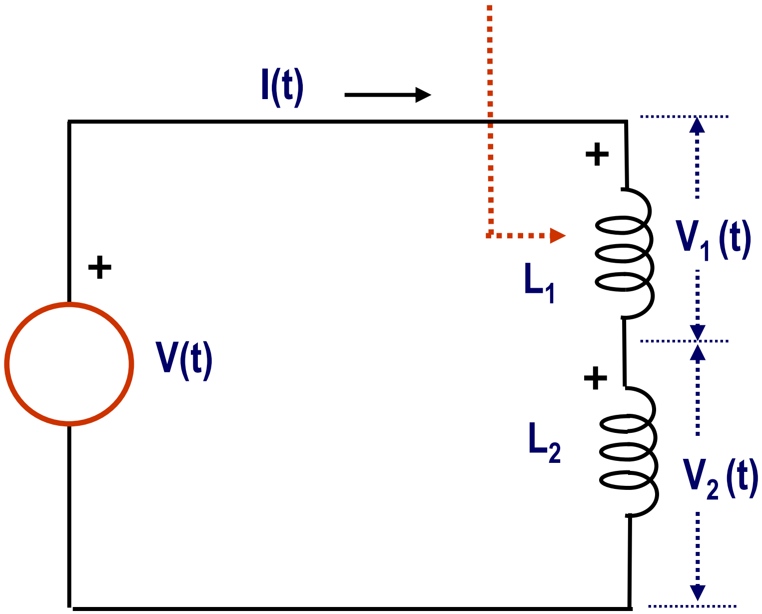
$$= L_{tot} \, d\,I(t) / dt$$

where

$$L_{tot} = L_1 + L_2$$

is the total inductance

Series connected inductances



AC Circuits

Series and Shunt Connected Inductors

Shunt connected inductances are combined in the same way as in shunt connected resistances

$$I_1(t) = (1/L_1) \int V(t)dt$$

$$I_2(t) = (1/L_2) \int V(t)dt$$

$$\overline{+\qquad\qquad\qquad\qquad}$$

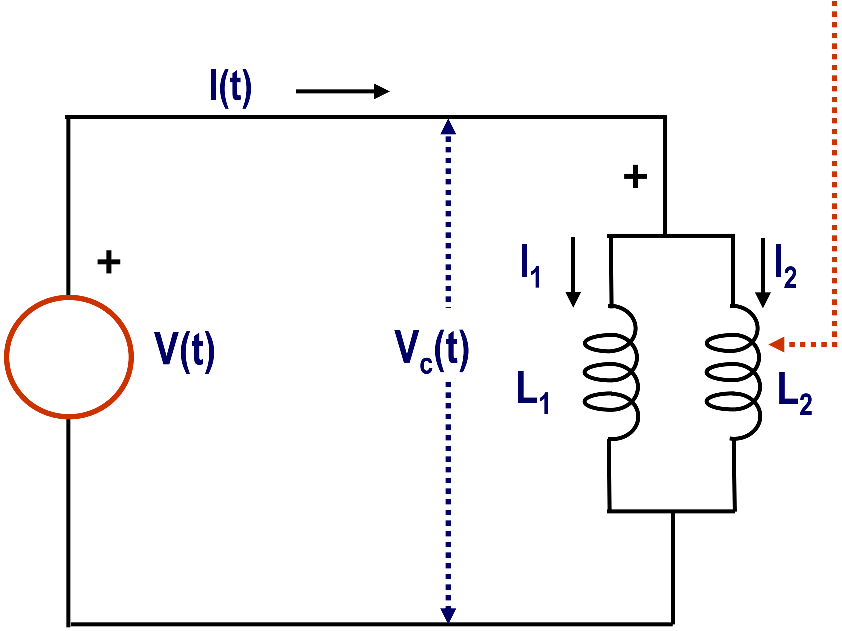
$$I(t) = [\,(1/L_1) + (1/L_2)\,] \int V(t)dt$$

$$= (1/L_{tot}) \int V(t)dt$$

Hence,

$$L_{tot} = \frac{1}{(1/L_1) + (1/L_2)}$$

Shunt connected inductances



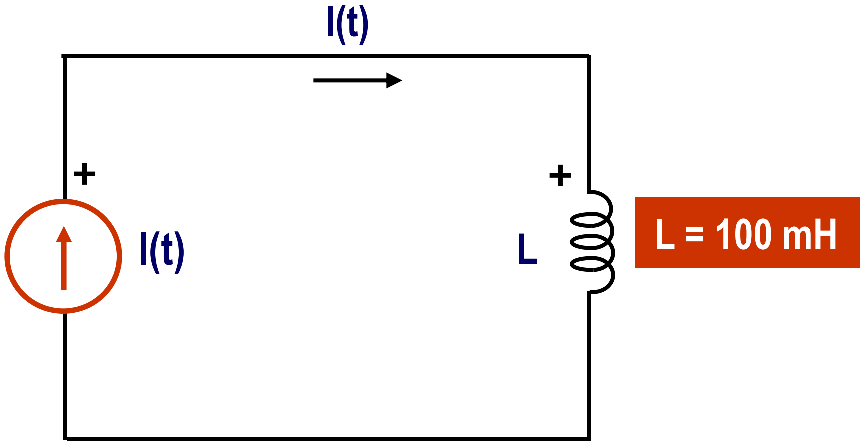
AC Circuits

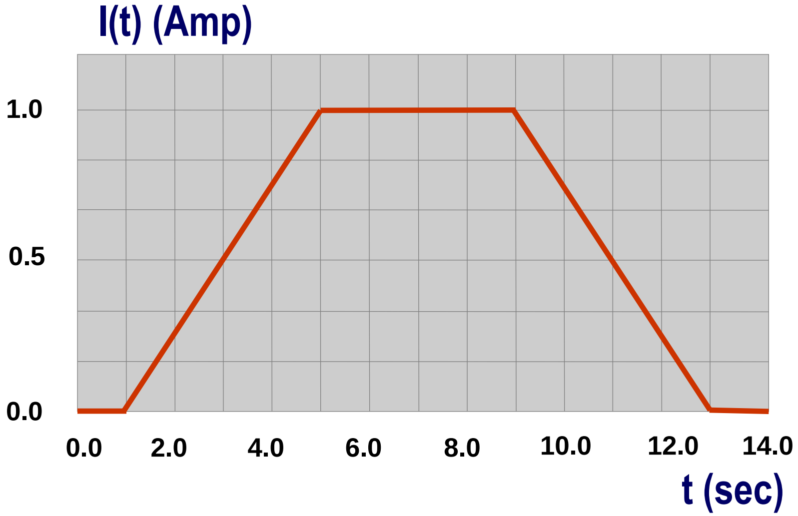
Example - 3

Problem

Calculate the voltage across the 100 mH inductor with the current shown in the figure on the RHS

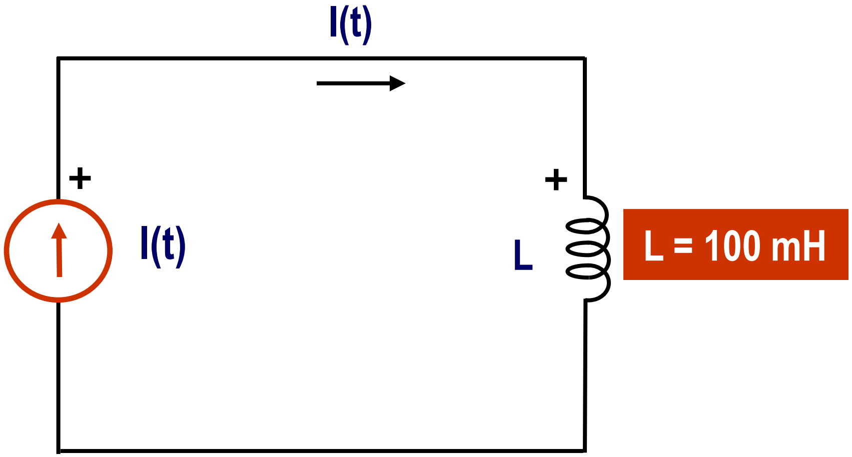
$I(t) = 0$ $t < 1$ s
$I(t) = 1/((5-1))\,(t - 1) = \frac{1}{4}(t - 1)$ $1 \leq t \leq 5$ s
$I(t) = 1$ $5 \leq t \leq 9$ s
$I(t) = -1/((5-1))\,(t - 13) = -\frac{1}{4}(t - 13)$ $9 \leq t \leq 13$ s
$I(t) = 0$ $t \geq 13$ s





AC Circuits

Example - 3



Solution

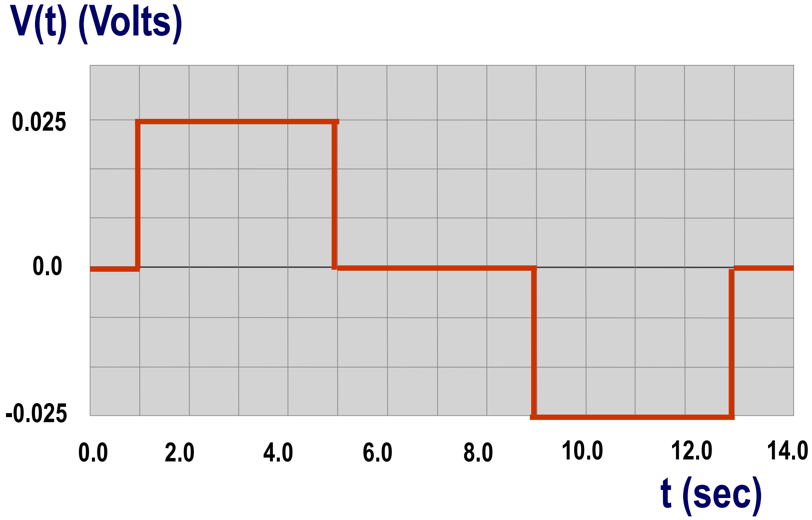
$V(t) = L\, d\, I(t) / dt$

Differentiating the expression for current waveform

$d/\, I(t)\, dt = d\, (¼\, (t - 1)) /dt = ¼$

and multiplying by the inductance L ($L = 10^{-1}$ H);

$V(t) = 0$ $\quad t < 1$ s
$V(t) = 10^{-1} \times ¼ = 0.025$ V $\quad 1 \leq t \leq 5$ s
$V(t) = 0$ $\quad 5 \leq t \leq 9$ s
$V(t) = -10^{-1} \times ¼ = -0.025$ V $\quad 9 \leq t \leq 13$ s
$V(t) = 0$ $\quad t \geq 13$ s



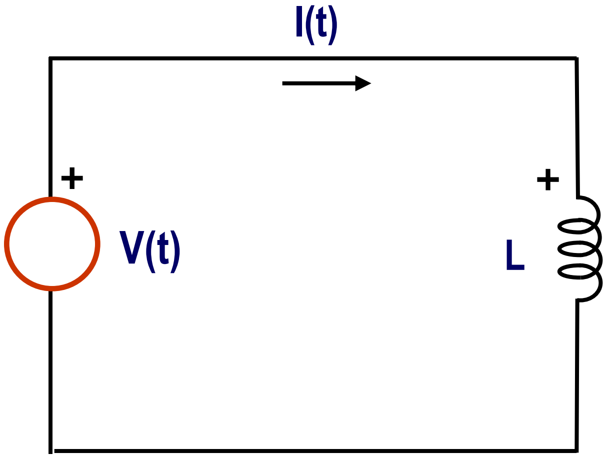
AC Circuits

Example - 4

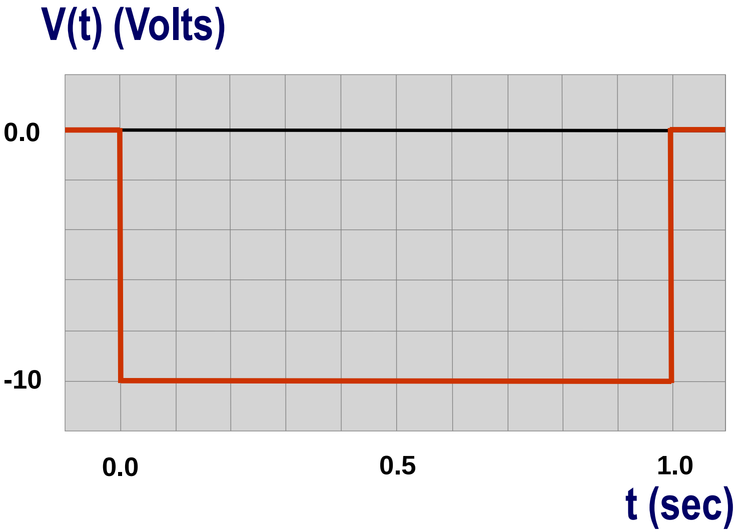
Problem

Assume that the inductor shown on the RHS is connected to a voltage source with the waveform shown in the figure
Determine the inductor current waveform by assuming that the initial current in the inductor is zero

$V(t) = 0 \qquad t < 0$ sec
$V(t) = -10$ V $\qquad 0 \leq t \leq 1$ sec
$V(t) = 0 \qquad t \geq 1$ sec



L = 100 mH



AC Circuits

Example - 4

Solution

$I(t) = (1/L) \int V(t)\,dt + I(0)$

Integrating the voltage expression;

$$I(t) = (1/L) \int V(t)\,dt + I(0)$$
$$= 1/(100 \times 10^{-3}) \int V(t)\,dt$$

$I(t) = 0$ $\quad t < 0$ sec

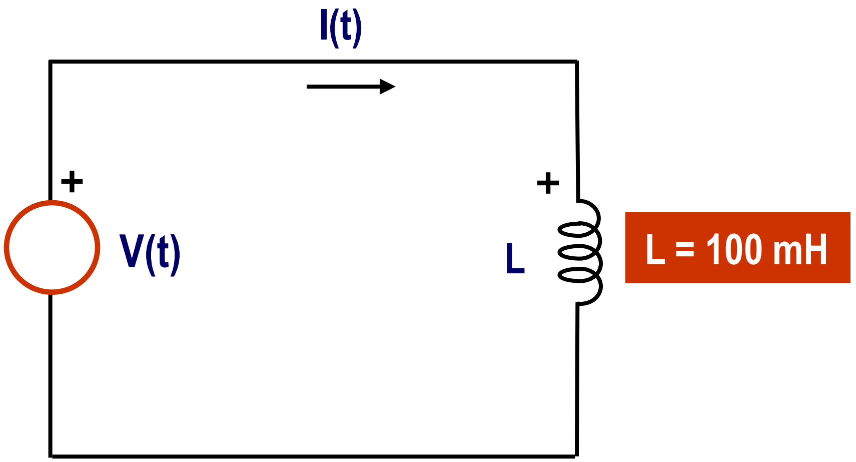
$I(t) = 1/(100 \times 10^{-3}) \int V(t)\,dt$

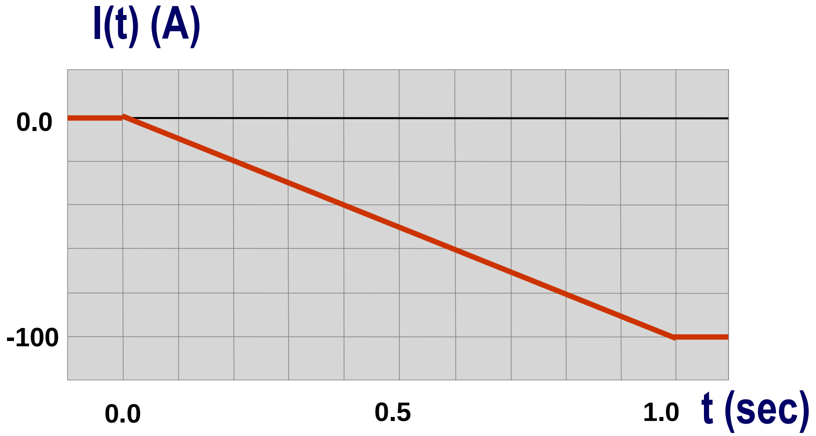
$= 10 \int V(t)\,dt$

$= 10 \int -10\,dt$

$= -100\ t$ $\quad 0 \leq t \leq 1$ sec

$I(t) = -100$ A $\quad t \geq 1$ sec





AC Circuits

Energy Stored in an Inductor

Problem

Calculate the instantaneous energy stored in an inductor with an inductance L and an instantaneous voltage $V_L(t)$

$$P(t) = V_L(t)\, I(t)$$

$$W_L(t) = \int P(t)\, dt$$

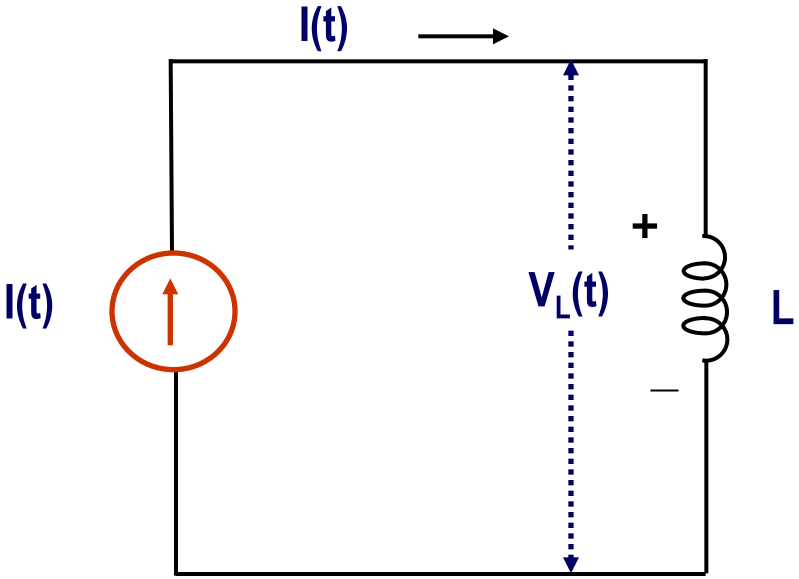
$$= \int V_L(t)\, I(t)\, dt$$

$$= \int I(t)\, L\, dI(t) / dt \;\; dt$$

$$= L \int I(t)\, dI(t)$$

or

$$W_L(t) = ½\; L\, I^2(t)$$



AC Circuits

Example – 5

Problem

Find the instantaneous energy in the inductor for the current shown in the figure

