MODELING OF CONTROL SYSTEMS

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Outline

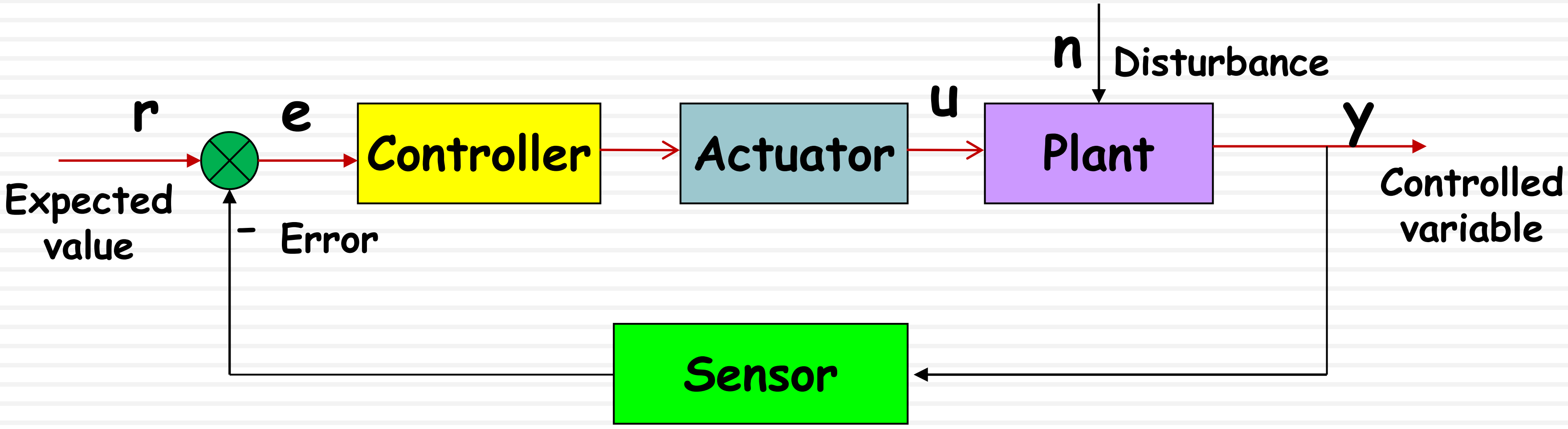
- Introduction
- Differential equations and Linearization of nonlinear mathematical models
- Transfer function and impulse response function
- Laplace transform review
- Block diagram and signal flow graph

- **Reference:**
  - **Chapter 2: "Modern Control Systems", Richard Dorf, Robert Bishop**

The materials of this presentation are based on the Lecture note slides of the Control System Engineering (Fall 2008) course offered by Prof. Bin Jiang and Dr. Ruiyun QI, Nanjing University of Aeronautics and Astronautics (NUAA), China

Introduction

How to analyze and design a control system



- The first thing is to establish system model (mathematical model)

Introduction (2)

- A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the system accurately
- The dynamics of many systems may be described in terms of differential equations obtained from physical laws governing a particular system
- In obtaining a mathematical model, we must make a compromise between the simplicity of the model and the accuracy of the results of the analysis
- In general, in solving a new problem, it is desirable to build a simplified model so that we can get a general feeling for the solution

