Modern Control

“Linear System Theory and Design”, Chapter 1

http://zitompul.wordpress.com

2 0 1 4

- **The class materials of the modern control part are based on the Lecture note slides of the Modern Control course offered by Dr.-Ing. Erwin Sitompul, President University, Indonesia.**

https://zitompul.wordpress.com/1-ee-lectures/6-modern-control/

- Textbook used in this part is:

“Linear System Theory and Design”, Third Edition, Chi-Tsong Chen

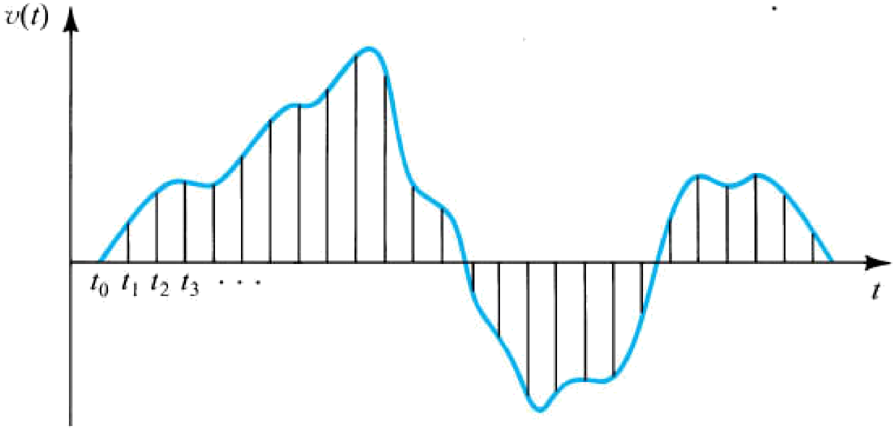
Classical Control and Modern Control

Classical Control	Modern Control
SISO (Single Input Single Output)	MIMO (Multiple Input Multiple Output)
Low order ODEs	High order ODEs, PDEs
Time-invariant	Time-invariant and time variant
Fixed parameters	Changing parameters
Linear	Linear and non-linear
Time-response approach	Time- and frequency response approach
Continuous, analog	Tends to be discrete, digital
Before 80s	80s and after

- The difference between classical control and modern control originates from the different modeling approach used by each control.
- The modeling approach used by modern control enables it to have new features not available for classical control.

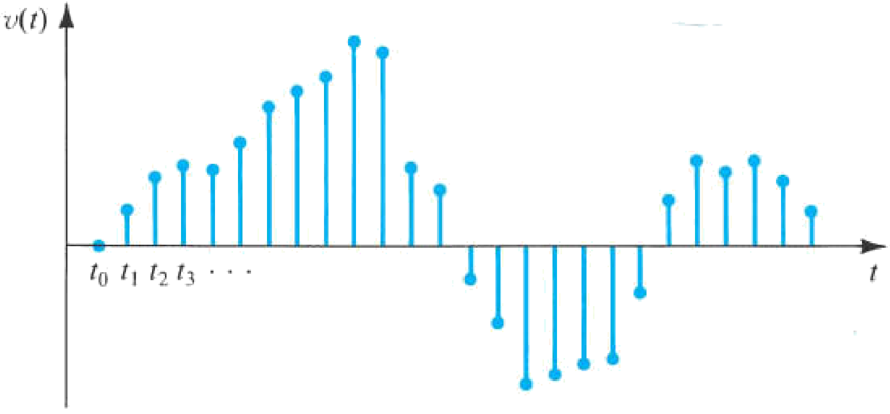
Signal Classification

- Continuous signal



(a)

- Discrete signal



(b)

Classification of Systems

- Systems are classified based on:
    - *The number of inputs and outputs*: single-input single-output (SISO), multi-input multi-output (MIMO), MISO, SIMO.
    - *Existence of memory*: if the current output depends on the current input only, then the system is said to be memoryless, otherwise it has memory → purely resistive circuit vs. RLC-circuit.
    - *Causality*: a system is called causal or non-anticipatory if the output depends only on the present and past inputs and independent of the future unfed inputs.
    - *Dimensionality*: the dimension of system can be finite (lumped) or infinite (distributed).
    - *Linearity*: superposition of inputs yields the superposition of outputs.
    - *Time-Invariance*: the characteristics of a system with the change of time.