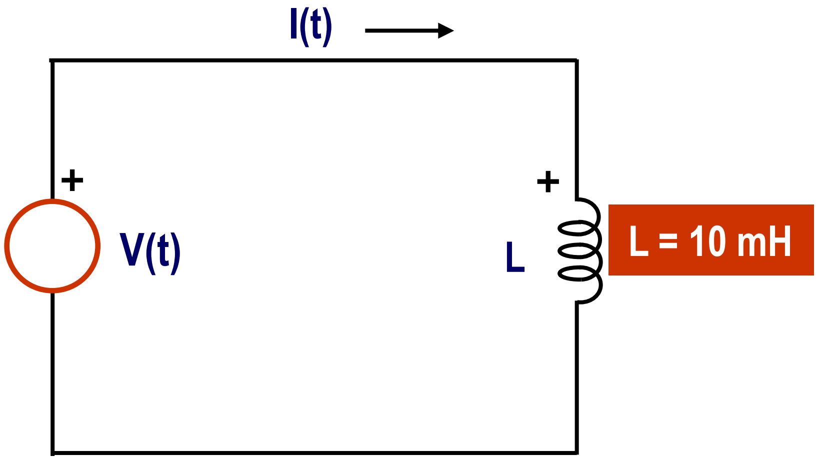
$I(t) = 0$ $t < 1$ s

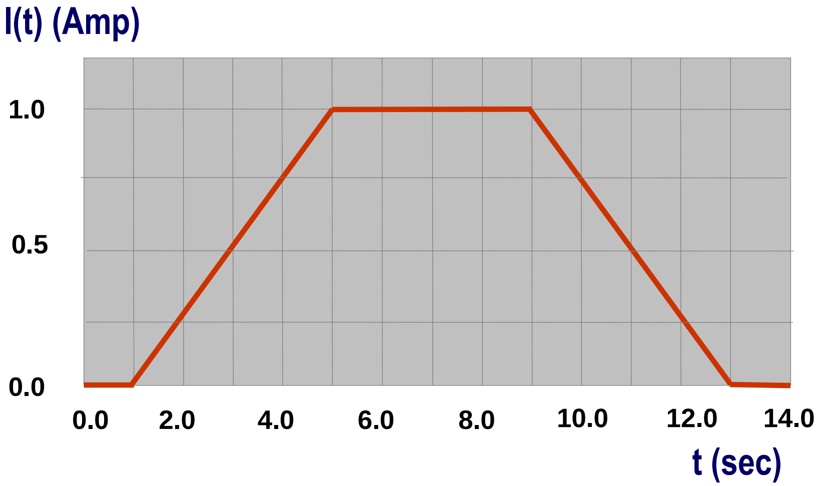
$I(t) = ¼ (t - 1)$ Amp $1 \leq t \leq 5$ s

$I(t) = 1$ Amp $5 \leq t \leq 9$ s

$I(t) = - ¼ (t - 13)$ Amp $9 \leq t \leq 13$ s

$I(t) = 0$ $t \geq 13$ s





AC Circuits

Example – 5

Solution

$$W_L(t) = \tfrac{1}{2}\, L\, I^2(t)$$

By using the above formula

$W(t) = 0$ Joule $\quad t < 1$ s

$W(t) = \tfrac{1}{2}\, 10^{-2}\ \text{x}\ (\tfrac{1}{4}\,(t - 1))^2$

$= 0.3125\ \text{x}\ 10^{-3}\ \text{x}\ (t-1)^2$ Joules $\quad 1 \le t \le 5$ s

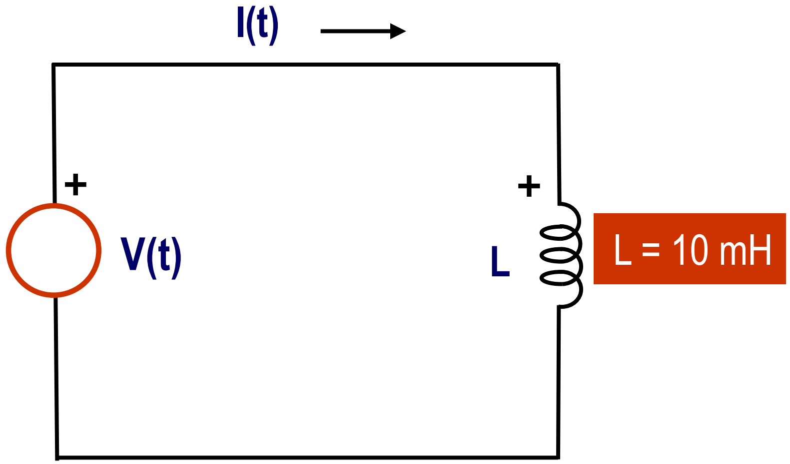
$W(t) = 0.01 / 2\ \text{x}\ 1^2 = 0.005$ Joules $\quad 5 \le t \le 9$ s

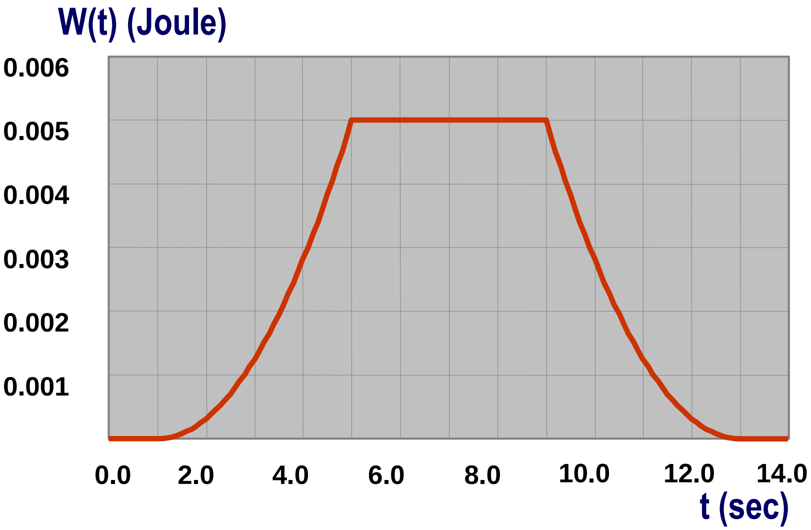
$W(t) = \tfrac{1}{2}\, 10^{-2}\ \text{x}\ (\tfrac{1}{4}\,(t - 13))^2$

$= 0.3125\ \text{x}\ 10^{-3}\ \text{x}\ (t - 13)^2$ Joules $\quad 9 \le t \le 13$ s

$W(t) = 0 \quad t \ge 13$ s

Please note that 1 Joule = 1 Watt x 1 sec





AC Circuits

R-L Circuits

Problem

Solve the R-L circuit shown on the RHS which consists of a resistance in series with an inductance for current waveform when the switch is turned “on” at time: t = 0 sec

Solution

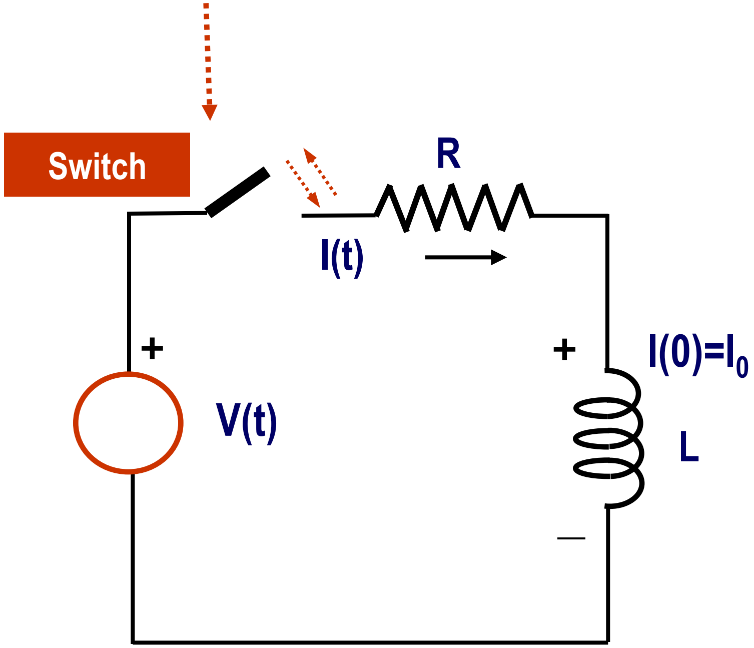
Writing down KVL for the circuit

$$V(t) = R\, I(t) + L\, dI(t) / dt$$

or

$$dI(t) / dt + (R/L)\, I(t) = (1/L)\, V(t)$$

Switch is turned “on” at: t = 0 sec



A first order ordinary differential equation

AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

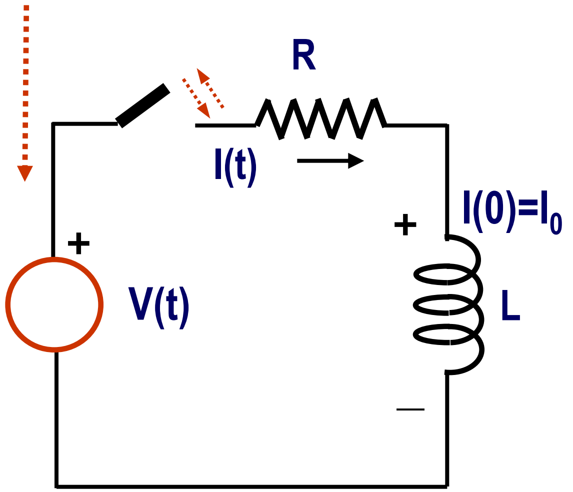
Solution

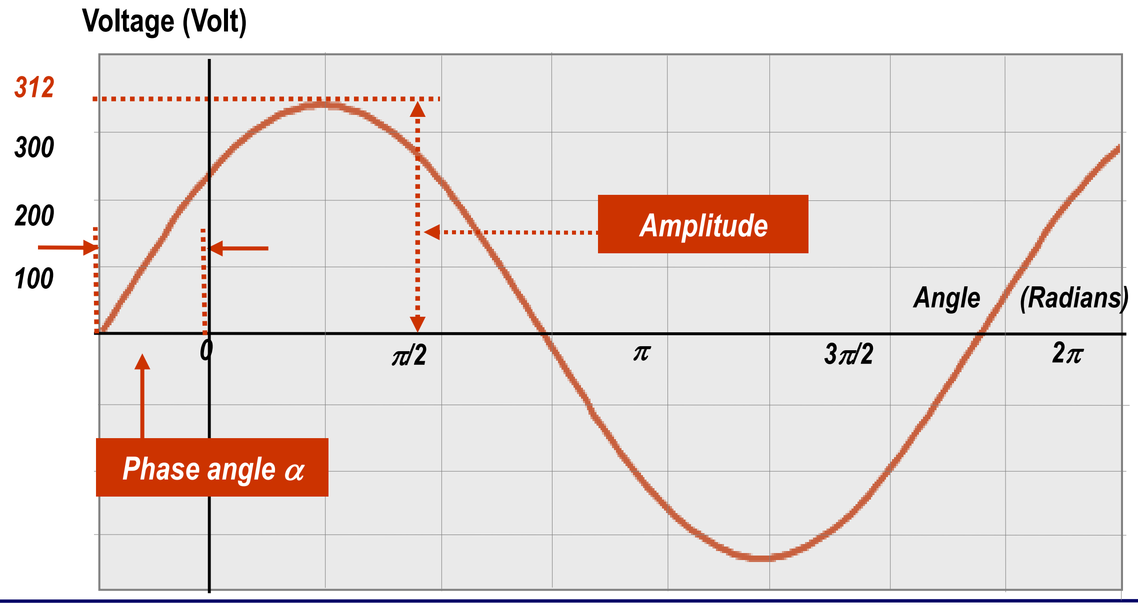
Solve the resulting first order ordinary differential equation (ODE)

$$dI(t) / dt + (R/L) I(t) = (1/L) V(t)$$

$$dI(t) / dt + (R/L) I(t) = (1/L) \hat{V} \sin (wt + \alpha)$$

$V(t) = \hat{V} \sin (wt + \alpha)$





AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

Solution

Solve the resulting first order ordinary differential equation (ODE)

$$dI(t) / dt + (R/L)\, I(t) = (\hat{V}/L) \sin (wt + \alpha)$$

Define an integration factor $\mu(t) = e^{\,t R/L}$

Multiply both sides of the above ODE by this factor;

$$\mu(t)\, dI(t)/dt + I(t)\, (R/L)\, \mu(t) = \mu(t)\, (\hat{V} / L) \sin (wt + \alpha)$$

$$\mu(t)\, dI(t)/dt + I(t)\, d/dt\, \mu(t) = (\hat{V} / L)\, \mu(t) \sin (wt + \alpha)$$

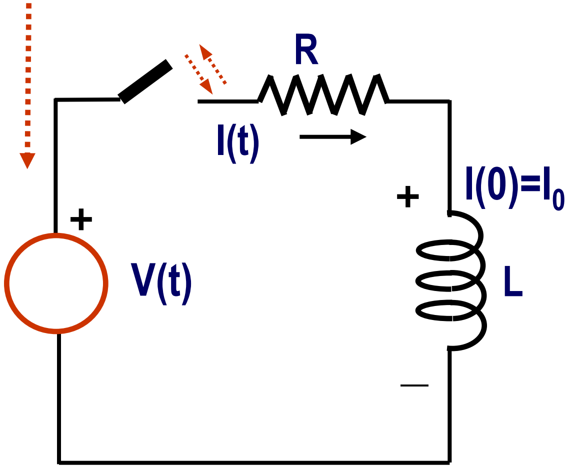
$$d/dt\, [\mu(t)\, I(t)] = (\hat{V} / L)\, \mu(t) \sin (wt + \alpha)$$

$$\int d/dt\, [\mu(t)\, I(t)]\, dt = (\hat{V} / L) \int \mu(t) \sin (wt + \alpha)\, dt + I(0)$$

$$\mu(t)\, I(t) = (\hat{V} / L) \int \mu(t) \sin (wt + \alpha)\, dt + I(0)$$

$$I(t) = \hat{I}\, \mu(t)^{-1} \int \mu(t) \sin (wt + \alpha)\, dt + \mu(t)^{-1} I(0)$$

$$V(t) = \hat{V} \sin (wt + \alpha)$$



$$d/dt\, \mu(t) = (R/L)\, \mu(t)$$

AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

Solution (Continued)

Subsituting the integration factor $\mu(t) = e^{t\,R/L}$ into the above solution;

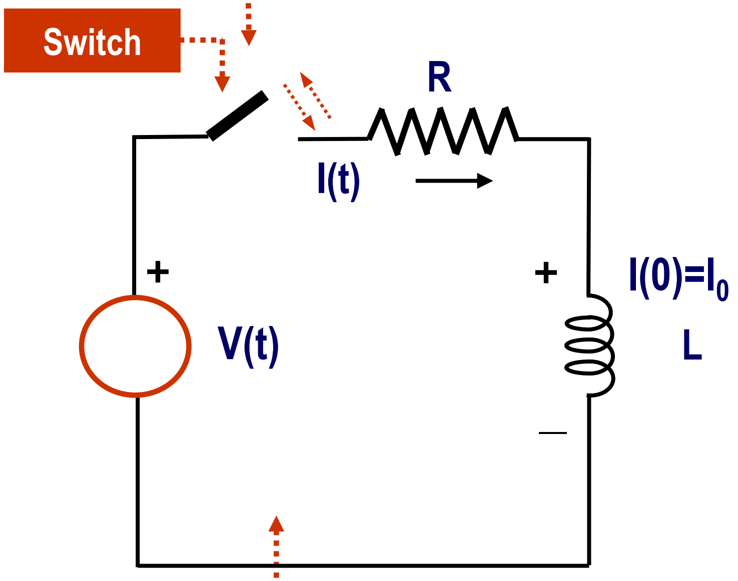
$$I(t) = \hat{I}\, e^{-t\,R/L} \int e^{t\,R/L} \sin(wt+\alpha)\, dt + e^{-t\,R/L}\, I(0)$$

Let $\alpha = 0$ (for simplicity)

$$\int e^{t\,R/L} \sin wt\, dt = e^{t\,R/L}\, \frac{(R/L)\sin wt - w\cos wt}{(R/L)^2 + w^2}$$

Taken from the Reference: Calculus and Analytic Geometry, Thomas, Addison Wesley, Third Ed. 1965, pp. 369

Switch is turned "on" at: t = 0 sec



$V(t) = \hat{V} \sin(wt + \alpha)$

AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

Solution (Continued)

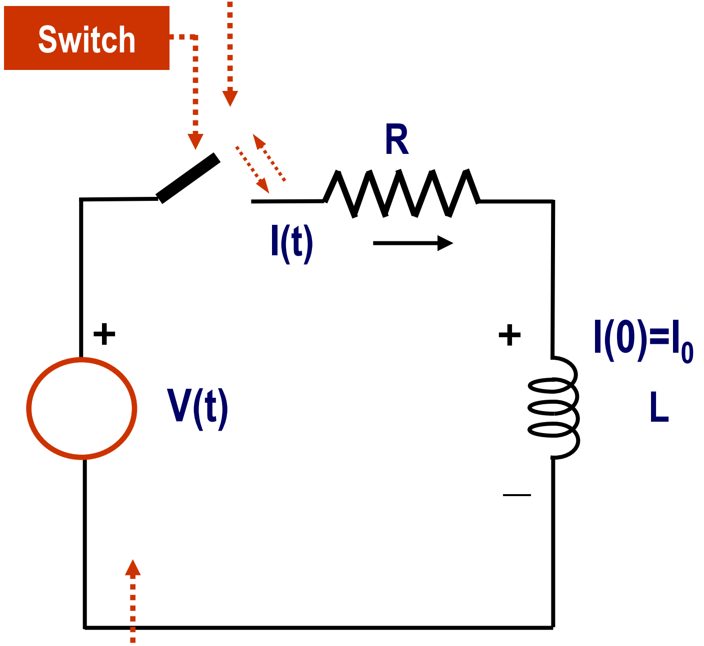
Subsituting the above term into the solution;

$$I(t) = \hat{I}\, \cancel{e^{-t\,R/L}}\, \cancel{e^{t\,R/L}}\, \frac{(R/L)\ \sin wt\ -\ w\ \cos wt}{(R/L)^2 + w^2} + e^{-t\,R/L}\, I(0)$$

$$I(t) = \hat{I}\, \frac{(R/L)\ \sin wt\ -\ w\ \cos wt}{(R/L)^2 + w^2} + e^{-t\,R/L}\, I(0)$$

$$I(t) = \underbrace{\frac{\hat{I}}{(wL/R)^2 + 1}\left(\sin wt\ -\ (wL/R)\ \cos wt \right)}_{\text{Steady-State Term}} + \underbrace{e^{-t\,R/L}\, I(0)}_{\text{Transient Term}}$$

Switch is turned "on" at: t = 0 sec



$V(t) = \hat{V} \sin(wt + \alpha)$

AC Circuits

Solution of the resulting First Order Ordinary Differential Equation

Numerical Example

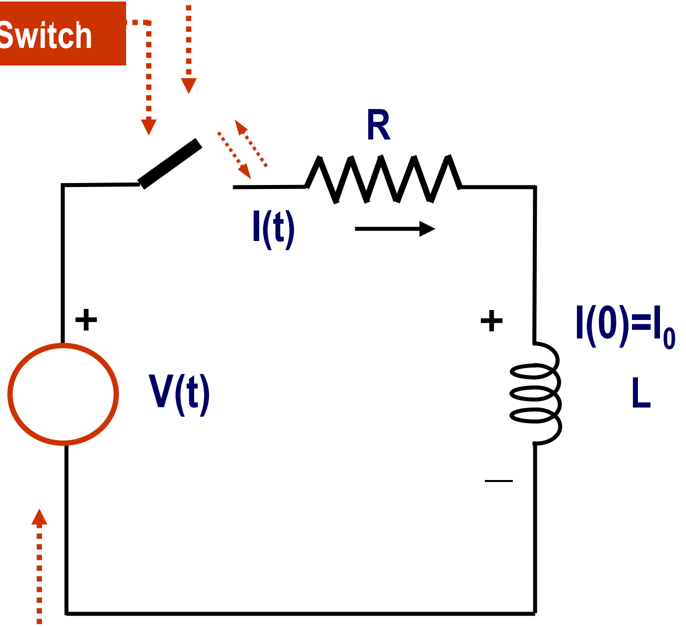
Now assume that the parameters of the circuit on the RHS are as follows;