System Model

System Model is

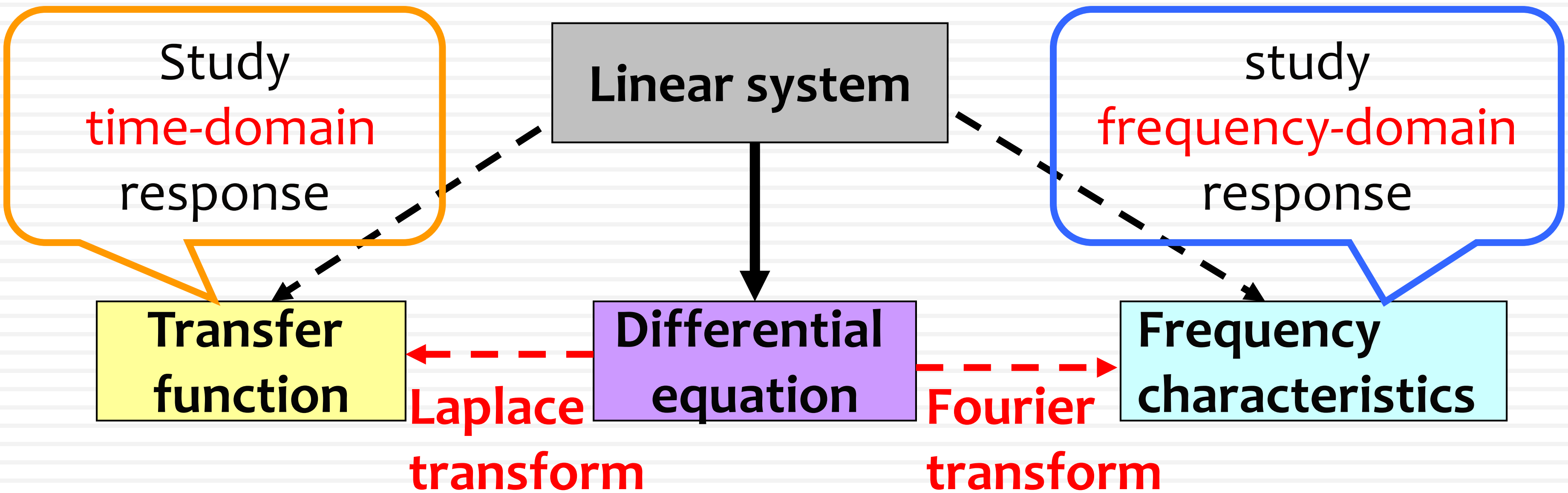
A mathematical expression of dynamic relationship between input and output of a system.

A mathematical model is the foundation to analyze and design automatic control systems

No mathematical model of a physical system is exact. We generally strive to develop a model that is adequate for the problem at hand but without making the model overly complex.

System Model (2)

- Differential equation
- Transfer function
- Frequency characteristic



Modeling Methods

Analytic method

According to

A. Newton's Law of Motion
B. Law of Kirchhoff
C. System structure and parameters

the mathematical expression of system input and output can be derived.

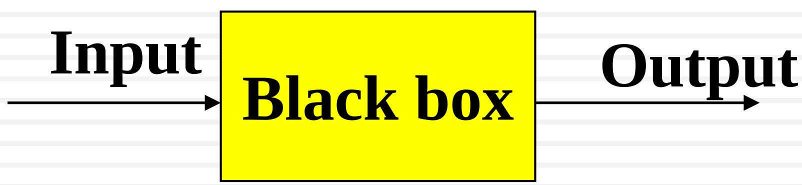
Thus, we build the mathematical model (suitable for simple systems).

Modeling Methods(2)

System identification method

Building the system model based on the system input—output charecteristics

This method is usually applied when there are little information available for the system.