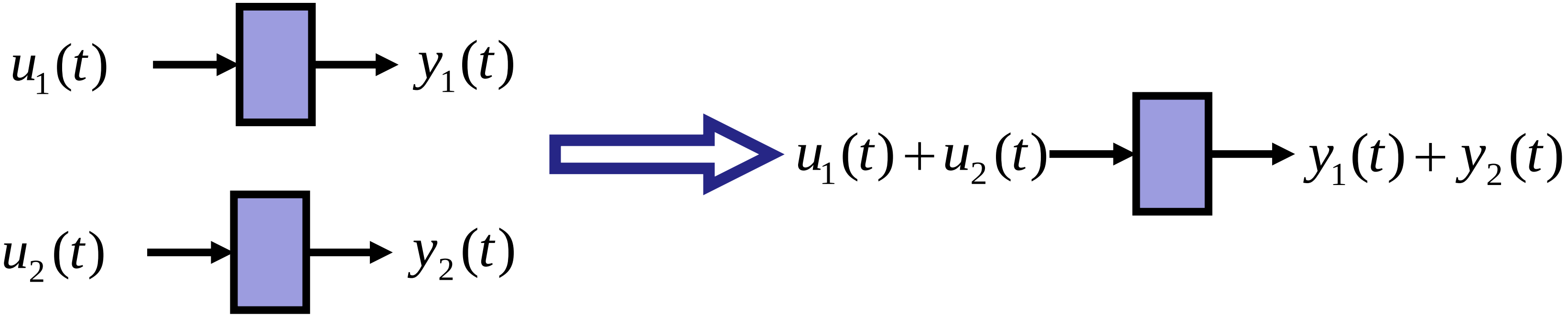
Classification of Systems

- Finite-dimensional system (lumped-parameters, described by differential equations
  - Linear and nonlinear systems
  - Continuous- and discrete-time systems
  - Time-invariant and time-varying systems

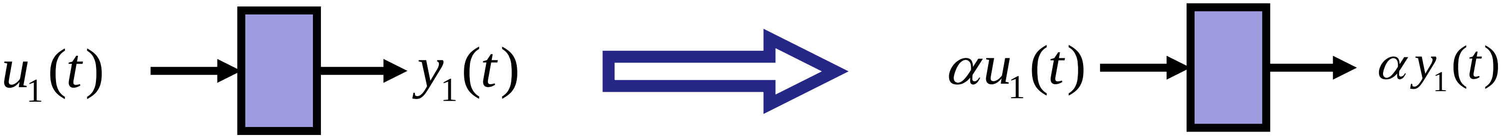
- Infinite-dimensional system (distributed-parameters, described by partial differential equations)
  - Heat conduction
  - Power transmission line
  - Antenna
  - Fiber optics

Linear Systems

- A system is said to be linear in terms of the system input *u*(t) and the system output *y*(t) if it satisfies the following two properties of superposition and homogeneity.
  - Superposition

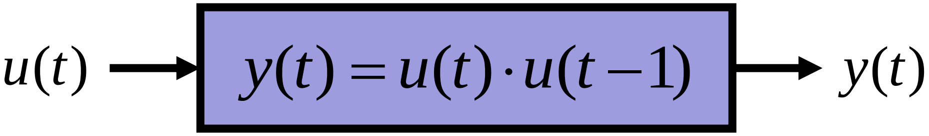


  - Homogeneity



Example: Linear or Nonlinear

Check the linearity of the following system.