V(t) = 312 sin wt Volts

R = 1 Ohms

L = 10 mHenry

Switch is turned "on" at: t = 0 sec



$$V(t) = \hat{V} \sin(wt + \alpha)$$

$$I(t) = \underbrace{\frac{\hat{I}}{(wL/R)^2 + 1}\left(\sin wt - (wL/R)\cos wt\right)}_{\text{Steady-State Term}} + \underbrace{e^{-tR/L}\, I(0)}_{\text{Transient Term}}$$

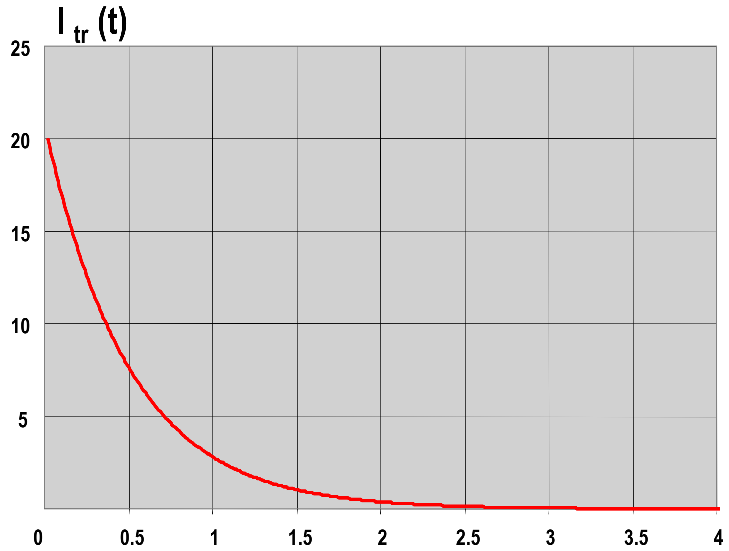
METU

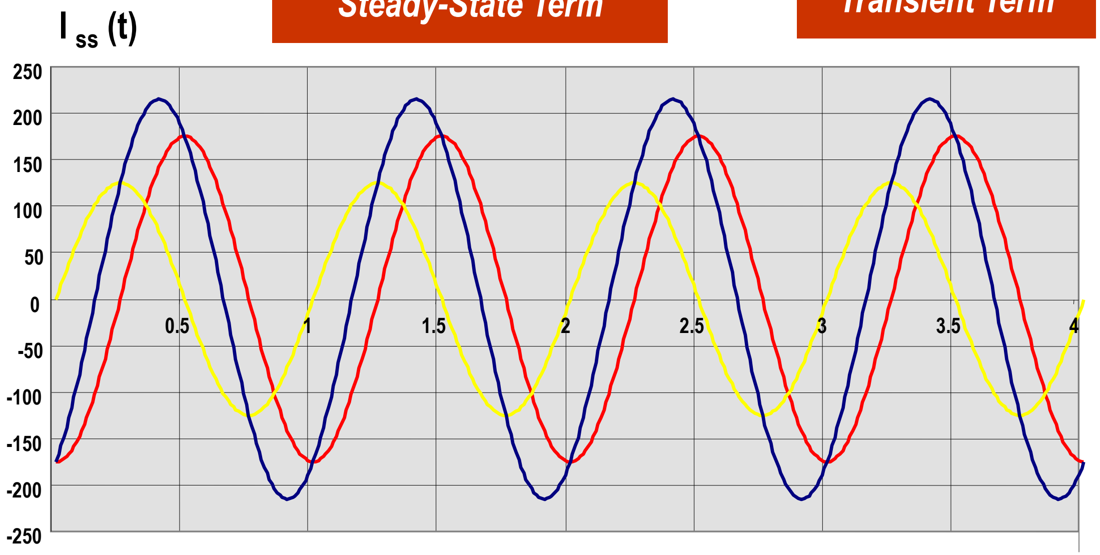
AC Circuits

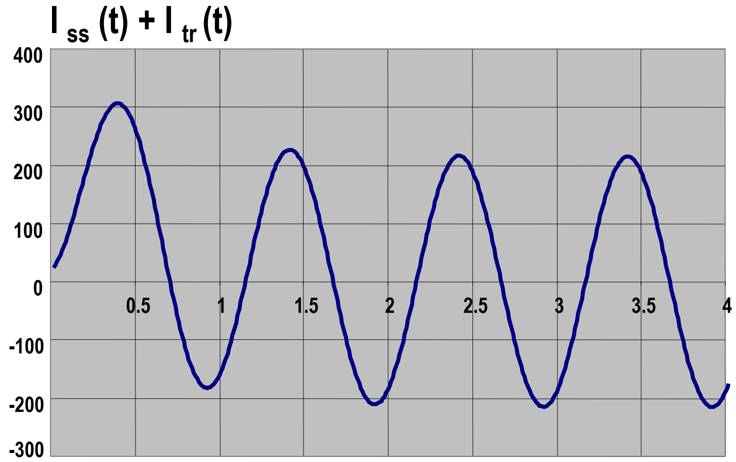
Solution of the resulting First Order Ordinary Differential Equation

Numerical Example

$$I(t) = \underbrace{\frac{\hat{I}}{(wL/R)^2 + 1}\left(\sin wt - (wL/R) \cos wt \right)}_{\text{Steady-State Term}} + \underbrace{e^{-t\,R/L}\, I(0)}_{\text{Transient Term}}$$







AC Circuits

Solution for DC Voltage - Two Simple Rules

Rule - 1

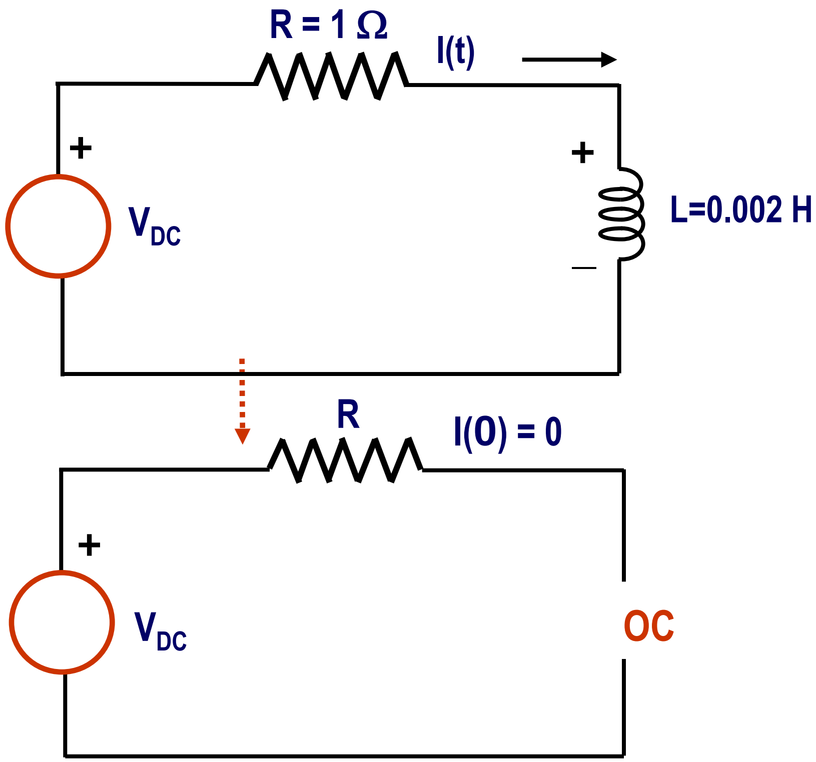
An inductor with no initial current acts as an open circuit to a DC voltage source initially

$$I(0) = (1 / L) \int_{-\infty}^{0} V(t)\, dt = I_0 = 0 \qquad (OC)$$

Then, the initial value of current will be;

$I(0) = 0$

$I(0) = 0$



$I(0) = 0$

The current waveform will then be;

$$I(t) = I(\infty) + [\, I(0) - I(\infty)\,]\, e^{-t/\tau}$$

$\tau = L / R$ = Time Constant: The time required for the inductor to reach 63 % of full current = 2 mH / 1 Ω = 0.002 / 1 = 0.002 Sec

Solution for DC Voltage - Two Simple Rules

Rule - 2

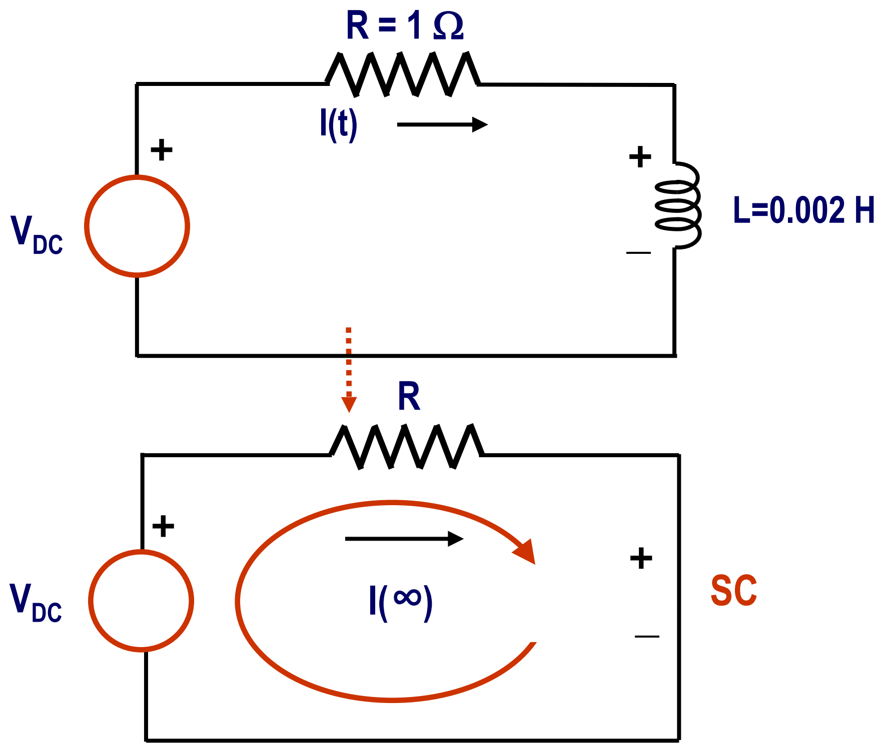
An inductor acts as a short circuit to a DC current (or voltage) finally

$V(t) = L\, d/dt\, I(t) = L\, d/dt\, (ct) = 0$ (SC)

Then, the final value of current will be;

$I(\infty) = V / R$

$I(\infty) = V / R$



$I(\infty) = V / R$ | $I(0) = 0$

The current waveform will then be;

$$I(t) = I(\infty) + [\, I(0) - I(\infty)\,]\, e^{-t/\tau}$$

$\tau = L / R$ = Time Constant: The time required for the inductor to reach 63 % of full current
= 2 mH / 1 Ω = 0.002 / 1 = 0.002 Sec

Solution for DC Voltage - Two Simple Rules

The current waveform will then be;

$$I(t) = I(\infty) + [\, I(0) - I(\infty)\,]\; e^{-t/\tau}$$

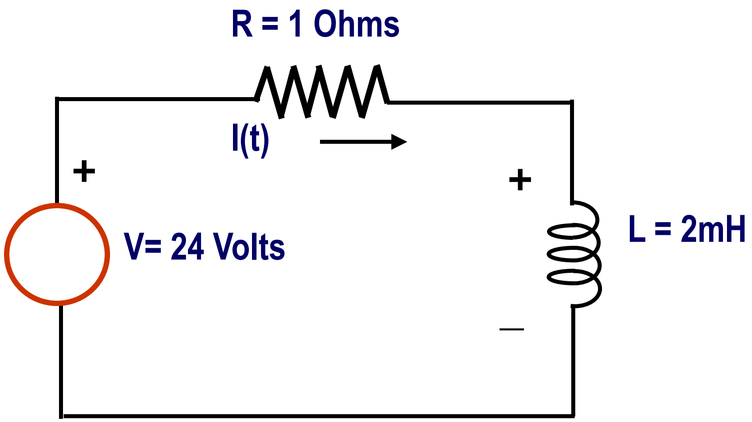
$I(\infty) = V / R$

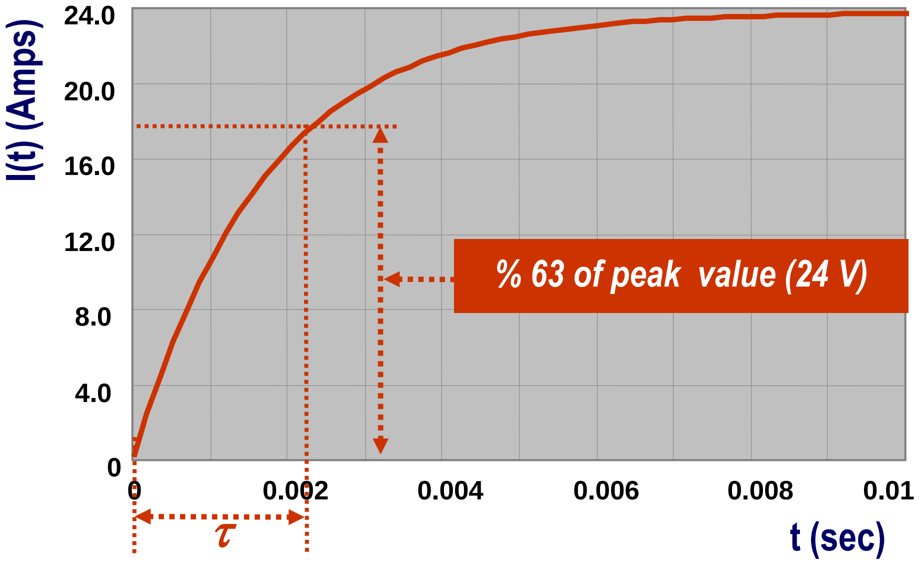
$I(0) = 0$

or

$$I(t) = V/R \;-\; V/R \;\; e^{-t/\tau}$$
$$= V/R\;(1 - e^{-t/\tau})$$

τ = L / R = Time Constant: The time required for
the inductor to reach 63 % of full current
= 2 mH / 1 Ω = 0.002 / 1 = 0.002 Sec





AC Circuits

Example - 6

Problem

Find the current waveform in the 2 mH inductor with 6 Amps initial current connected to a 24 Volt DC voltage source through a wire with 1 Ohm resistance as shown on the RHS

Inductor has 6 Amp initial current;

$I(0) = I_0 = 6\ Amp$

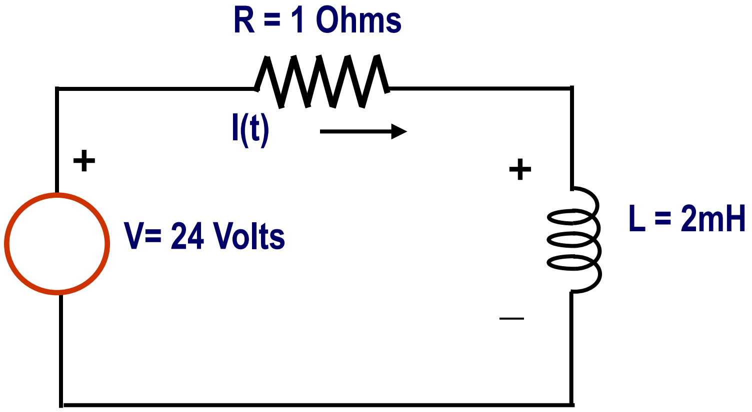
Inductor will be SC at the end, hence;

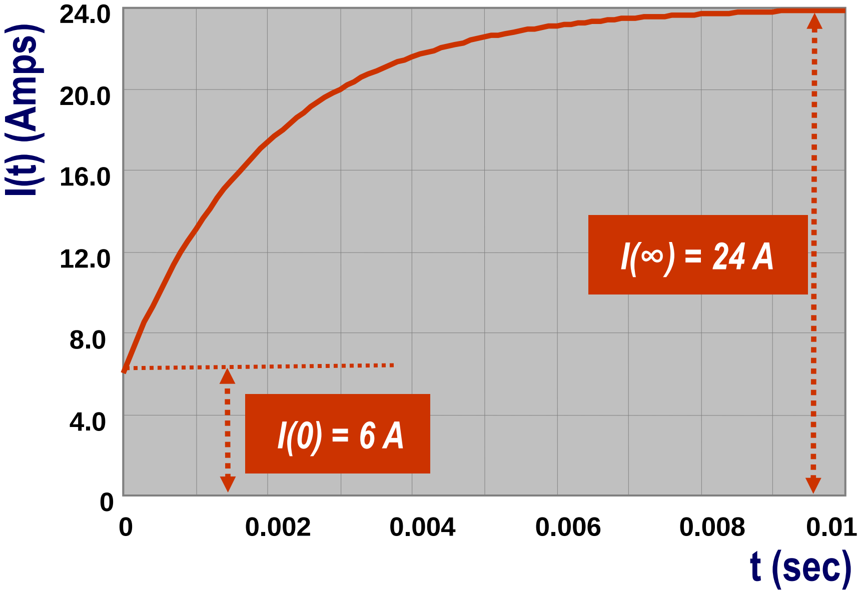
$I(\infty) = V / R = 24 / 1 = 24\ Amps$

The current waveform will then be;

$I(t) = I(\infty) + [\ I(0) - I(\infty)\ ]\ e^{-t/\tau}$

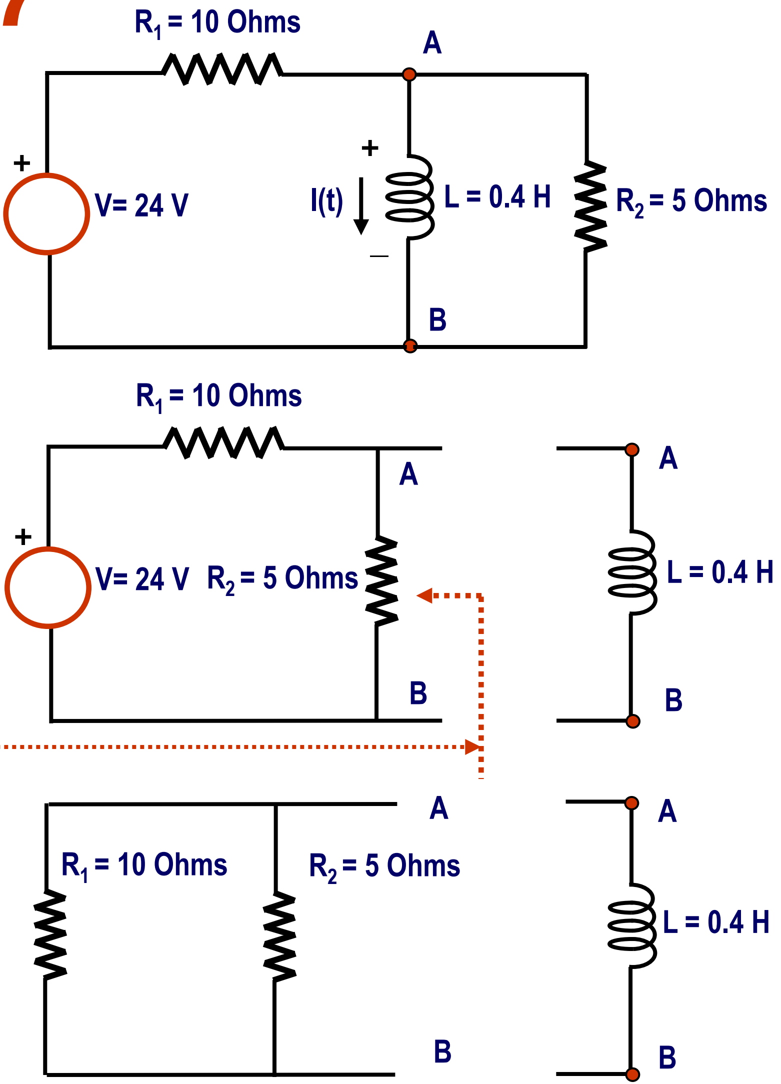
$I(t) = 24 + (6 - 24)\ e^{-t/0.002} = 24 - 18\ e^{-t/0.002}\ \ Amps$