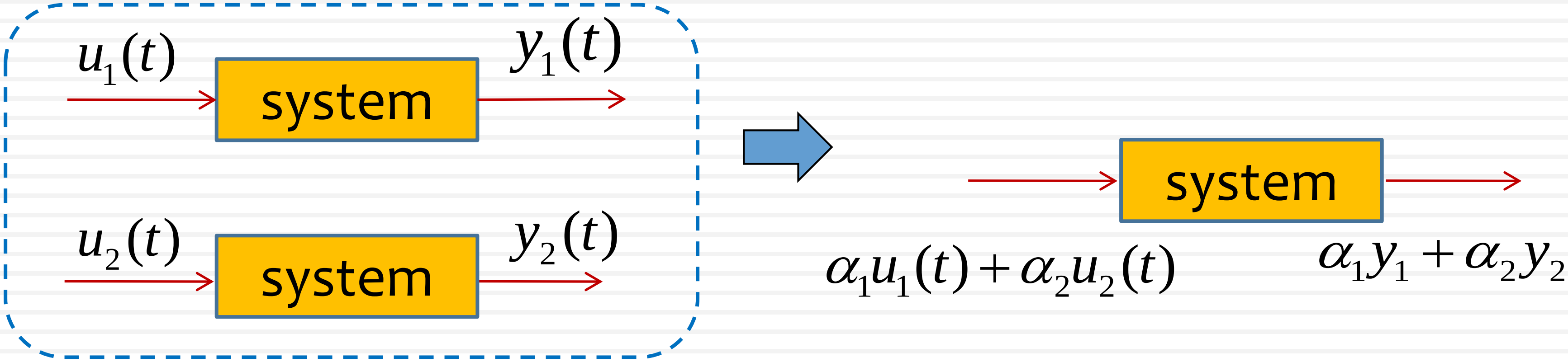
Neural Networks,
Fuzzy Systems

Black box: the system is totally unknown.

Grey box: the system is partially known.

Modeling Methods (3)

- Why Focus on Linear Time-Invariant (LTI) System?
- What is linear system?
  - A system is called linear if the principle of superposition applies



Is y(t)=u(t)+2 a linear system?

Modeling Methods (4)

- The overall response of a linear system can be obtained by
  - Decomposing the input into a sum of elementary signals
  - Figuring out each response to the respective elementary signal
  - Adding all these responses together

Thus, we can use typical elementary signal (e.g. unit step ,unit impulse, unit ramp) to analyze system for the sake of simplicity.

Modeling Methods (4)

- What is time-invariant system?
  - A system is called time-invariant if the parameters are stationary with respect to time during system operation
  - Examples?

2.2 Establishment of differential equation and linearization

Differential equation

Linear ordinary differential equations

--- A wide range of systems in engineering are modeled mathematically by differential equations

--- In general, the differential equation of an *n*-th order system is written

$$a_0 c^{(n)}(t) + a_1 c^{(n-1)}(t) + \cdots + a_{n-1} c^{(1)}(t) + c(t) = b_0 r^{(m)}(t) + \cdots + b_{m-1} r^{(1)}(t) + b_m r(t)$$

Time-domain model

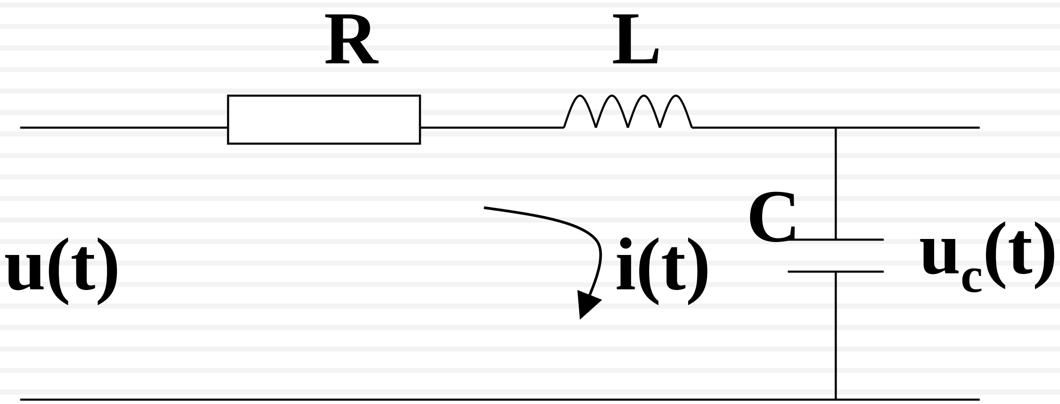
How to establish ODE of a control system