Let $u(t)=u_1(t)$, then $y_1(t)=u_1(t)\cdot u_1(t-1)$

Let $u(t)=\alpha u_1(t)$, then $y_1(t)=\alpha u_1(t)\cdot \alpha u_1(t-1)$

$$=\alpha^2 u_1(t)\cdot u_1(t-1)$$

Thus $f(\alpha u(t))\neq \alpha y(t)$ → The system is **nonlinear**

Example: Linear or Nonlinear

Check the linearity of the following system (governed by ODE).

$$u(t) \longrightarrow \boxed{y''(t)+2y'(t)+y(t)=u'(t)+3u(t)} \longrightarrow y(t)$$

Let $y_1''(t)+2y_1'(t)+y_1(t)=u_1'(t)+3u_1(t)$

$y_2''(t)+2y_2'(t)+y_2(t)=u_2'(t)+3u_2(t)$

Then

$$\begin{aligned}
&[\alpha u_1(t)+\beta u_2(t)]'+3[\alpha u_1(t)+\beta u_2(t)]\\
&=\alpha u_1'(t)+\beta u_2'(t)+\alpha 3u_1(t)+\beta 3u_2(t)\\
&=\alpha[u_1'(t)+3u_1(t)]+\beta[u_2'(t)+3u_2(t)]\\
&=\alpha[y_1''(t)+2y_1'(t)+y_1(t)]+\beta[y_2''(t)+2y_2'(t)+y_2(t)]\\
&=[\alpha y_1(t)+\beta y_2(t)]''+2[\alpha y_1(t)+\beta y_2(t)]'+[\alpha y_1(t)+\beta y_2(t)]
\end{aligned}$$