AC Circuits

Example - 7



Problem

Solve the circuit shown on the RHS for current waveform flowing in the inductor

Solution

First take out the branch containing inductor, and find the Thevenin Equivalent circuit of the part shown on the LHS seen from the terminals A and B

Kill the voltage source, and find R_{eq}.

$$\begin{aligned} R_{eq} &= 10 \,//\, 5 \\ &= 10 \times 5 /(10 + 5) \\ &= 10 / 3 \text{ Ohms} \end{aligned}$$

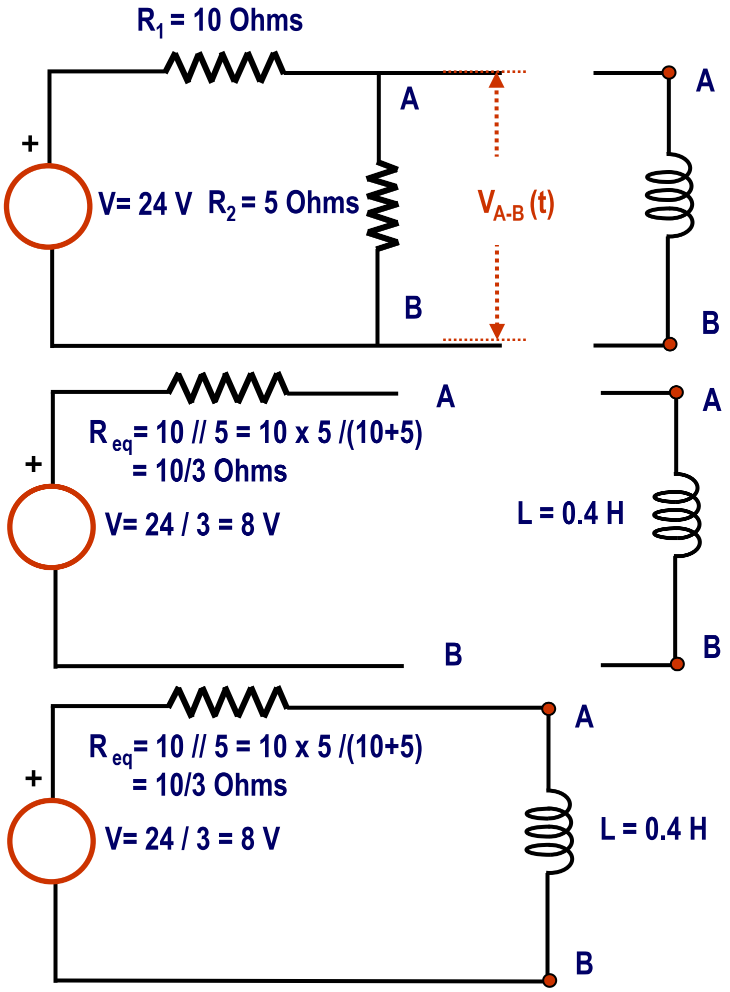
Example – 7 (Continued)

Solution (Continued)

Open circuit terminals A – B and find V_{AB}

$$V = 24\ V \times R_2 / (R_1 + R_2)$$
$$= 24 \times 5 / 15 = 24 / 3$$
$$= 8\ V$$

- Form the resulting Thevenin equivalent Circuit,
- Connect the inductance to the resulting Thevenin equivalent circuit,
- Solve the resulting circuit by using the straightforward method described in Example 6



R-L-C Circuits

Problem

Solve the following circuit for current waveform, which consists of a resistance, an inductance and a capacitance connected in series

Solution

Writing down KVL for the circuit;

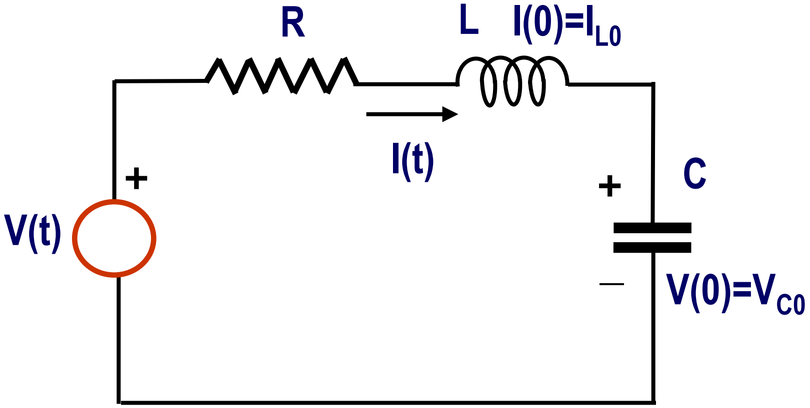
$$V(t) = R\, I(t) + V_L(t) + V_C(t)$$
$$= R\, I(t) + L\, dI(t)/dt + (1/C) \int I(t)dt + V_C(0)$$

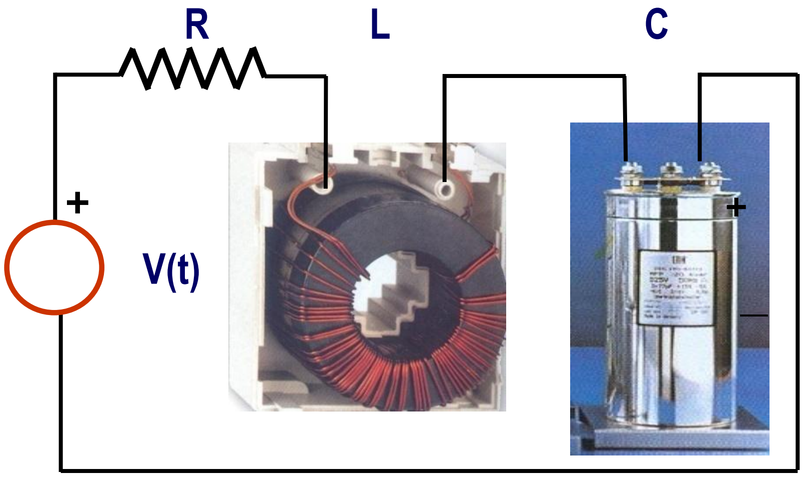
Differentiating both sides wrt time once;

$$dV(t)/dt = R\, dI(t)/dt + L d^2I(t)/dt^2 + (1/C)I(t)$$

or

$$d^2I(t)/dt^2 + (R/L)dI(t)/dt + (1/LC)\, I(t) = (1/L)\, dV(t)/dt$$





A second order ordinary differential equation

AC Circuits

Initial Conditions

Differential Equation

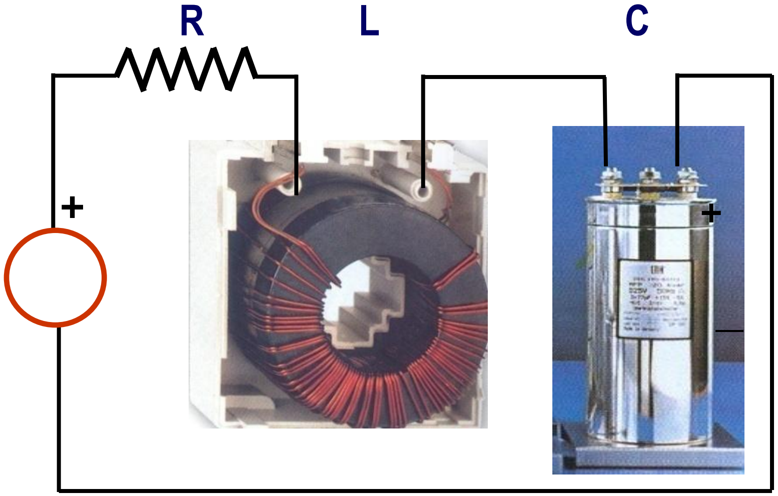
$$d^2I(t)/dt^2 + (R/L)dI(t)/dt + (1/LC)\, I(t) = (1/L)\, dV(t)/dt$$

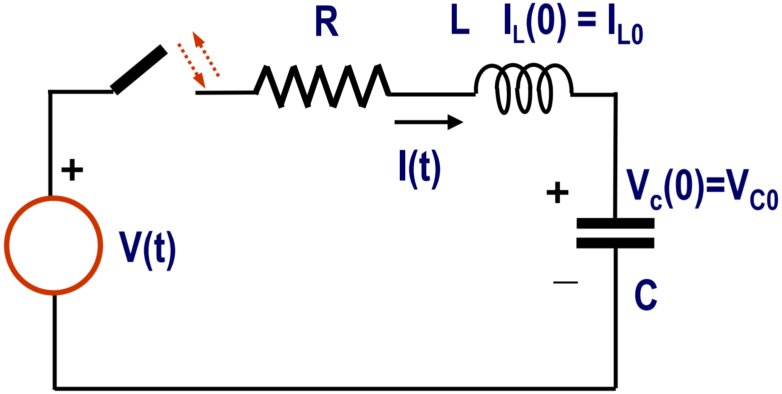
Initial Conditions

$$I_L(0) = I_{L0} \quad \boxed{1}$$

$$V_C(0) = V_{C0} \quad \boxed{2}$$

Please note that a differential equation needs initial conditions in number equal to its order, i.e. two here





Initial Conditions

Differential Equation

$$d^2I(t)/dt^2 + (R/L)dI(t)/dt + (1/LC)\,I(t) = (1/L)\,dV(t)/dt$$

Initial Conditions

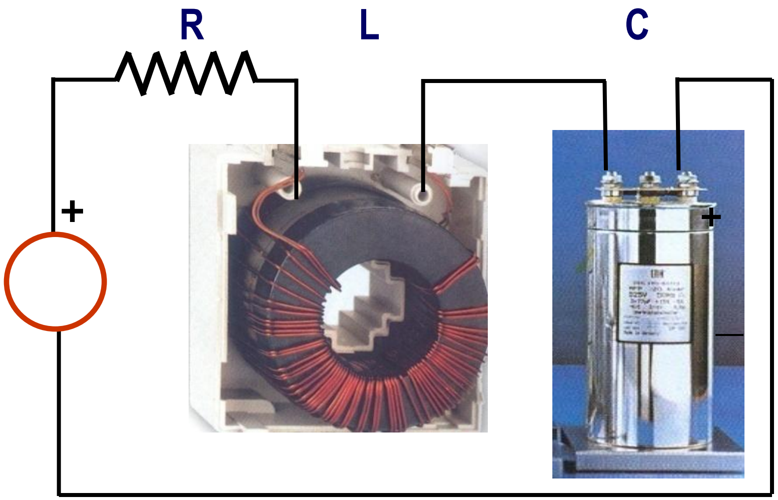
The voltage initial condition $V_C(0)$ may also be written as,

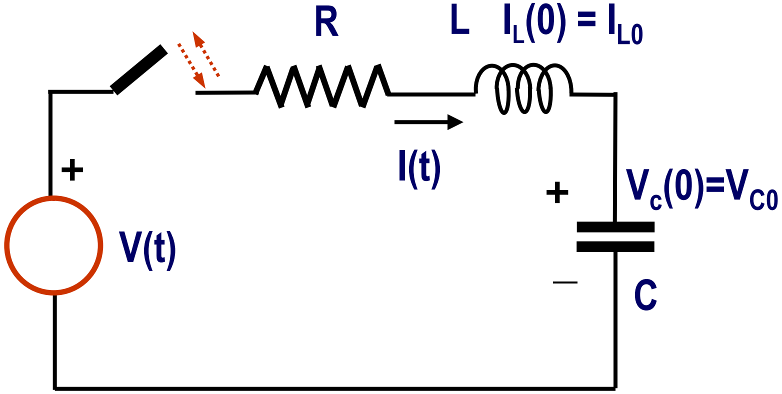
$$V_C(0) = V(0) - V_L(0) - V_R(0)$$

$$= V(0) - L\,d/dt\,I_L(0) - R\,I_L(0)$$

or

$$d/dt\,I_L(0) = I_L'(0) = (1/L)\,[\,V(0) - V_C(0) - R\,I_L(0)\,] \quad \boxed{2}$$





AC Circuits

Initial Conditions

Differential Equation

$$d^2I(t)/dt^2 + (R/L)dI(t)/dt + (1/LC)\ I(t) = (1/L)\ dV(t)/dt$$

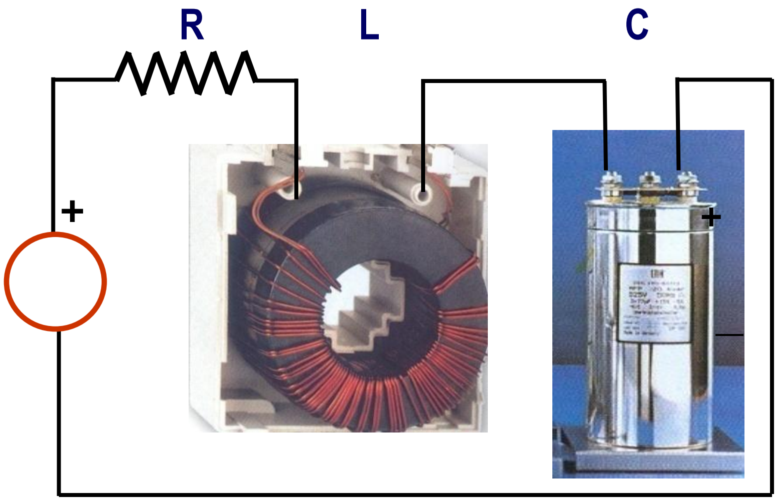
Initial Conditions

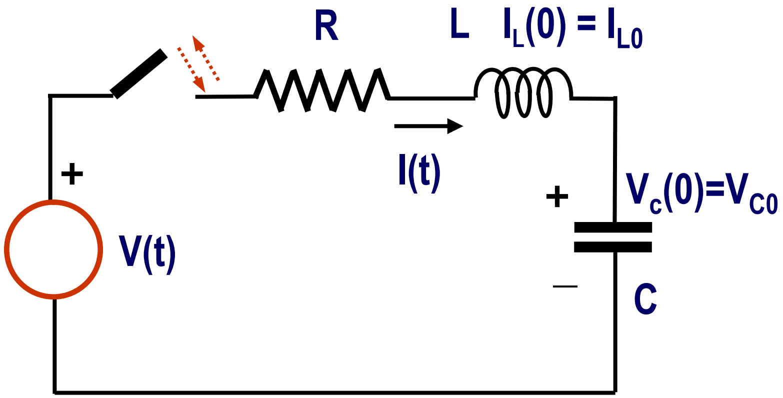
Thus the initial conditions in terms of the solution variable I(t) may now be written as,

$$I_L(0) = I_{L0} \quad \boxed{1}$$

$$I_L'(0) = (1/L)\ [\ V(0) - V_C(0) - R\ I_L(0)\ ] \quad \boxed{2}$$

Please note that a differential equation needs initial conditions in number equal to its order, i.e. two here





AC Circuits

Example

R-L-C Circuit

Solve the R-L-C circuit with the given parameters shown on the RHS for current I(t)

Writing down KVL for the circuit shown on the RHS the following ODE is obtained

$$d^2I(t)/dt^2 + (R/L)dI(t)/dt + (1/LC)\, I(t) = (1/L)\, dV(t)/dt$$

$2 / 1 = 2$

$1 / (LC) = 1/(1x\ 2.494\ x\ 10^{-3}) = 401$

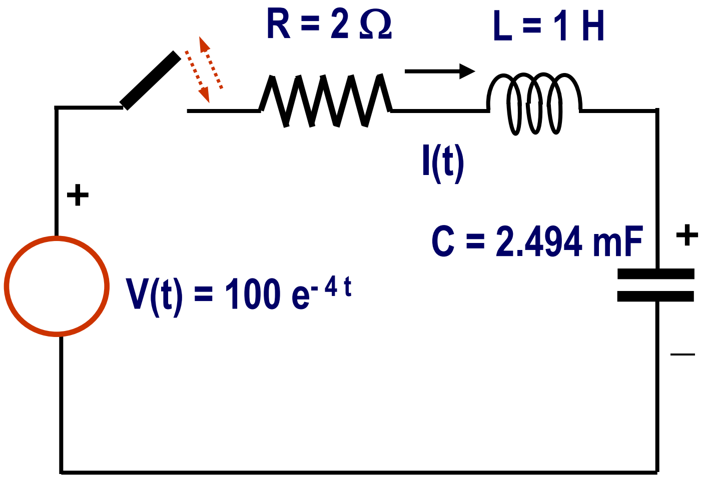
$$d^2I(t)/dt^2 + (2/1)\, dI(t)/dt + 401\, I(t) = (1/1)\, dV(t)/dt$$

$$d^2I(t)/dt^2 + 2\, dI(t)/dt + 401\, I(t) = d/dt\, (100\, e^{-4t}) = -\, 400\, e^{-4t}$$

Initial Conditions

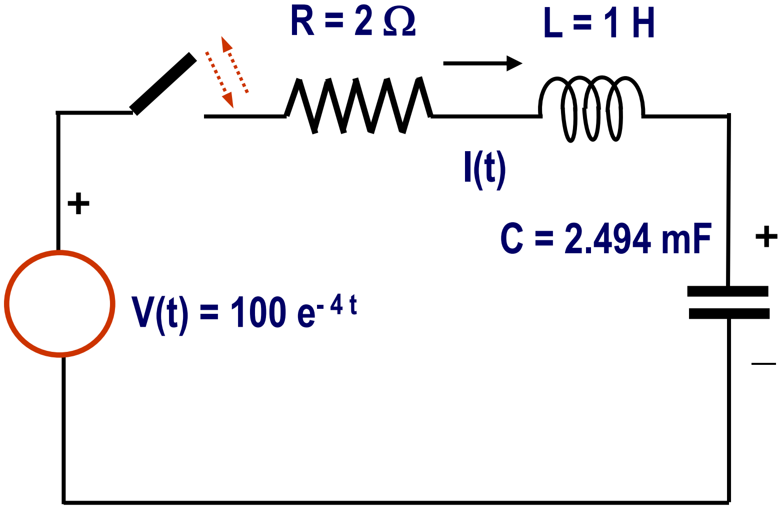
$V_C(0) = V_{C0} = 87\ Volts$

$I_L(0) = I_{L0} = 0$



AC Circuits

Solution



R-L-C Circuit

First, obtain the homogeneous equation by setting the RHS source function to zero

$$d^2I(t)/dt^2 + 2\ dI(t)/dt + 401\ I(t) = 0$$

Then, solve the characteristic equation

$$s^2 + 2\ s + 401 = 0$$

a =1 | b = 2 | c = 401

$$s_1, s_2 = (- b \mp \sqrt{b^2 - 4 \times a \times c\ )} / (2\ a)$$

$$= \ -1 \mp\ j\ 20$$

Eigenvalues of the differential equation

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AC Circuits

Solution

R-L-C Circuit

Then, the homogeneous solution becomes

$$I(t) = k_1\, e^{s1\,t} + k_2\, e^{s2\,t}$$

$$= k_1\, e^{(-1-j20)\,t} + k_2\, e^{(-1+j20)\,t}$$

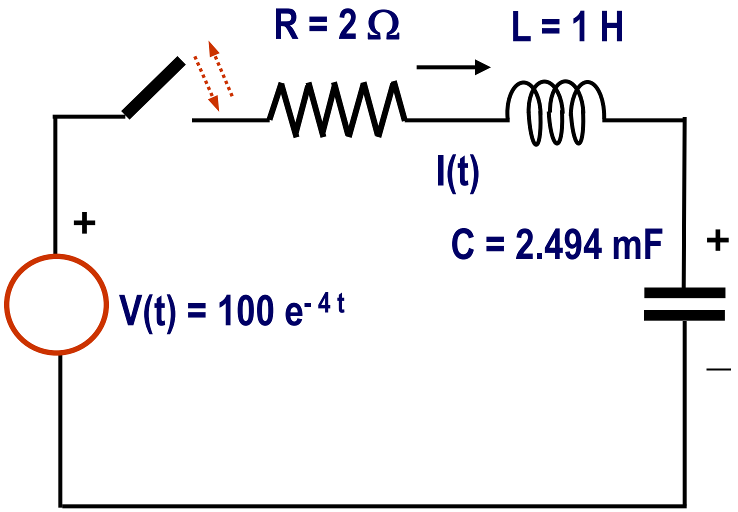
$$= k_1\, e^{-t} \times e^{-j20\,t} + k_2\, e^{-t} \times e^{j20\,t}$$

$$= e^{-t}\,(k_1\, e^{-j20\,t} + k_2\, e^{j20\,t})$$

$k_1(\cos\theta - j\sin\theta)$ $k_1(\cos\theta + j\sin\theta)$

$$= e^{-t}\,[\,k_1\,(\cos 20\,t - j\sin 20\,t\,)$$

$$+ k_2\,(\cos 20\,t + j\sin 20\,t\,)\,]$$



Taken from the Reference: Calculus and Analytic Geometry, Thomas, Addison Wesley, Third Ed. 1965, pp. 867