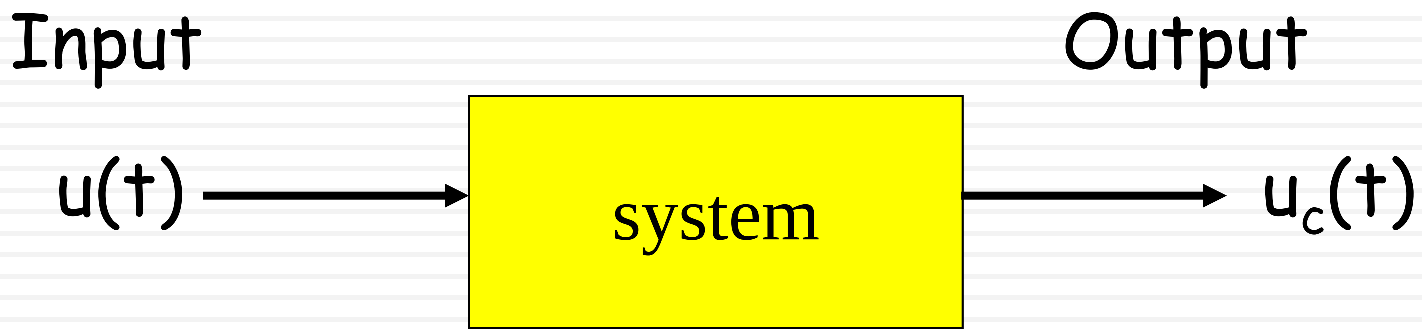
--- list differential equations according to the physical rules of each component;

--- obtain the differential equation sets by eliminating intermediate variables;

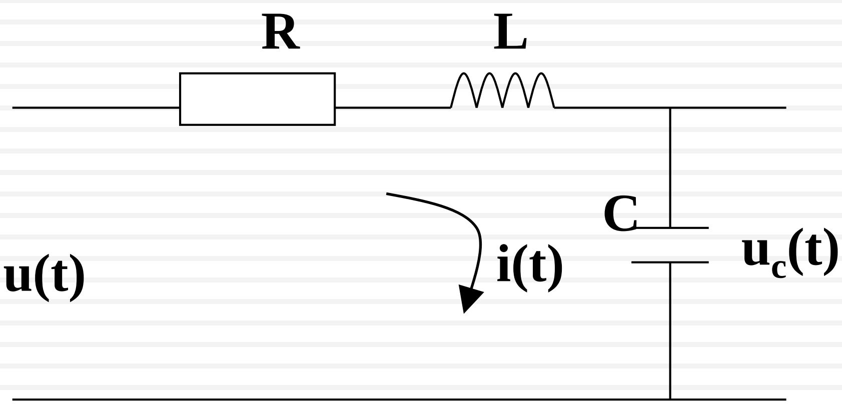
--- get the overall input-output differential equation of control system.

Examples-1 RLC circuit





Defining the input and output according to which cause-effect relationship you are interested in.



According to Law of Kirchhoff in electricity

$$u(t) = Ri(t) + L\frac{di(t)}{dt} + u_c(t) - - - (1)$$

$$u_C(t) = \frac{1}{C}\int i(t)dt - - - (2)$$

$$\Rightarrow i(t) = C\frac{du_C(t)}{dt}$$

$$u(t) = RC\frac{du_C(t)}{dt} + LC\frac{d^2u_C(t)}{dt^2} + u_C(t)$$

- It is re-written as in standard form

$$LC\ddot{u}_C(t) + RC\dot{u}_C(t) + u_C(t) = u(t)$$

Generally, we set

- the output on the left side of the equation
- the input on the right side
- the input is arranged from the highest order to the lowest order