→ The system is **linear**

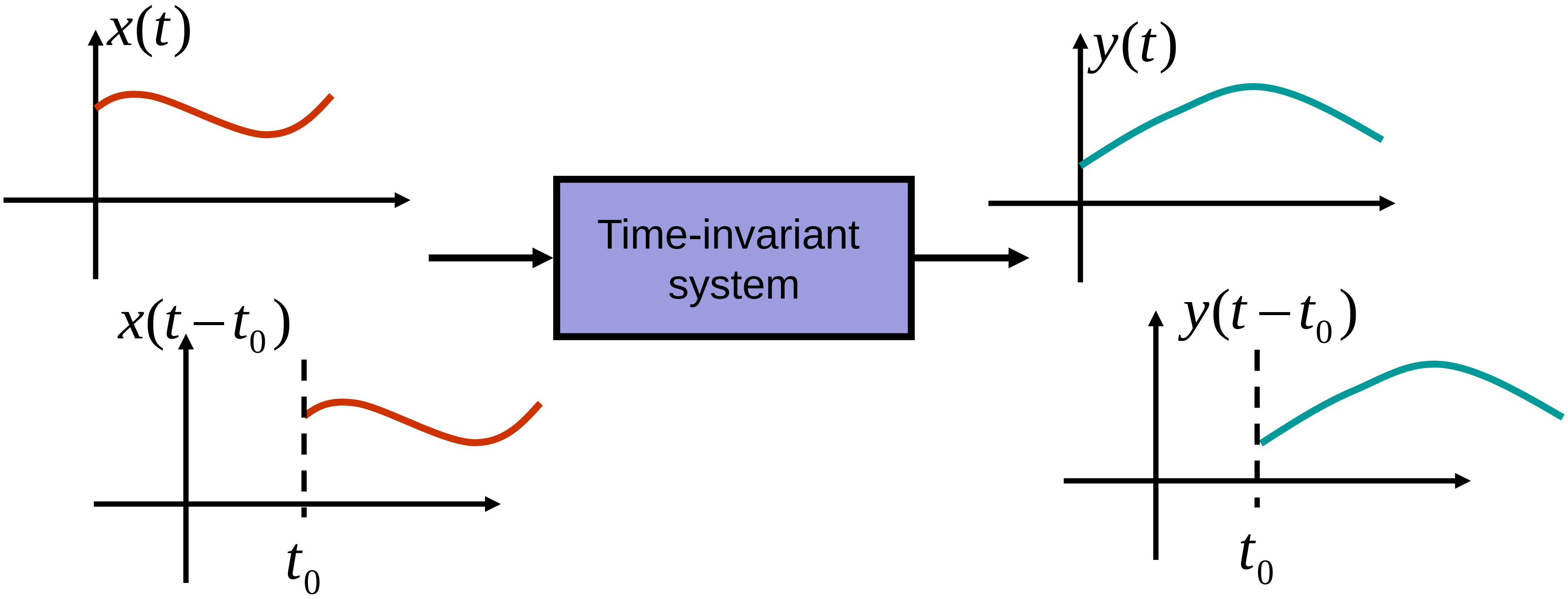
Properties of Linear Systems

- For linear systems, if input is zero then output is zero.
- A linear system is causal if and only if it satisfies the condition of initial rest:

$$u(t) = 0 \text{ for } t \le t_0 \rightarrow y(t) = 0 \text{ for } t \le t_0$$

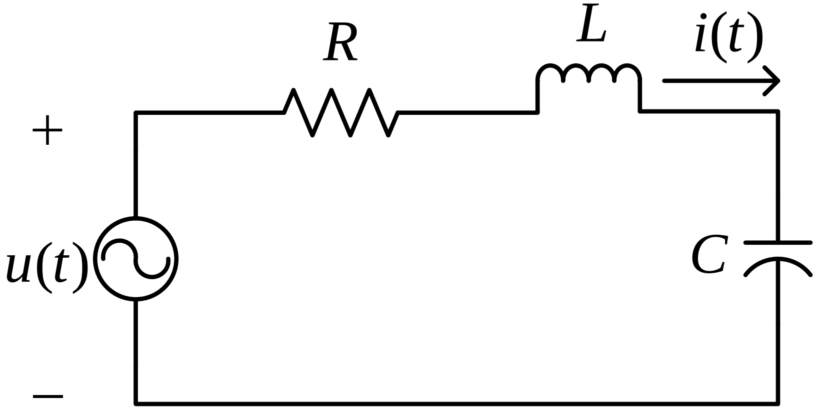
Time-Invariance

- A system is said to be time-invariant if a time delay or time advance of the input signal leads to an identical time shift in the output signal.

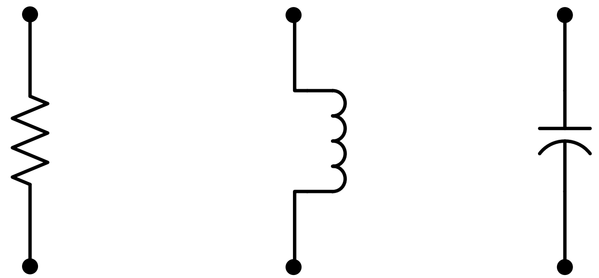


- A system is said to be time-invariant if its parameters do not change over time.

Laplace Transform Approach



***RLC* Circuit**



$v=Ri$	$v=L\frac{di}{dt}$	$i=C\frac{dv}{dt}$
$V(s)=R\cdot I(s)$	$V=sL\cdot I(s)$	$I(s)=sC\cdot V(s)$
Resistor	**Inductor**	**Capacitor**

Input variables:
- Input voltage $u(t)$

Output variables:
- Current $i(t)$

$$Ri(t)+L\frac{di(t)}{dt}+\frac{1}{C}\int_0^t i(\tau)d\tau+v_0=u(t)$$

$$RI(s)+L(sI(s)-i_0)+\frac{1}{Cs}I(s)+\frac{v_0}{s}=U(s)$$

Laplace Transform Approach

$$RI(s)+L(sI(s)-i_0)+\frac{1}{Cs}I(s)+\frac{v_0}{s}=U(s)$$

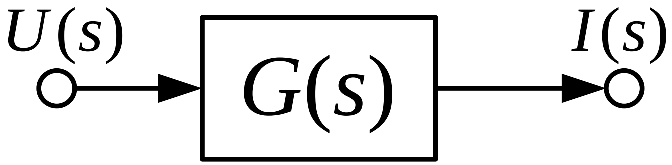
$$\left(Ls+R+\frac{1}{Cs}\right)I(s)=U(s)+Li_0-\frac{v_0}{s}$$

$$I(s)=\underbrace{\frac{Cs}{LCs^2+RCs+1}U(s)}_{\textbf{Current due to input}}+\underbrace{\frac{LCsi_0-Cv_0}{LCs^2+RCs+1}}_{\textbf{Current due to initial condition}}$$