Euler's Identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$\theta = 20\,t$

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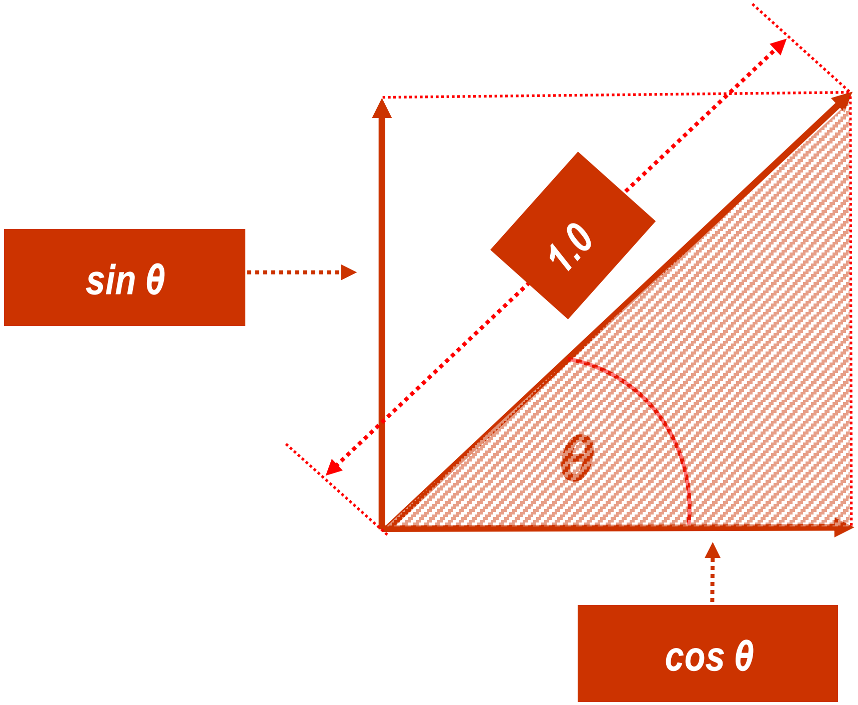
Euler's Identity

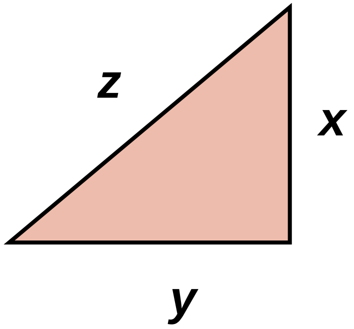
Definition

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\left| e^{j\theta} \right| = \left| \cos\theta + j\sin\theta \right|$$
$$= \sqrt{\cos^2\theta + \sin^2\theta}$$
$$= 1$$

Graphical Representaion





$$x^2 + y^2 = z^2$$
$$z = \sqrt{x^2 + y^2}$$

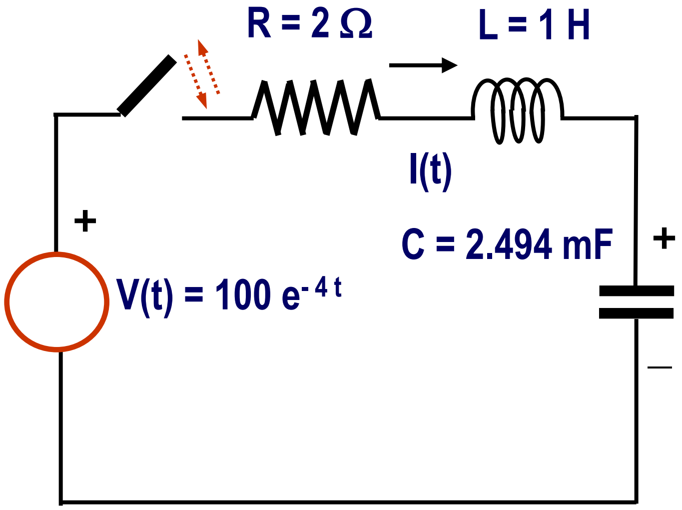
AC Circuits

Solution

R-L-C Circuit

Rearranging the terms

$$
\begin{aligned}
I(t) &= e^{-t}\,[\,k_1\,(\cos 20t - j \sin 20t\,) \\
&+ \quad k_2\,(\cos 20t + j \sin 20t\,)\,] \\
&= e^{-t}\,[\,\underbrace{(k_1 + k_2)}_{A} \cos 20t + \underbrace{(k_2 - k_1)}_{B} \sin 20t\,]
\end{aligned}
$$



Unknown coefficients to determined

Hence, the homogeneous solution (decaying sinusoidal term) becomes;

$$I(t) = e^{-t}\,(\,A \cos 20t + B \sin 20t\,)$$

AC Circuits

Solution

Nonhomogeneous Solution (Transient Term)

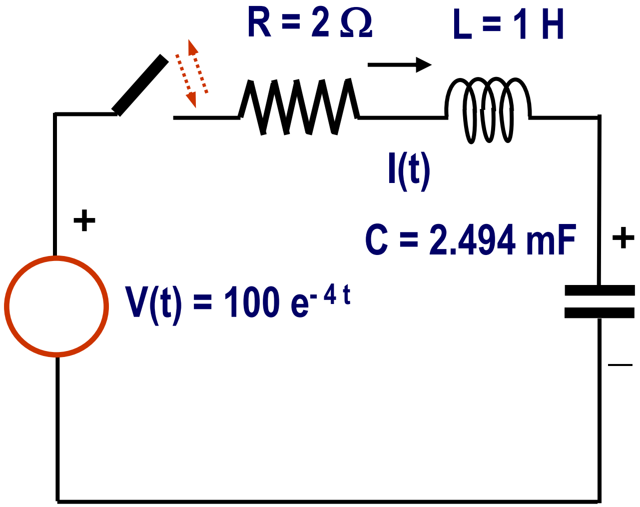
Now the nonhomogeneous solution (transient term) is to be determined

Definition of the Nonhomogeneous Solution

General form of the nonhomogeneous solution (transient term) may be expressed as

$$I_n(t) = c\, e^{-4t}$$

where, $I_n(t)$ is the nonhomogeneous solution,
c is an unknown coefficient to be determined



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AC Circuits

Solution

Nonhomogeneous Solution (Transient Term)

Substitute the nonhomogeneous solution;

$$I_n(t) = c\, e^{-4t}$$

to the given differential equation;

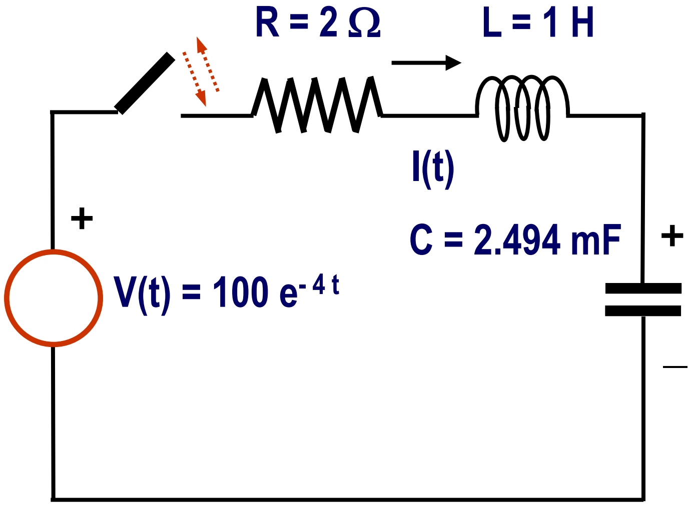
$$d^2I(t)/dt^2 + 2\, dI(t)/dt + 401\, I(t) = -\,400\, e^{-4t}$$

and solve it for the unknown coefficient c

$$d^2(ce^{-4t})/dt^2 + 2\, d(ce^{-4t})/dt + 401\, ce^{-4t} = -400\, e^{-4t}$$

$$16c\,\cancel{e^{-4t}} + 2c(-4\,\cancel{e^{-4t}}) + 401\, c\,\cancel{e^{-4t}} = -\,400\,\cancel{e^{-4t}}$$

These terms cancel



AC Circuits

Solution

Nonhomogeneous Solution (Transient Term)

These terms cancel

$$d^2(ce^{-4t})/dt^2 + 2\,d(ce^{-4t})/dt + 401\,ce^{-4t} = -400\,e^{-4t}$$

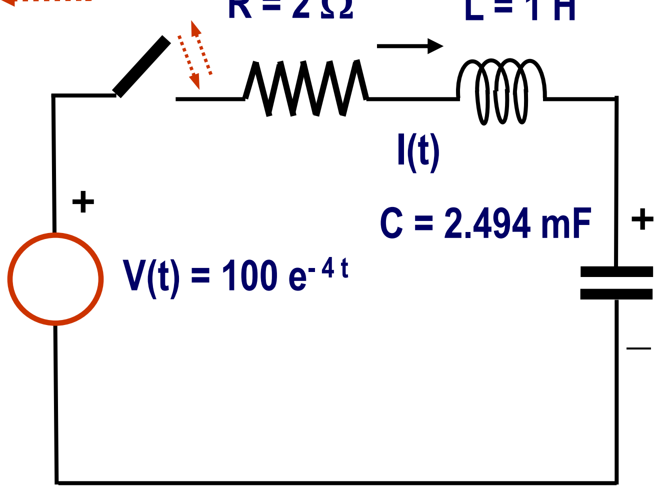
$$16c\,\cancel{e^{-4t}} + 2c(-4\,\cancel{e^{-4t}}) + 401\,c\,\cancel{e^{-4t}} = -400\,\cancel{e^{-4t}}$$

$$16\,c - 8\,c + 401\,c = -400$$

$$409\,c = -400 \quad \text{or} \quad c = -400/409 = -0.97799$$

Thus, the nonhomogeneous solution becomes;

$$I_n(t) = -0.97799\,e^{-4t}$$



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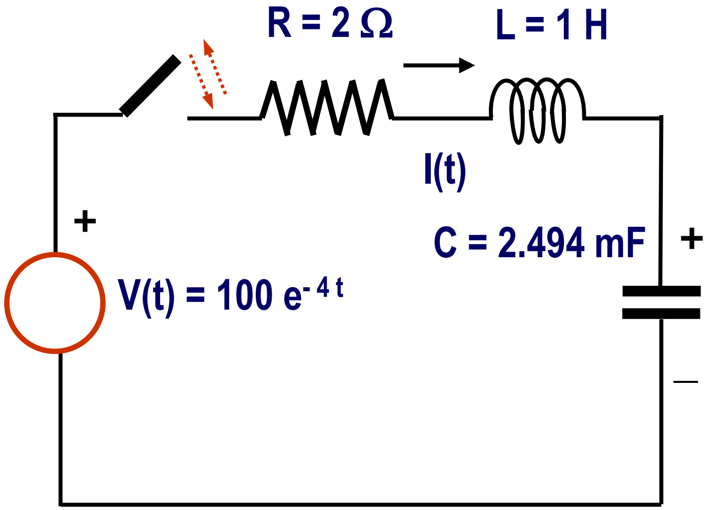
Solution

Complete Solution

Complete solution is the summation of the homogeneous (decaying sinusoidal) and nonhomogeneous solutions (transient term)



$$I(t) = \overbrace{-0.97799\, e^{-4t}}^{\textit{Transient Term}} + \overbrace{e^{-t}\,(A\cos 20t + B\sin 20t\,)}^{\textit{Decaying Sinosoidal Term}}$$

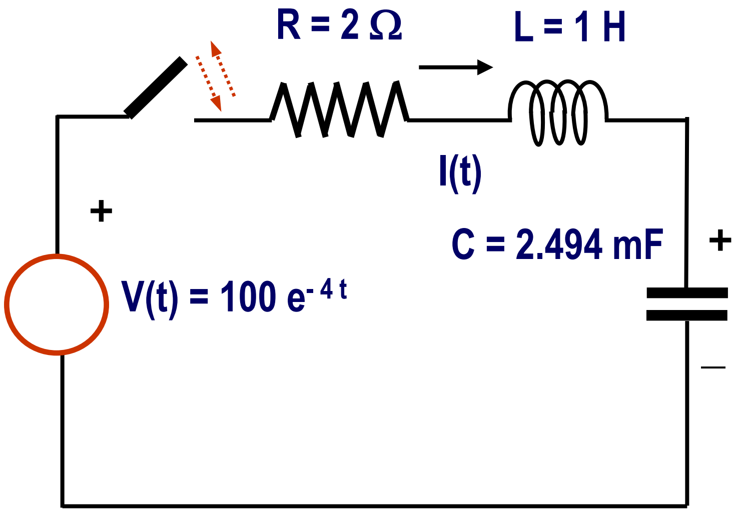
Unknown coefficients to determined

AC Circuits

Solution

Determination of the Unknown Coefficients

$$\underbrace{I(t) = -\,0.97799\, e^{-4t}}_{\text{Transient Term}} + \underbrace{e^{-t}\,(\,A \cos 2t + B \sin 2t\,)}_{\text{Decaying Sinosoidal Term}}$$



The above solution must satisfy the given initial conditions;

$I_L(0) = I_{L0} = 0$

$V_C(0) = V_{C0} = 87$ Volts

AC Circuits

Solution

Determination of the Unknown Coefficients

$I_L(0) = I_{L0} = 0$

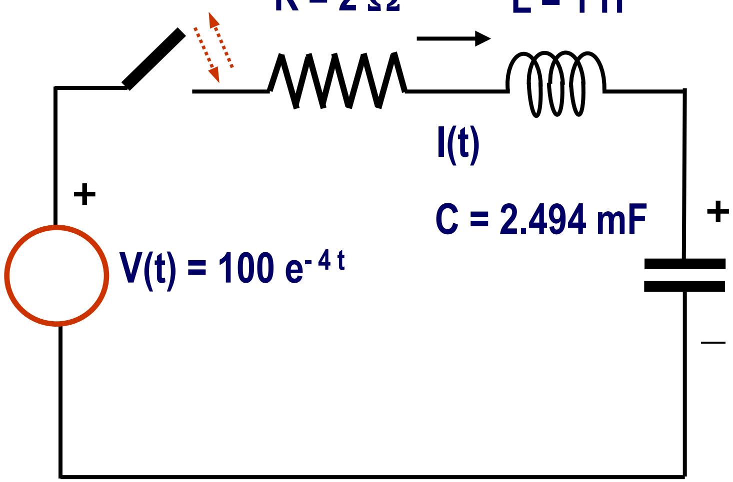
$V_C(0) = V_{C0} = 87$ Volts

Substitute the given initial conditions into the complete solution equation and solve for the unknown coefficients A and B;

$$I(t) = -\,0.97799\, e^{-4t} + e^{-t}\,(A \cos 2t + B \sin 2t)$$

Transient Term

Decaying Sinosoidal Term



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AC Circuits

Solution

Determination of the Unknown Coefficients

$I_L(0) = I_{L0} = 0$
$V_C(0) = V_{C0} = 87$ Volts

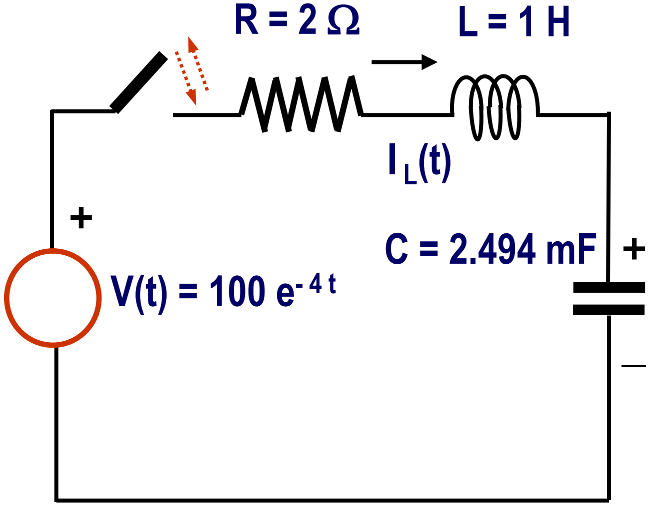
Substitute the given initial conditions into the complete solution equation and solve for the unknown coefficients A and B;

$$I_L(t) = -0.97799\, e^{-4t} + e^{-t}\,(A \cos 20t + B \sin 20t)$$

$$I_L(0) = -0.97799\, \overset{1}{\cancel{e^{0}}} + \overset{1}{\cancel{e^{0}}}\,(A\, \overset{1}{\cancel{\cos 0}} + \overset{0}{\cancel{B \sin 0}})$$

$$= -0.97799 + A = 0$$

$$A = 0.97799$$



AC Circuits

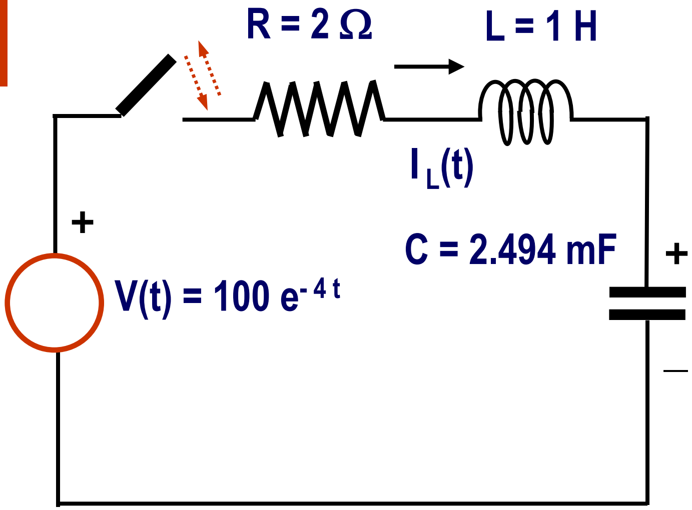
Solution

Determination of the Unknown Coefficients

$I_L(0) = I_{L0} = 0$

$V_C(0) = V_{C0} = 87$ Volts

$I_L(t) = -0.97799\, e^{-4t} + e^{-t}\,(A \cos 20t + B \sin 20t)$



$V(0) = 100\, e^{-4t}$

$= 100\, e^{-4 \times 0}$

$= 100$ V

87 V

0 V

$$\frac{d}{dt} I_L(0) = I_L'(0) = (1/L)\,[\,\underbrace{V(0)} - \underbrace{V_C(0)} - \underbrace{R\, I_L(0)}\,]$$