Examples-2 Mass-spring-friction system

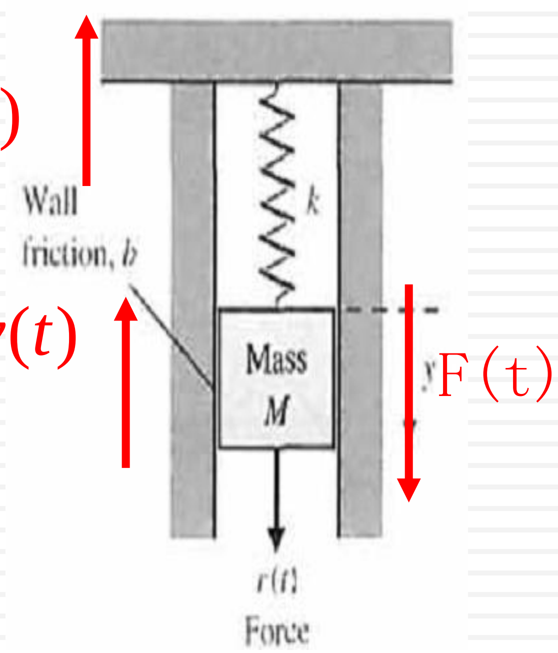
Gravity is neglected.

We are interested in the relationship between external force F(t) and mass displacement x(t)

Define: input—F(t); output---x(t)

$F_1 = kx(t)$

$F_2 = fv(t)$



$$\sum F = ma$$

$$ma = F - F_1 - F_2$$

$$v = \frac{dx(t)}{dt}, \quad a = \frac{d^2x(t)}{dt}$$

By eliminating intermediate variables, we obtain the overall input-output differential equation of the mass-spring-friction system.

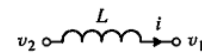
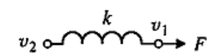
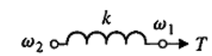
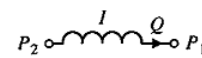
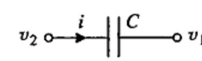
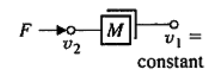
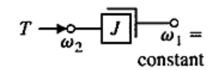
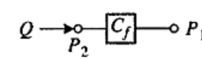
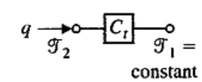
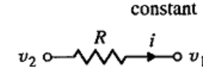
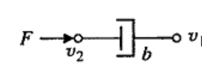
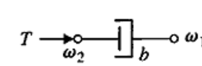
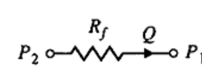
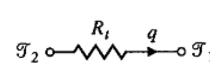
$$m\ddot{x}(t) + f\dot{x}(t) + kx(t) = F(t)$$

Recall the RLC circuit system

$$LC\ddot{u}_c(t) + RC\dot{u}_c(t) + u_c(t) = u(t)$$

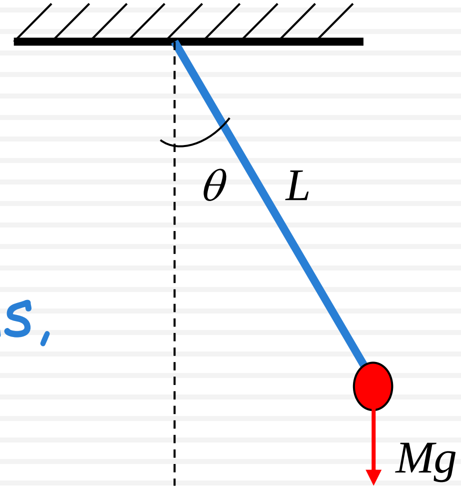
These formulas are similar, that is, we can use the same mathematical model to describe a class of systems that are physically absolutely different but share the same Motion Law.