- For zero initial conditions (v_0 = 0, i_0 = 0),

$$I(s)=G(s)U(s)$$

where $G(s)=\dfrac{Cs}{LCs^2+RCs+1}$



Transfer function

State Space Approach

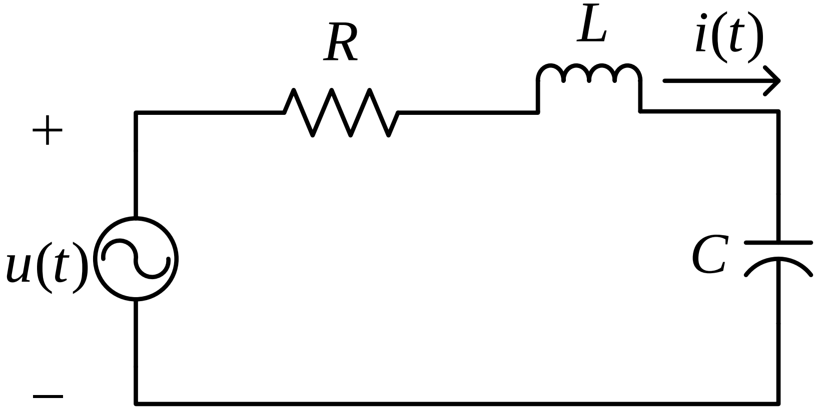
- Laplace Transform method is not effective to model time-varying and non-linear systems.
- The state space approach to be studied in this course will be able to handle more general systems.
- The state space approach characterizes the properties of a system without solving for the exact output.
- Let us now consider the same RLC circuit and try to use state space to model it.

State Space Equations as Linear System

- A system $\underline{\mathbf{y}}(t) = f(\underline{\mathbf{x}}(t),\underline{\mathbf{u}}(t))$ is said to be linear if it follows the following conditions:
  - If $f\left(\underline{\mathbf{x}}_1(t),\underline{\mathbf{u}}_1(t)\right)=\underline{\mathbf{y}}_1(t)$,
    then $f\left(\alpha\underline{\mathbf{x}}(t),\alpha\underline{\mathbf{u}}_1(t)\right)=\alpha\underline{\mathbf{y}}_1(t)$
  - If $f\left(\underline{\mathbf{x}}_1(t),\underline{\mathbf{u}}_1(t)\right)=\underline{\mathbf{y}}_1(t)$
    and $f\left(\underline{\mathbf{x}}_2(t),\underline{\mathbf{u}}_2(t)\right)=\underline{\mathbf{y}}_2(t)$
    then $f\left(\underline{\mathbf{x}}_1(t)+\underline{\mathbf{x}}_2(t),\underline{\mathbf{u}}_1(t)+\underline{\mathbf{u}}_2(t)\right)=\underline{\mathbf{y}}_1(t)+\underline{\mathbf{y}}_2(t)$
  - Then, it can also be implied that

$$f\left(\alpha\underline{\mathbf{x}}_1(t)+\beta\underline{\mathbf{x}}_2(t),\alpha\underline{\mathbf{u}}_1(t)+\beta\underline{\mathbf{u}}_2(t)\right)=\alpha\underline{\mathbf{y}}_1(t)+\beta\underline{\mathbf{y}}_2(t)$$