$$= (1/1)\,(100 - 87 - 2 \times 0)$$

$$= 13 \text{ Amp/sec}$$

AC Circuits

Solution

Determination of the Unknown Coefficients

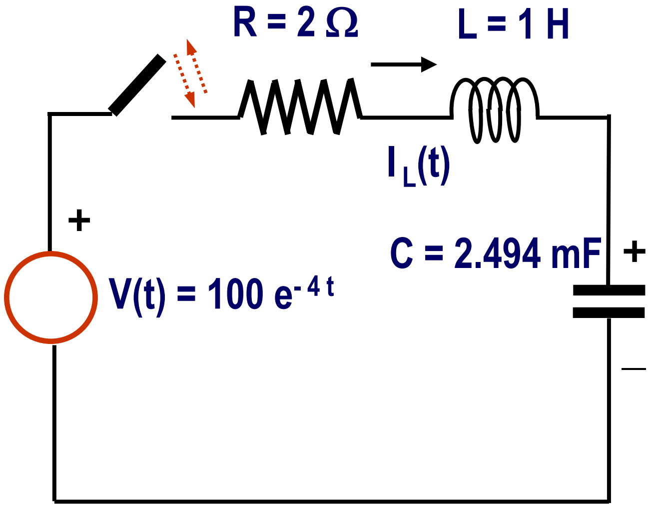
$$\frac{d}{dt} I_L(t) = 0.97799 \text{ x } 4\, e^{-4t} - e^{-t} (A \cos 20t + B \sin 20t) + e^{-t} (-20 A \sin 20 t + 20 B \cos 20t)$$

$$\frac{d}{dt} I_L(0) = 0.97799 \text{ x } 4 \overset{1}{e^{0}} - \overset{1}{e^{0}} (A \overset{1}{\cos 0} + \overset{0}{B \sin 0}) + \overset{1}{e^{0}} (-20 A \overset{0}{\sin 0} + 20 B \overset{1}{\cos 0}) = 13 \text{ Amp/sec}$$

$$= 0.97799 \text{ x } 4 - A + 20 B = 13 \text{ Amp/sec}$$

$$= 0.97799 \text{ x } 4 - 0.97799 + 20 B = 13 \text{ Amp/sec}$$

$$B = (13 - 3 \text{ x } 0.9799) / 20 = 0.5033$$

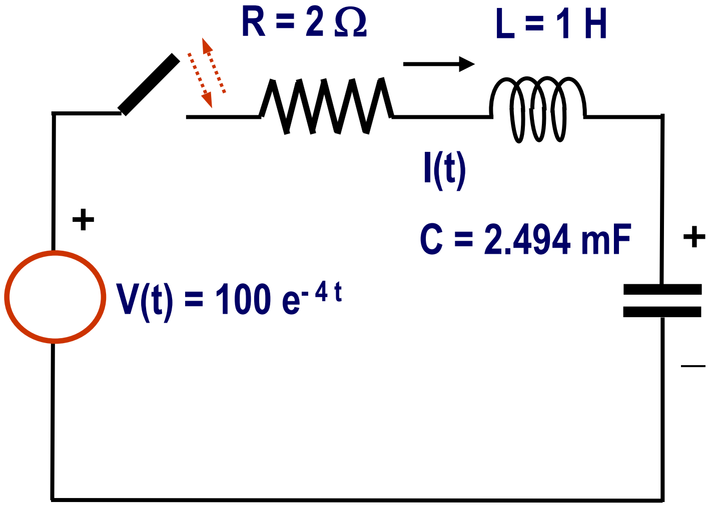


AC Circuits

Solution Terms

General form of the Solution

$$\underbrace{I_L(t) = -\,0.9799\, e^{-4t}}_{\text{Transient Term}} + \underbrace{e^{-t}\,(\,0.97799 \cos 20t + 0.5033 \sin 20t\,)}_{\text{Decaying sinosoidal Term}}$$



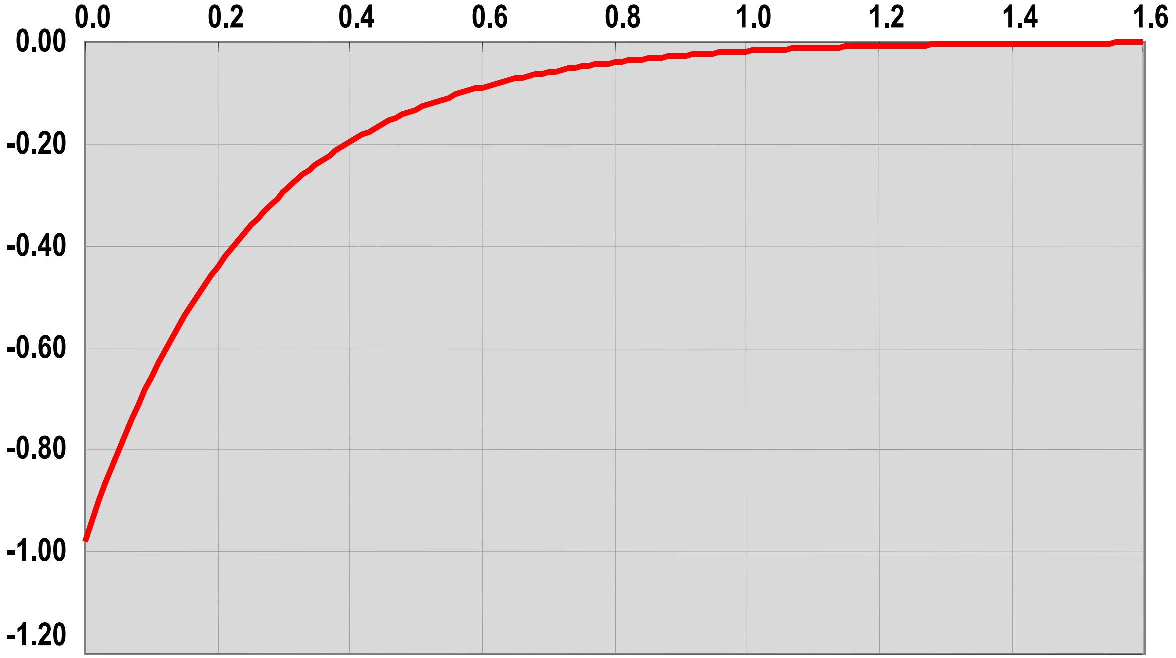
AC Circuits

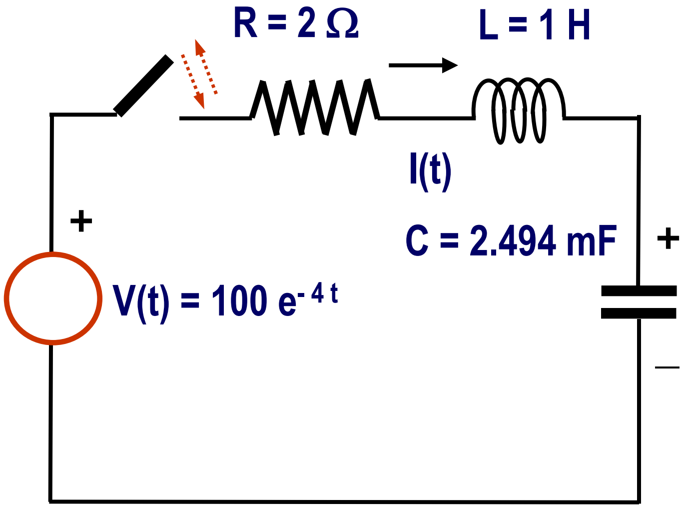
Solution Terms

Transient Term

$$I_L(t) = \boxed{- 0.9799\, e^{-4t}} + e^{-t}\,(\, 0.97799 \cos 20t + 0.5033 \sin 20t\,)$$

Transient Term





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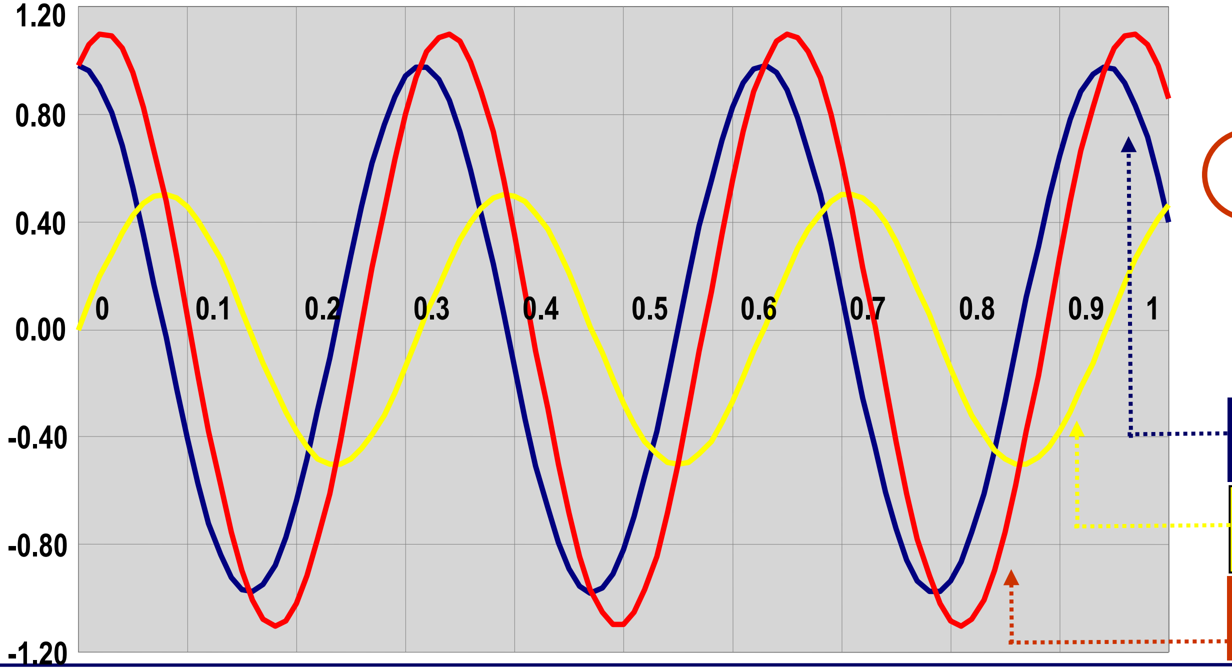
AC Circuits

Solution Terms

Sinusoidal Terms

$$I_L(t) = -\ 0.9799\ e^{-4t}\ + e^{-t}\left(\ 0.97799 \cos 20t + 0.5033 \sin 20t\ \right)$$