ODEs of Some Electrical and Mechanical systems

Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Inductive storage	Electrical inductance	$v_{21} = L\frac{di}{dt}$	$E = \frac{1}{2}Li^2$	
	Translational spring	$v_{21} = \frac{1}{k}\frac{dF}{dt}$	$E = \frac{1}{2}\frac{F^2}{k}$	
	Rotational spring	$\omega_{21} = \frac{1}{k}\frac{dT}{dt}$	$E = \frac{1}{2}\frac{T^2}{k}$	
	Fluid inertia	$P_{21} = I\frac{dQ}{dt}$	$E = \frac{1}{2}IQ^2$	
Capacitive storage	Electrical capacitance	$i = C\frac{dv_{21}}{dt}$	$E = \frac{1}{2}C{v_{21}}^2$	
	Translational mass	$F = M\frac{dv_2}{dt}$	$E = \frac{1}{2}M{v_2}^2$	
	Rotational mass	$T = J\frac{d\omega_2}{dt}$	$E = \frac{1}{2}J{\omega_2}^2$	
	Fluid capacitance	$Q = C_f\frac{dP_{21}}{dt}$	$E = \frac{1}{2}C_f{P_{21}}^2$	
	Thermal capacitance	$q = C_t\frac{d\mathcal{T}_2}{dt}$	$E = C_t\mathcal{T}_2$	
Energy dissipators	Electrical resistance	$i = \frac{1}{R}v_{21}$	$\mathcal{P} = \frac{1}{R}{v_{21}}^2$	
	Translational damper	$F = bv_{21}$	$\mathcal{P} = b{v_{21}}^2$	
	Rotational damper	$T = b\omega_{21}$	$\mathcal{P} = b{\omega_{21}}^2$	
	Fluid resistance	$Q = \frac{1}{R_f}P_{21}$	$\mathcal{P} = \frac{1}{R_f}{P_{21}}^2$	
	Thermal resistance	$q = \frac{1}{R_t}\mathcal{T}_{21}$	$\mathcal{P} = \frac{1}{R_t}\mathcal{T}_{21}$	

Examples-3 nonlinear system

In reality, most systems are indeed nonlinear, e.g. then pendulum system, which is described by nonlinear differential equations.



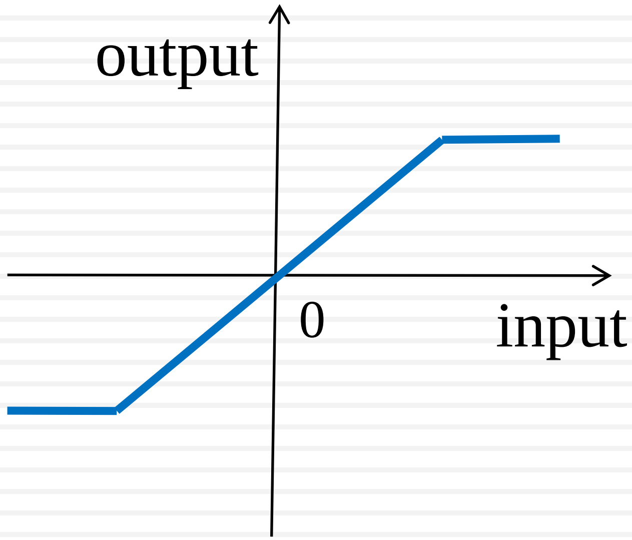
$$ML\frac{d^2\theta}{dt^2} + Mg\sin\theta(t) = 0$$

- It is difficult to analyze nonlinear systems, however, we can linearize the nonlinear system near its equilibrium point under certain conditions.

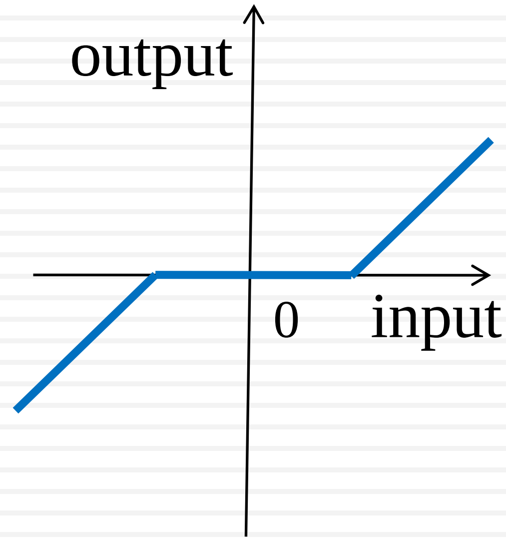
$$ML\frac{d^2\theta}{dt^2} + Mg\theta(t) = 0 \quad (\text{when } \theta \text{ is small})$$

Linearization of nonlinear differential equations

Several typical nonlinear characteristics in control system.