State Space Approach



***RLC* Circuit**

$$i_C = C\frac{dv_C}{dt} \Rightarrow \frac{dv_C}{dt} = \frac{1}{C} i_C$$

$$\boxed{\frac{dv_C}{dt} = \frac{1}{C} i_L}$$

$$v_L = L\frac{di_L}{dt} \Rightarrow \frac{di_L}{dt} = \frac{1}{L} v_L$$

$$\frac{di_L}{dt} = \frac{1}{L}(u - v_R - v_C)$$

$$\boxed{\frac{di_L}{dt} = \frac{1}{L}(u - Ri_L - v_C)}$$

State variables:
- Voltage across *C*
- Current through *L*

- **We now have two first-order ODEs**
- **Their variables are the state variables and the input**

State Space Approach

$$\boxed{\frac{dv_C}{dt} = \frac{1}{C} i_L}$$

$$\boxed{\frac{di_L}{dt} = \frac{1}{L}(u - Ri_L - v_C)}$$

- The two equations are called **state equations**, and can be rewritten in the form of:

$$\begin{bmatrix} dv_C/dt \\ di_L/dt \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u$$

- The output is described by an **output equation**:

$$i_L = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

State Space Approach

- The state equations and output equation, combined together, form the **state space** description of the circuit.

$$\begin{bmatrix} dv_C/dt \\ di_L/dt \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u$$

$$i_L = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

- In a more compact form, the state space can be written as:

$$\boxed{\begin{aligned} \dot{\underline{\mathbf{x}}} &= \underline{\mathbf{A}}\,\underline{\mathbf{x}} + \underline{\mathbf{B}}\,\underline{u} \\ \underline{y} &= \underline{\mathbf{C}}\,\underline{\mathbf{x}} \end{aligned}}$$

$$\underline{\mathbf{A}} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \qquad \underline{\mathbf{x}} = \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

$$\underline{\mathbf{B}} = \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \qquad \dot{\underline{\mathbf{x}}} = \begin{bmatrix} dv_C/dt \\ di_L/dt \end{bmatrix}$$

$$\underline{\mathbf{C}} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

State Space Approach

- The state of a system at t_0 is the information at t_0 that, together with the input u for $t_0 \leq t < \infty$, uniquely determines the behavior of the system for $t \geq t_0$.
- The number of state variables = the number of initial conditions needed to solve the problem.
- As we will learn in the future, there are infinite numbers of state space that can represent a system.

- The main features of state space approach are:
  - It describes the behaviors inside the system.
  - Stability and performance can be analyzed without solving for any differential equations.
  - Applicable to more general systems such as non-linear systems, time-varying system.
  - Modern control theory are developed using state space approach.

Chapter 2

Mathematical Descriptions of Systems

State Space Equations

- The state equations of a system can generally be written as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & \cdots & b_{1r} \\ \vdots & b_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{n1} & \cdots & \cdots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}$$

$x_1(t), x_2(t), \cdots, x_n(t)$ are the state variables

$u_1(t), u_2(t), \cdots, u_r(t)$ are the system inputs

- **State equations are built of *n* linearly-coupled first-order ordinary differential equations**

State Space Equations

- By defining:

$$\underline{\mathbf{x}}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \qquad \underline{\mathbf{u}}(t)=\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix},$$

we can write

$$\boxed{\dot{\underline{\mathbf{x}}}(t)=\underline{\mathbf{A}}\,\underline{\mathbf{x}}(t)+\underline{\mathbf{B}}\,\underline{\mathbf{u}}(t)}$$

State Equations

State Space Equations

- The outputs of the state space are the linear combinations of the state variables and the inputs:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & \cdots & c_{1n} \\ \vdots & c_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ c_{m1} & \cdots & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} d_{11} & \cdots & \cdots & d_{1r} \\ \vdots & d_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ d_{m1} & \cdots & \cdots & d_{mr} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}$$