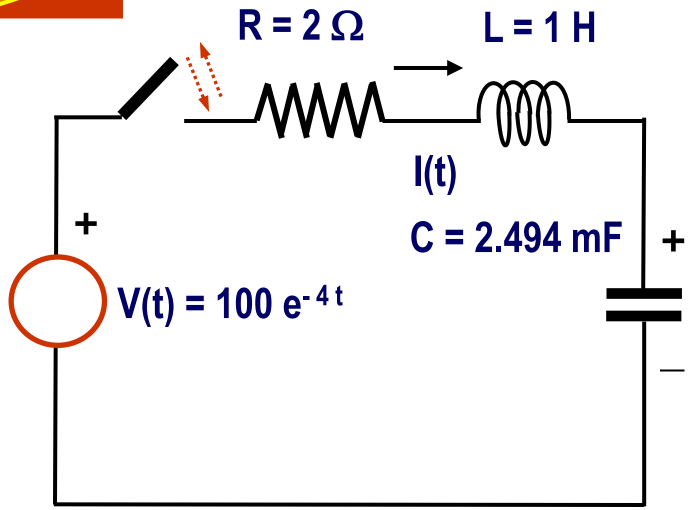
Sinosoidal Terms



0.9779 cos 20t

0.5033 sin 20t

0.9779 cos 20t + 0.5033 sin 20t

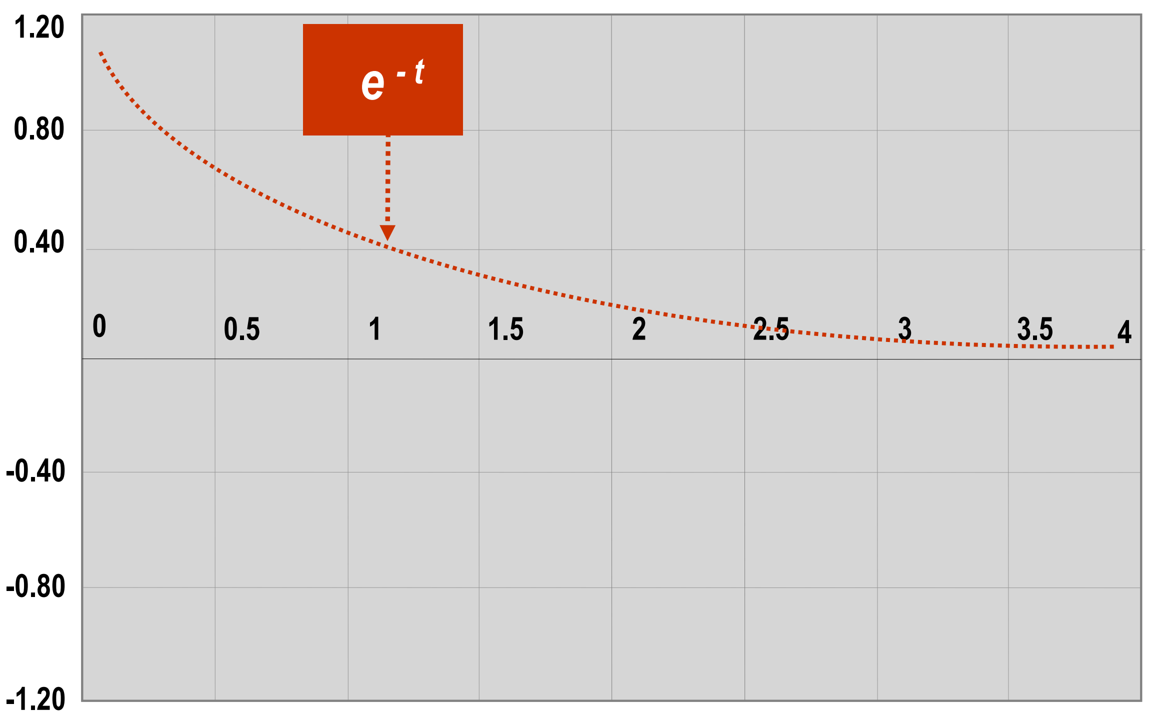


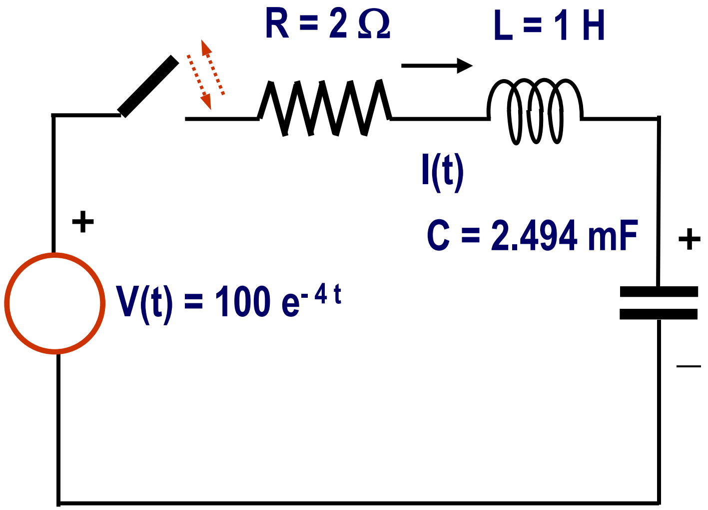
AC Circuits

Solution Terms

Exponentially Decaying Sinusoidal Term

$I_L(t) = - 0.9799\, e^{-4t} + e^{-t} (0.97799 \cos 20t + 0.5033 \sin 20t)$

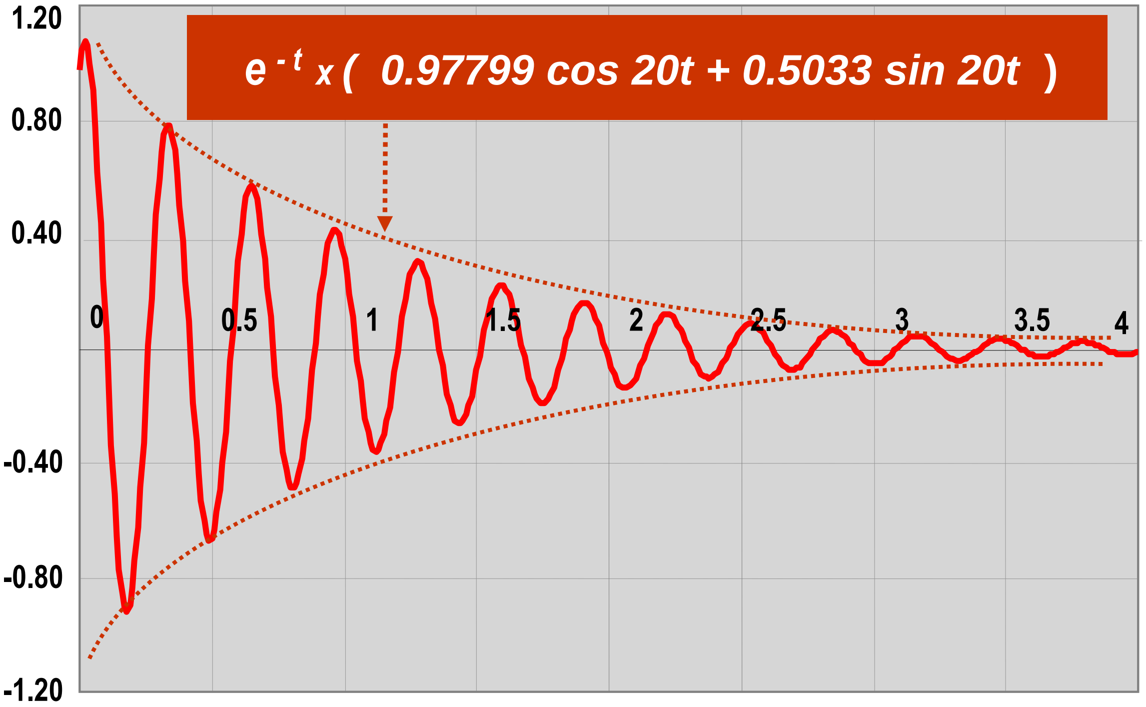


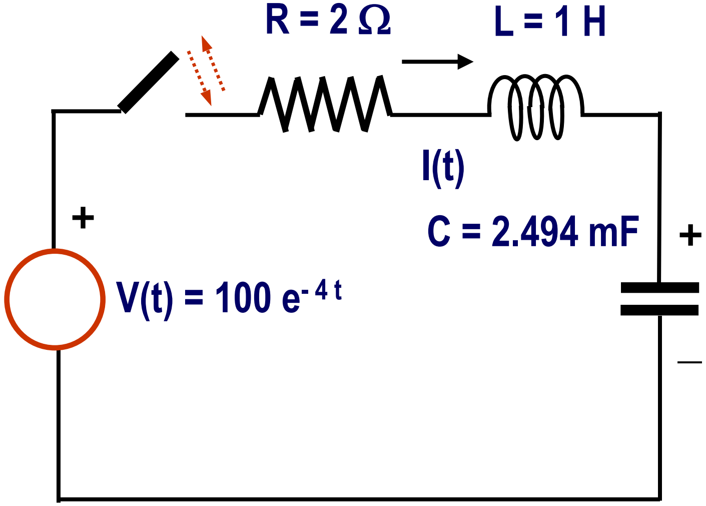


Solution Terms

Exponentially Decaying Sinusoidal Term

$I_L(t) = -0.9799\, e^{-4t} + e^{-t}(\ 0.97799 \cos 20t + 0.5033 \sin 20t\ )$



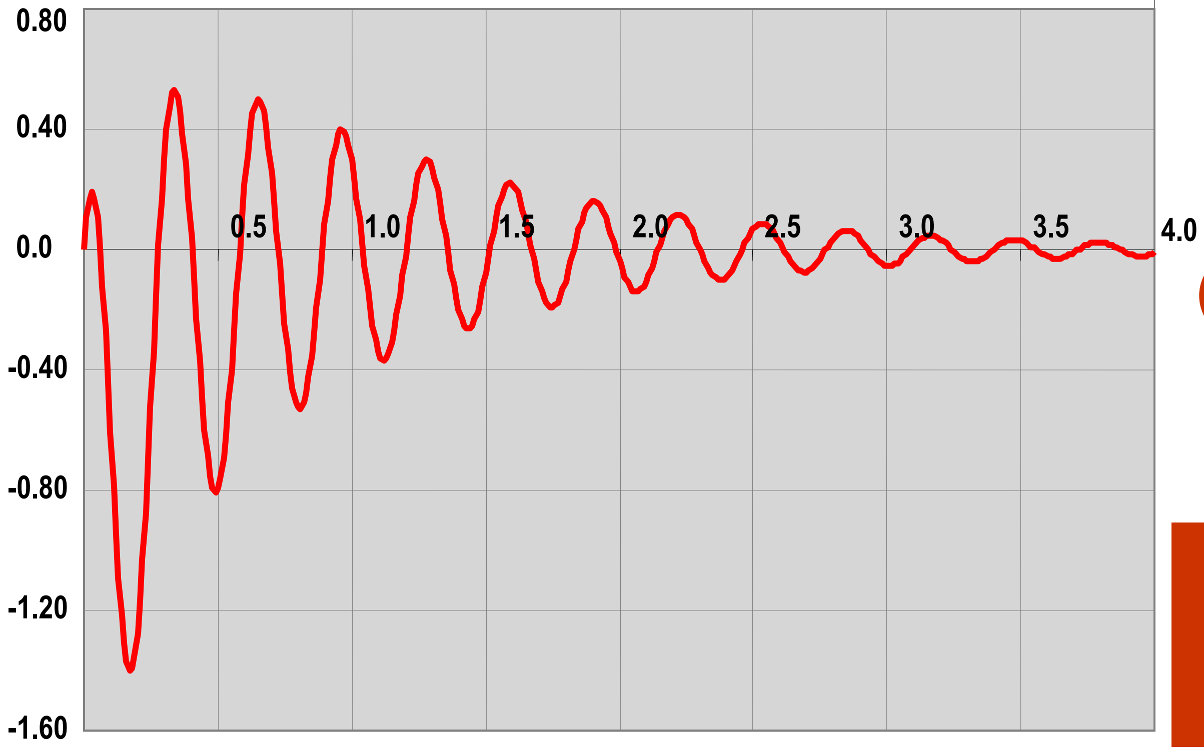


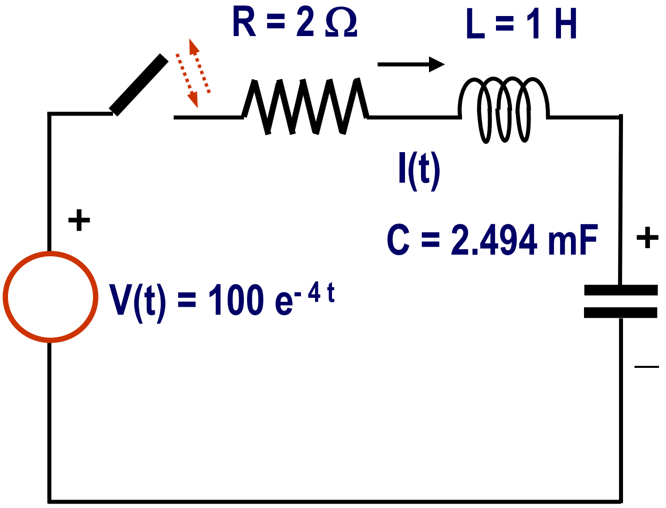
AC Circuits

Solution Terms

Overall Solution

$$I_L(t) = -\,0.9799\,e^{-4t} + e^{-t}\,(\;0.97799\cos 20t + 0.5033\sin 20t\;)$$





Homework:
Solve the same problem for the case that the voltage source has a sinusoidal waveform

AC Circuits

RMS Value

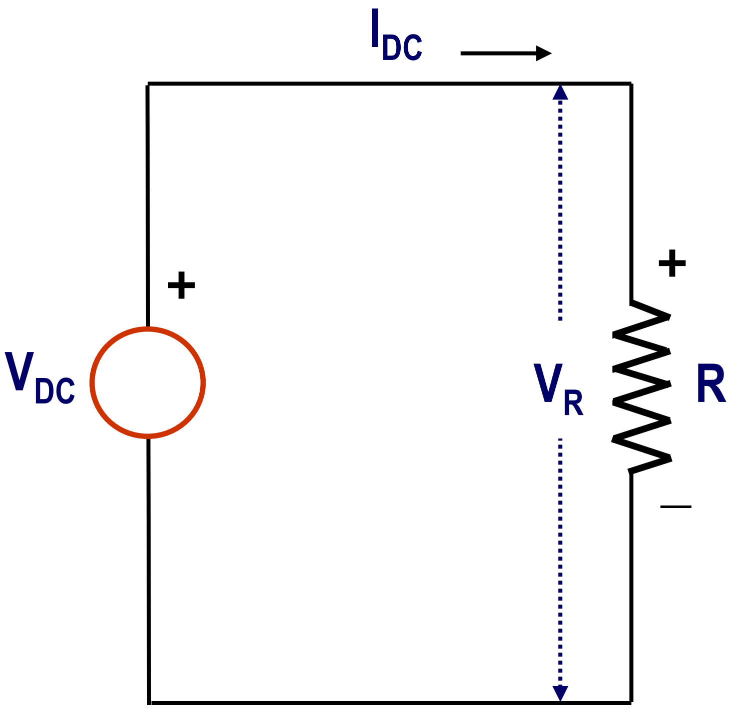
Definition: RMS (Root Mean Square)

RMS value of an AC current is the value of DC current that would dissipate the same amount of power on a resistance R

The above definition is based on the principle of equating the heating effects of the AC and DC currents calculated in both cases

First, calculate the power dissipated (heating effect) in the resistance R in the DC circuit shown on the RHS

$$
\begin{aligned}
V_R &= R \times I_{DC} \\
P_{DC} &= V_R \times I_{DC} \\
&= R \times I_{DC}^2
\end{aligned}
$$



AC Circuits

RMS Value

RMS (Root Mean Square)

Now, calculate the power dissipated (the heating effect) in the resistance R in the AC circuit shown on the RHS

$$V_R(t) = I(t) \times R$$

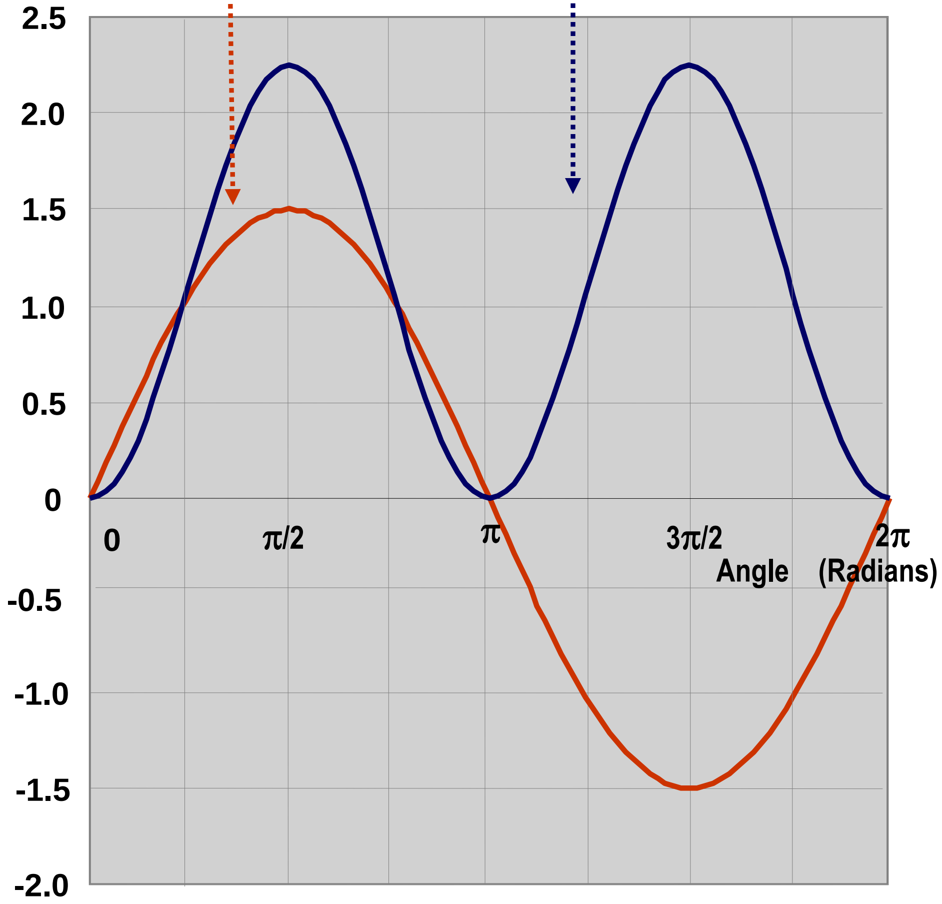
$$P_{AC}(t) = V(t) \times I(t)$$

$$= R\, I(t)^2 = R\,((V_{max} / R) \sin wt)^2$$

$$= R\,(I_{max} \sin wt)^2$$

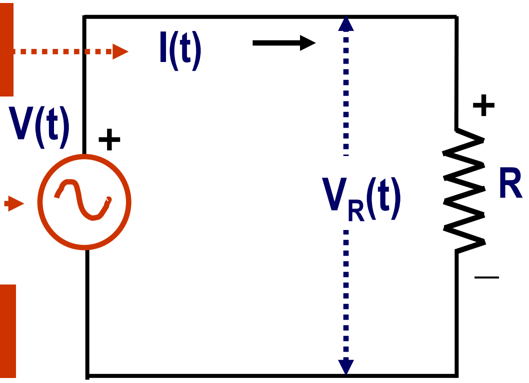
$I(t) = I_{max} \sin wt$

$I^2(t) = (I_{max} \sin wt)^2$



$$I(t) = \underbrace{(V_{max} / R)}_{I_{max}} \sin wt$$

$$V(t) = V_{max} \sin wt$$



AC Circuits

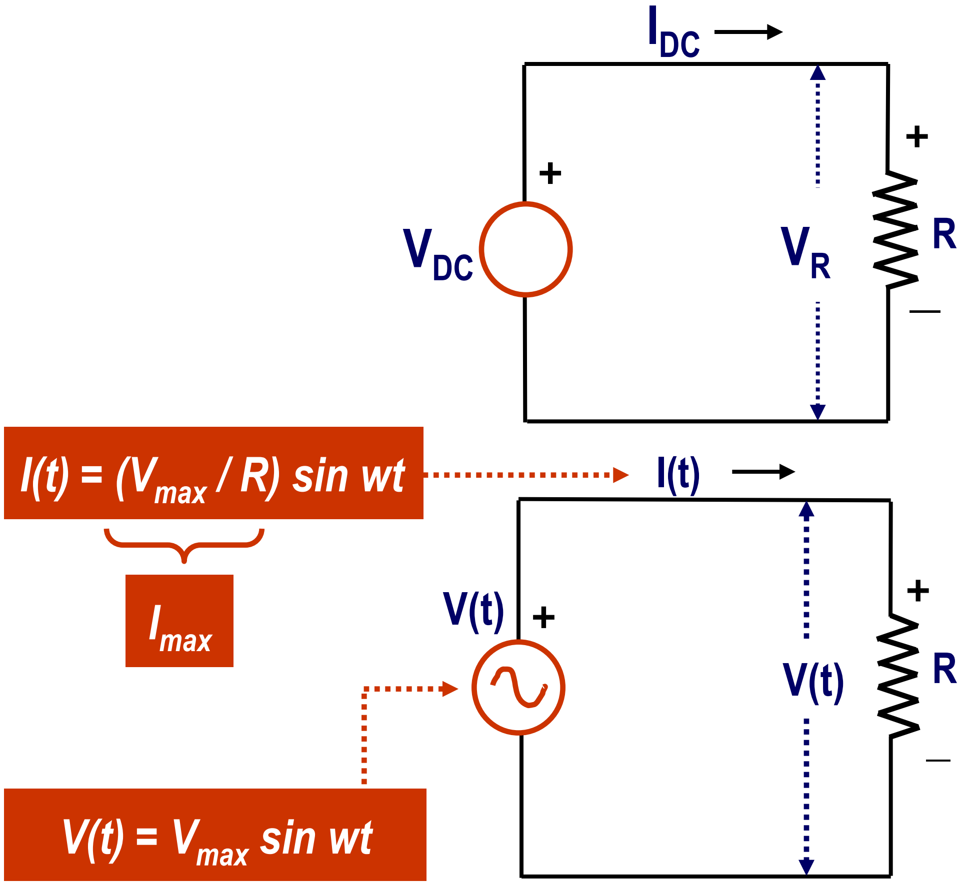
RMS Value

Definition

$$V_R = R \ x \ I_{DC}$$
$$P_{DC} = V_R \ x \ I_{DC}$$
$$= R \ x \ I_{DC}^2$$

$$V_R\,(t) = R \ x \ I(t)$$
$$P_{AC}\,(t) = V(t) \ x \ I(t)$$
$$= R \ I(t)^2 = R\ ((V_{max}\,/\,R) \sin wt\,)^2$$
$$= R\ (I_{\,max} \sin wt\,)^2$$

Now let us equate the power dissipations in the above cases



$$\underbrace{I(t) = (V_{max}\,/\,R)}_{I_{max}} \sin wt$$

$$V(t) = V_{max} \sin wt$$

AC Circuits

RMS Value

Definition

$V_R = R \times I_{DC}$

$P_{DC} = V_R \times I_{DC}$

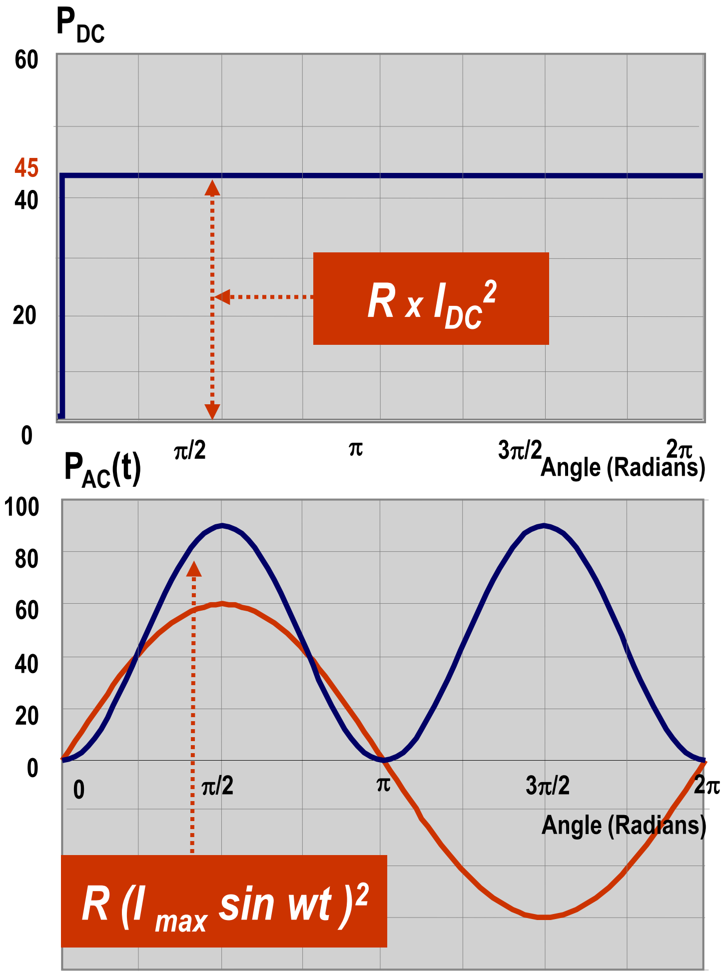
$= R \times I_{DC}^2$

$V_R(t) = R \times I(t)$

$P_{AC}(t) = V(t) \times I(t)$

$= R\, I(t)^2 = R\, ((V_{max} / R) \sin wt)^2$

$= R\, (I_{max} \sin wt)^2$



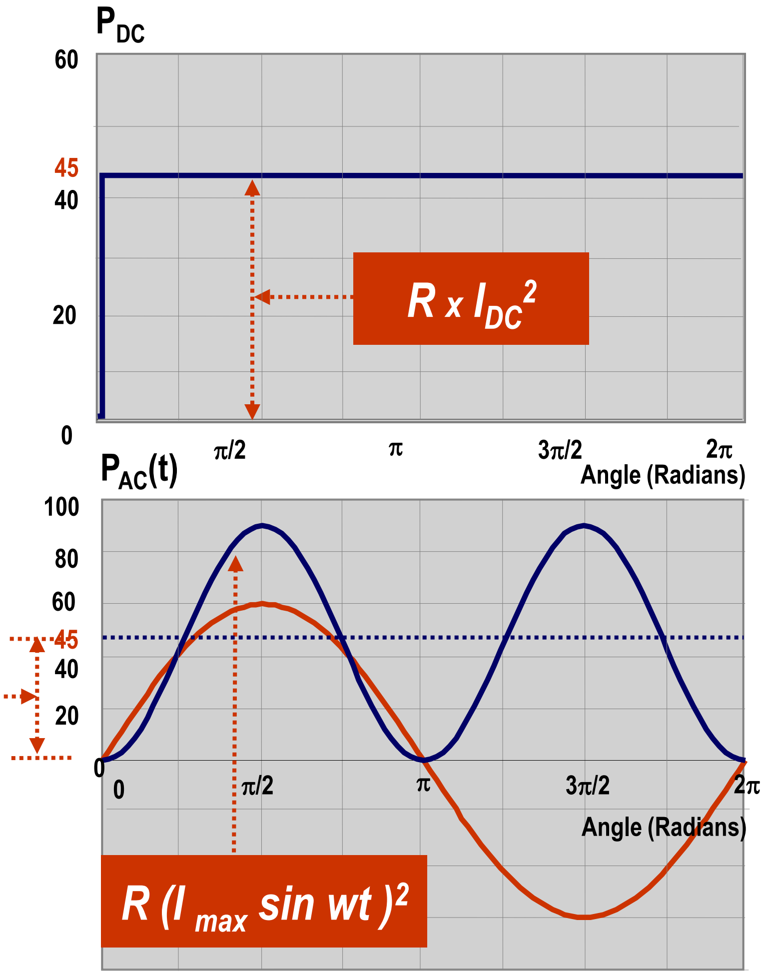
AC Circuits

RMS Value

Definition

In principle, the DC power waveform is NOT equal to the AC power waveform

They may however, be equal in terms of their averages, i.e. the mean of DC power transferred to the load over a period may be equated to that of AC power transferred to the load within a period