$y_1(t), y_2(t), \cdots, y_m(t)$ are the system outputs

State Space Equations

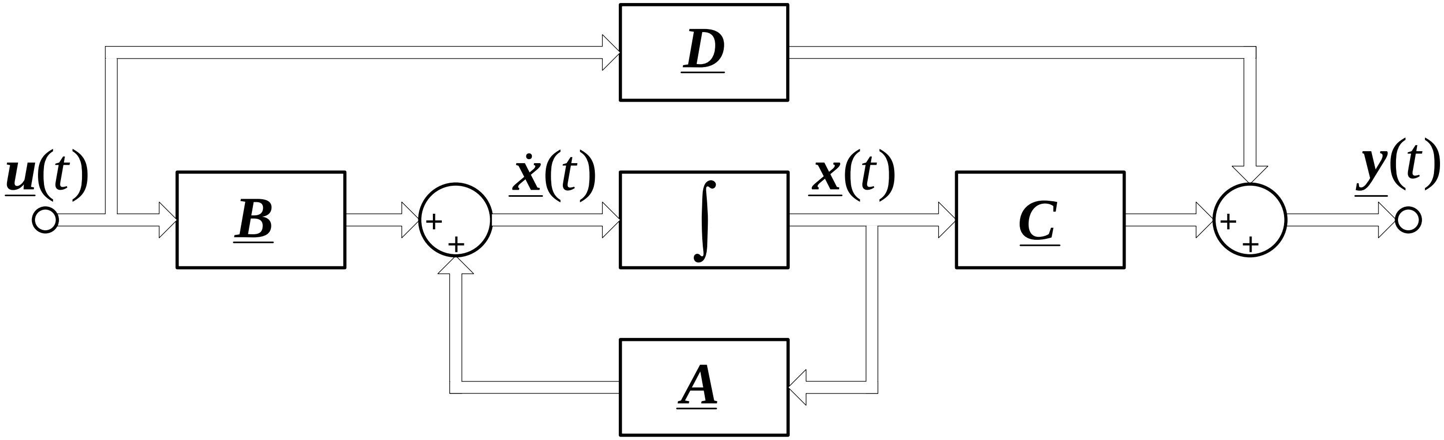
- By defining:

$$\underline{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix},$$

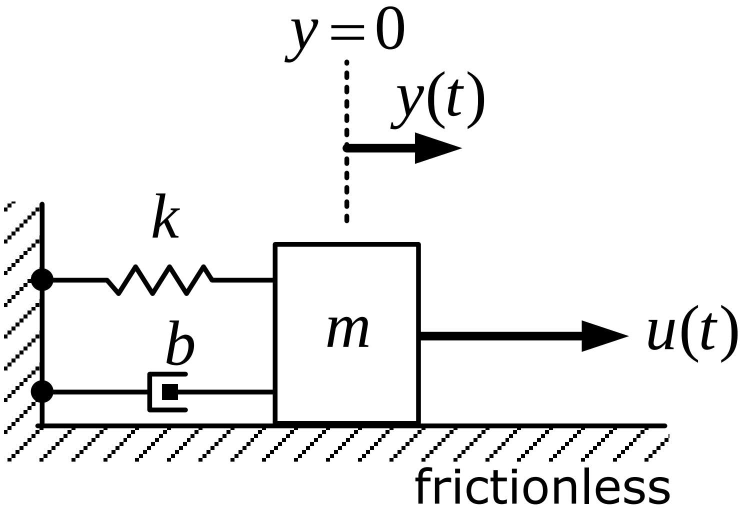
we can write

$$\boxed{\underline{y}(t) = \underline{C}\,\underline{x}(t) + \underline{D}\underline{u}(t)}$$

Output Equations



Example: Mechanical System



$$u(t)-ky(t)-b\frac{dy(t)}{dt}=m\frac{d^2y(t)}{dt^2}$$

State variables:

$$x_1(t)=y(t) \qquad \Rightarrow \dot{x}_1(t)=x_2(t)$$

$$x_2(t)=\frac{dy(t)}{dt} \qquad \Rightarrow \dot{x}_2(t)=\frac{d^2y(t)}{dt^2}$$

Input variables:

- Applied force $u(t)$

Output variables:

- Displacement $y(t)$

State equations:

$$\dot{x}_1(t)=x_2(t)$$

$$\dot{x}_2(t)=-\frac{k}{m}x_1(t)-\frac{b}{m}x_2(t)+\frac{1}{m}u(t)$$

Example: Mechanical System

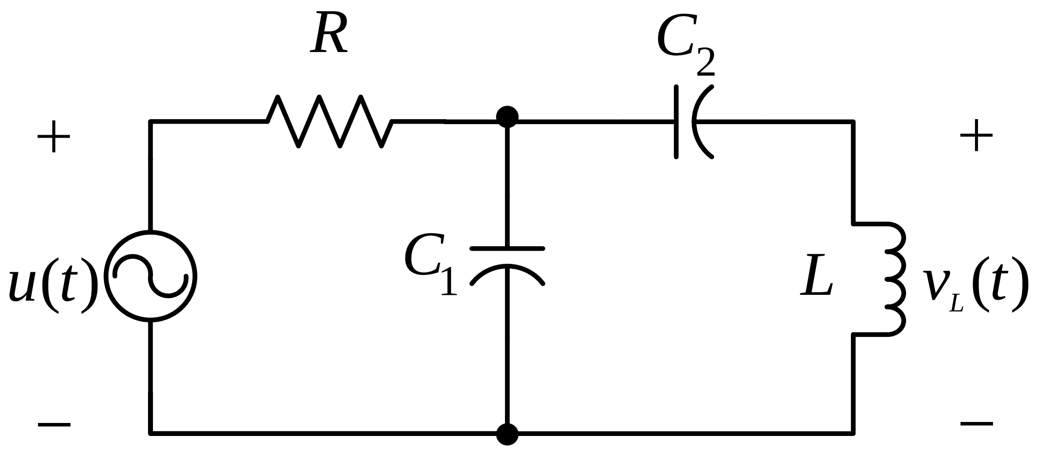
- The state space equations can now be constructed as below:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Homework 1: Electrical System

- Derive the state space representation of the following electric circuit:

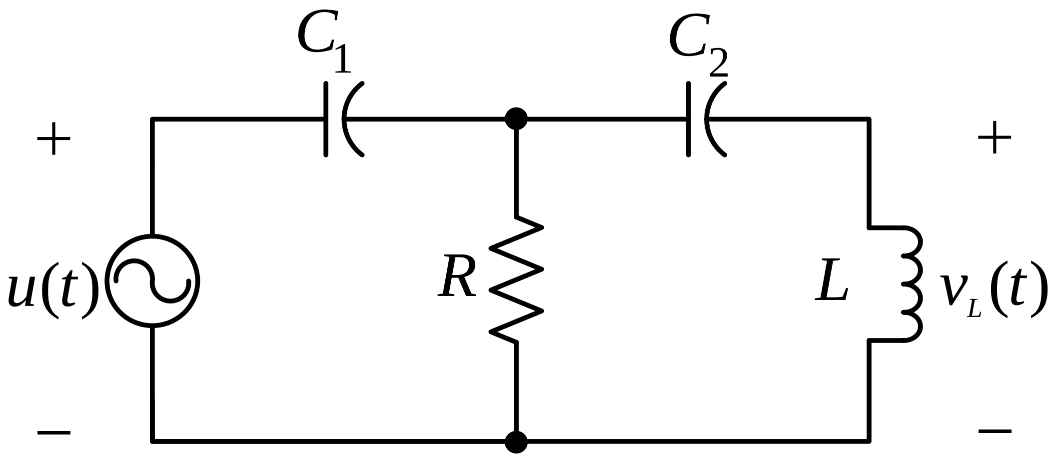


Input variables:
- Input voltage $u(t)$

Output variables:
- Inductor voltage $v_L(t)$

Homework 1A: Electrical System

- Derive the state space representation of the following electric circuit:



Input variables:
- Input voltage $u(t)$

Output variables:
- Inductor voltage $v_L(t)$