Average(P_{AC})

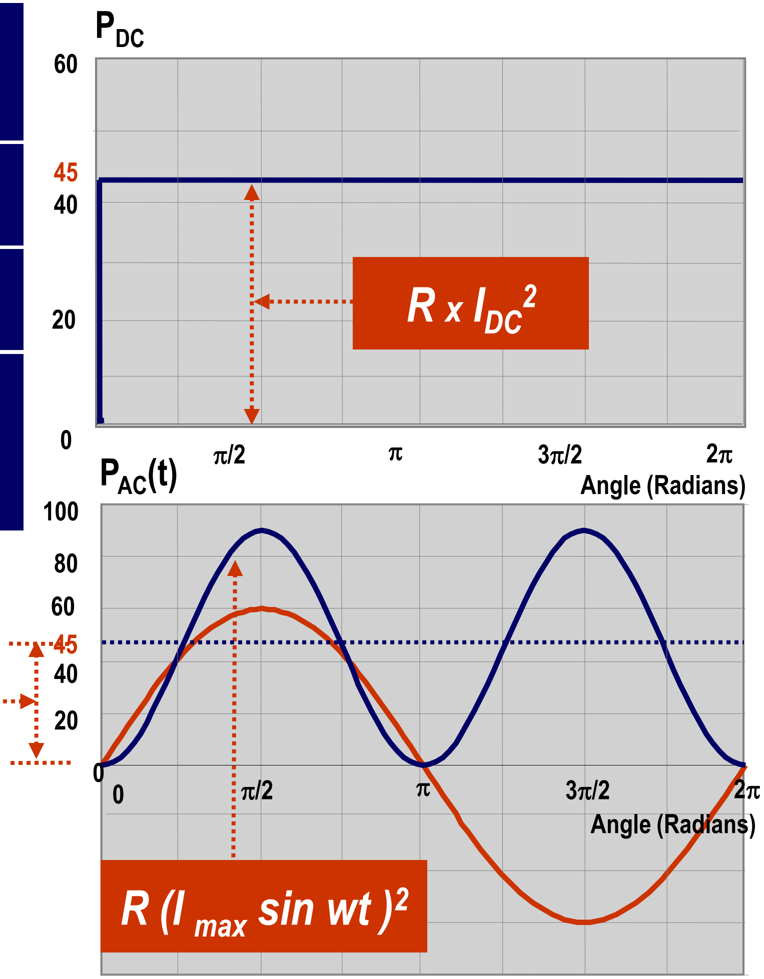
RMS Value

Definition

Hence, equating the averages of the two terms;

$$Average(\, P_{DC}) = Average(P_{AC})$$

$$P_{DC\ avg} = (1/T) \int_0^T R\, I_{DC}^2\, dt = (1/T)\, R\, I_{DC}^2\, t \Big|_0^T = R\, I_{DC}^{\ 2}$$



$Average(P_{AC})$

AC Circuits

RMS Value

Definition

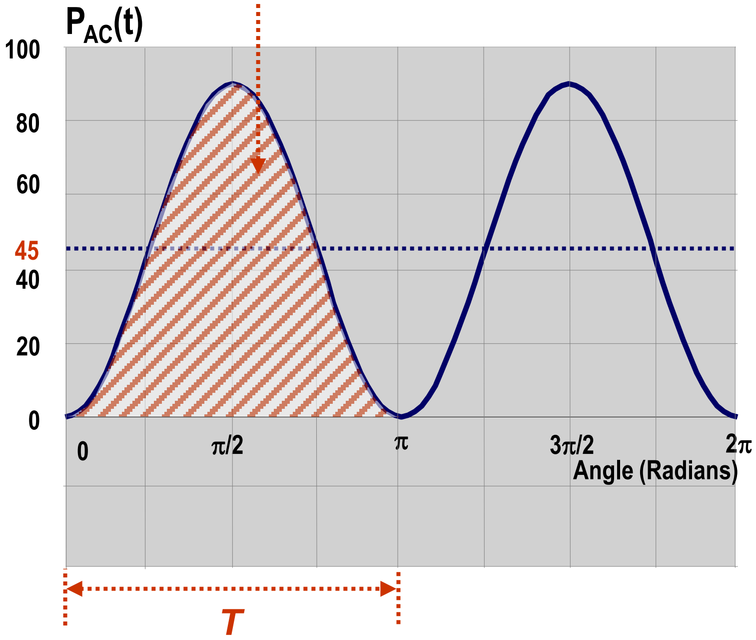
Please note that average value of an AC waveform is calculated as

$$P(t)_{avg} = (1/T) \int_0^T P_{AC}(t)\, dt$$

$$= (1/T) \int_0^T R\, I(t)^2\, dt$$

$$Average = \frac{\text{Area under the Curve}}{\text{Period}}$$

$$= (1/T) \int_0^T P_{AC}(t)\, dt$$



AC Circuits

RMS Value

Definition

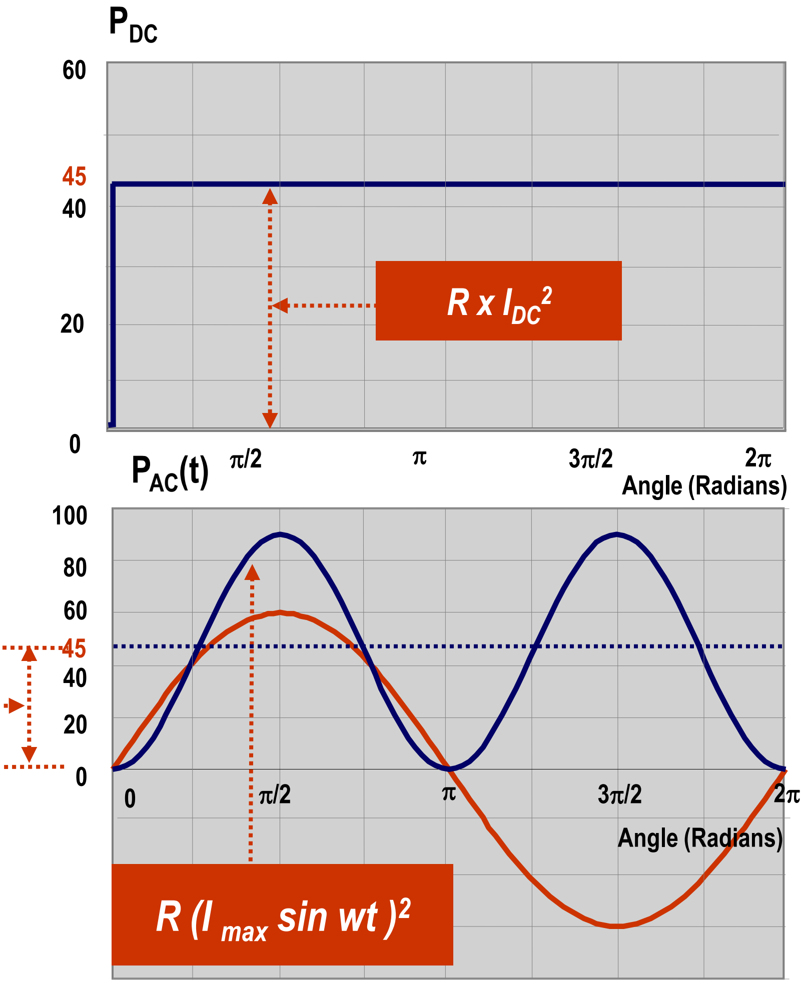
Hence, equating the average of the two terms;

$$\cancel{R}\, I_{DC}^{\;2} = (1/T)\,\cancel{R} \int_0^T I(t)^2 \; dt$$

$$I_{DC}^{\;2} = (1/T) \int_0^T I(t)^2 \; dt$$

$$I_{rms} = \sqrt{(\,1/T\,) \int I(t)^2 \, dt}$$

I_{rms}



METU

AC Circuits

Example

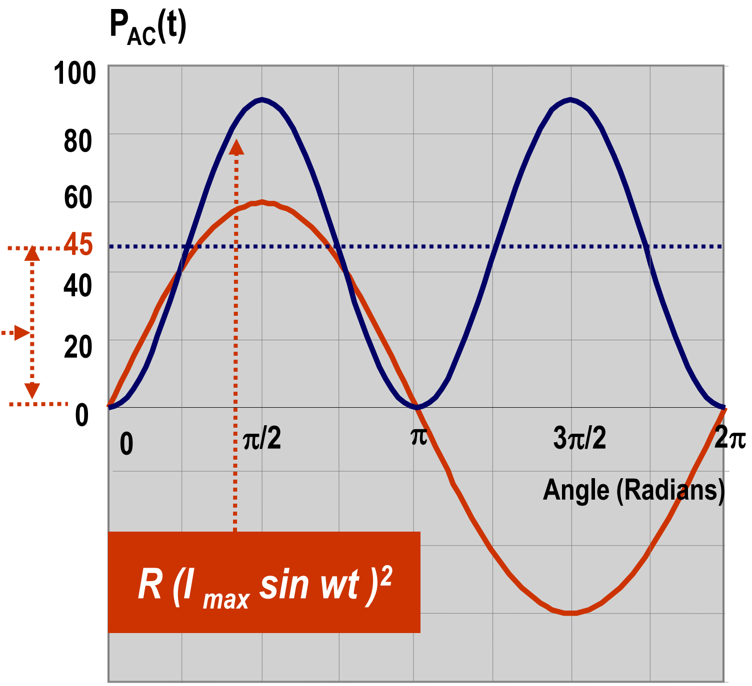
Problem

Find the RMS value of the sinusoidal current waveform;

$$I(t) = I_{max}\ \sin wt$$

shown on the RHS

I_{rms}



Solution

$$I_{rms} = \sqrt{(1/T) \int I(t)^2\, dt}$$

$$= \sqrt{(1/T) \int [\, I_{max} \sin wt\,]^2\, dt}$$

$$= I_{max} \sqrt{(1/T) \int \sin^2 wt\ \ dt}$$

METU

AC Circuits

RMS Value

Solution

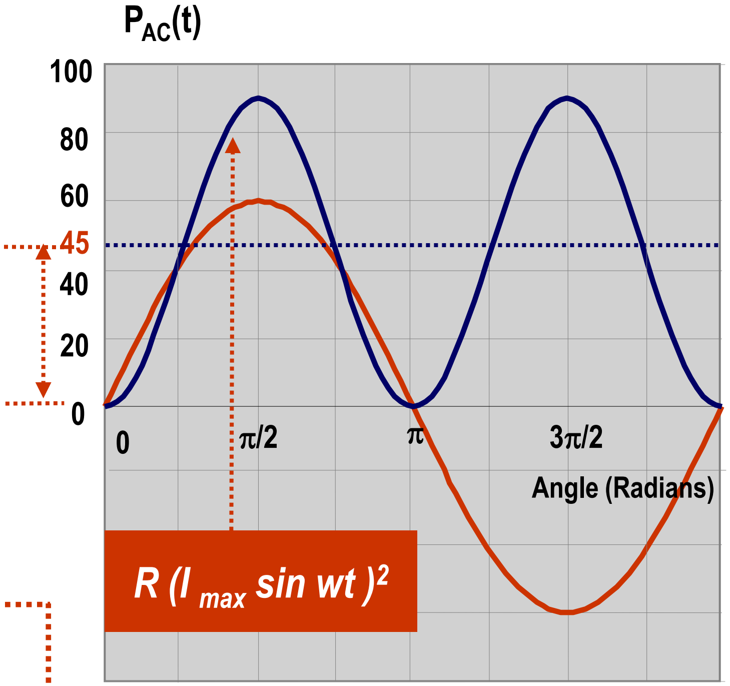
$$I_{RMS} = I_{max} \sqrt{(1/T) \int \sin^2 wt \;\; dt}$$

Now use;

$$\begin{aligned} \sin^2 wt &= 1 - \cos^2 wt \\ &= 1 - (1 + \cos 2wt)/2 \\ &= 1 - ½ - ½ \cos 2wt \\ &= ½ - ½ \cos 2wt \end{aligned}$$

$$I_{RMS} = I_{max} \sqrt{(1/T) \int (½ - ½ \cos 2wt)\, dt}$$

$$I_{RMS} = I_{max} \sqrt{(1/T) [\int ½\, dt \;-\; \underbrace{\int ½ \cos 2wt\, dt}_{= 0}]}$$



Taken from the Reference: Calculus and Analytic Geometry, Thomas, Addison Wesley, Third Ed. 1965, pp. 348

AC Circuits

RMS Value

Solution

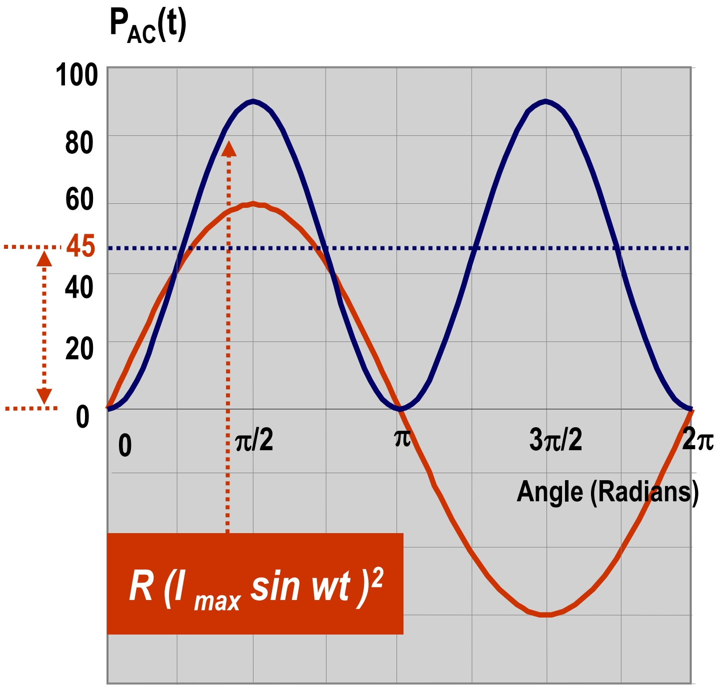
$$I_{RMS} = I_{max} \sqrt{(1/T)\left[\int \tfrac{1}{2}\,dt - \underbrace{\int \tfrac{1}{2}\cos 2wt\,dt}_{=\,0}\right]}$$

$$I_{RMS} = I_{max} \sqrt{(1/T)\ \tfrac{1}{2}\int dt} = I_{max}\sqrt{\tfrac{1}{2}\,(1 \cancel{/\ T})\ \cancel{T}}$$

$$I_{DC} = \sqrt{\tfrac{1}{2}}\ I_{max}$$

$$= I_{max} / \sqrt{2} = I_{rms} = I_{max} \times 0.7071$$

__Rule: RMS value of a sinusoidal waveform is 1/ √2 of its peak value__



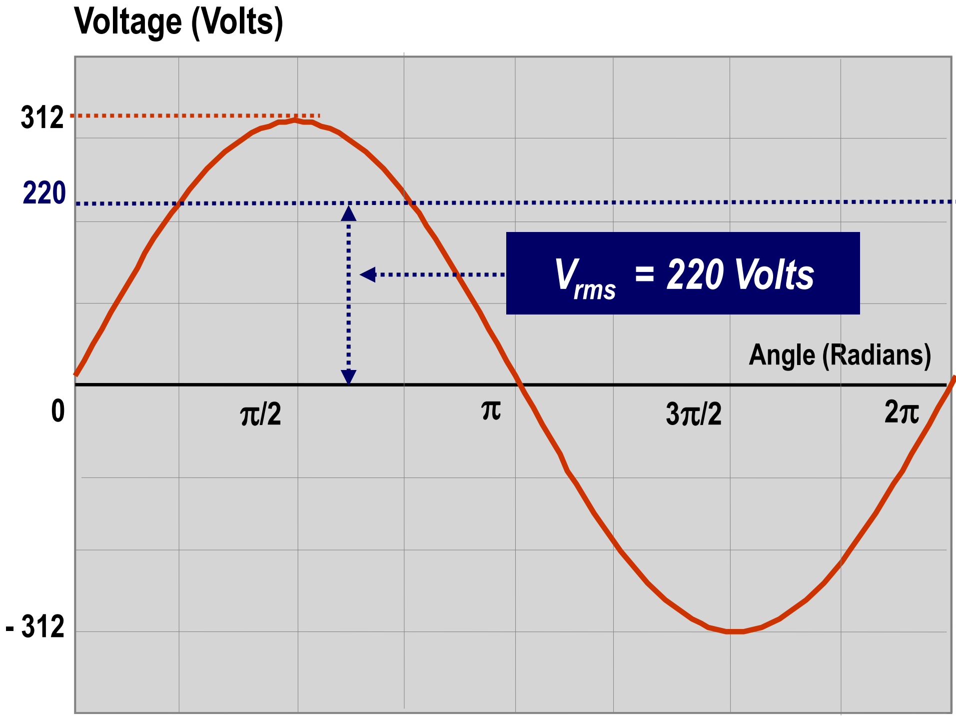
RMS Value of a Sinusoidal Waveform

Problem

Calculate the RMS value of the sinusoidal voltage waveform shown on the RHS

$$V_{rms} = V_{max} / \sqrt{2}$$
$$= V_{max} \times 0.7071$$
$$= 312 \times 0.7071 = 220 \text{ Volts}$$

RMS value of the domestic voltage in Turkey



V_{rms} = 220 Volts

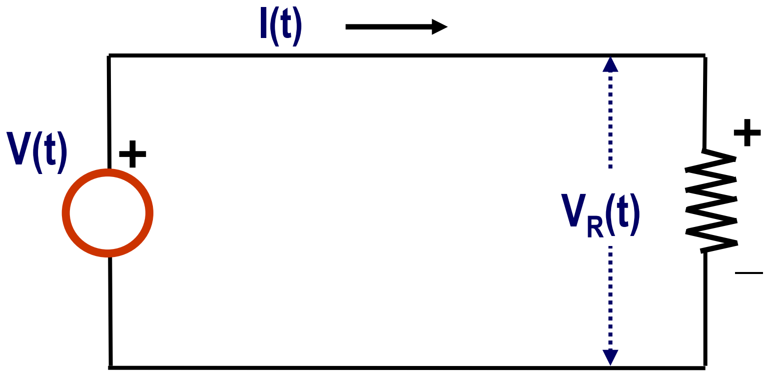
AC Circuits

Example - 8

Problem

$V_{rms} = \sqrt{(1/T) \int V(t)^2 dt}$

Calculate the RMS value of the rectangular voltage waveform shown on the RHS



$$V_{rms} = \sqrt{(1/T) \int V(t)^2 dt}$$

$$= \sqrt{(1/0.3) [\int_0^{0.1} 4^2 dt + \int_{0.1}^{0.3} (-4)^2 dt]}$$

= 4 Volts