Saturation (Amplifier)



Dead-zone (Motor)

Methods of linearization

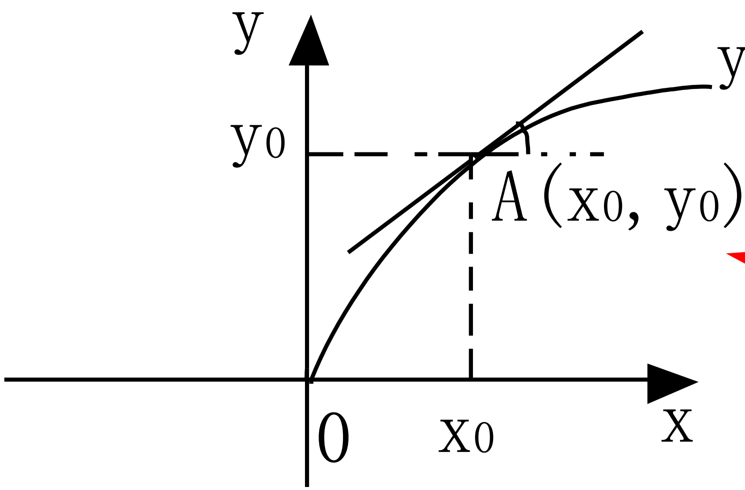
（1） Weak nonlinearity neglected

If the nonlinearity of the component is not within its linear working region, its effect on the system is weak and can be neglected.

（2） Small perturbation/error method

Assumption: In the system control process, there are just small changes around the equilibrium point in the input and output of each component.

This assumption is reasonable in many practical control system: in closed-loop control system, once the deviation occurs, the control mechanism will reduce or eliminate it. Consequently, all the components can work around the equilibrium point.



A(x0,y0) is equilibrium point. Expanding the nonlinear function y=f(x) into a Taylor series about A(x0,y0) yields

$$y = f(x) = y_0 + \left.\frac{dy}{dx}\right|_{x_0}(x - x_0) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_0}(x - x_0)^2 + \cdots\cdots$$

The input and output only have small variance around the equilibrium point.

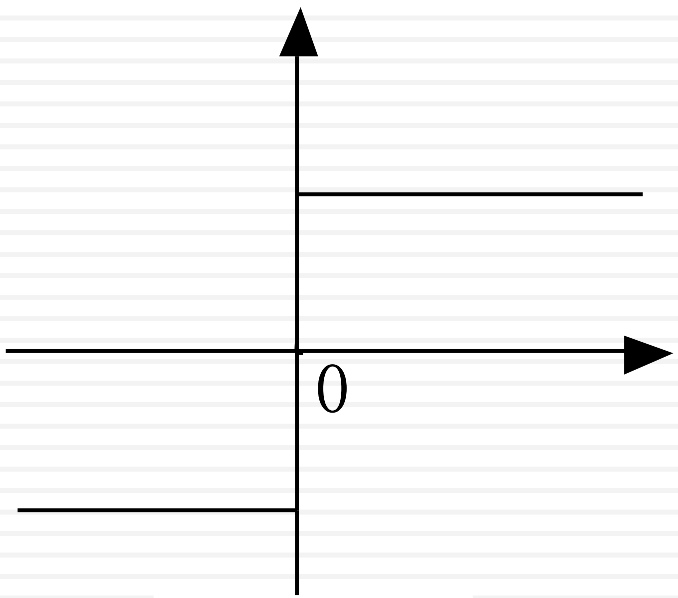
$$\Delta x = (x - x_0), (\Delta x)^n \to 0$$

$$y = y_0 + \left.\frac{dy}{dx}\right|_{x_0}(x - x_0)$$

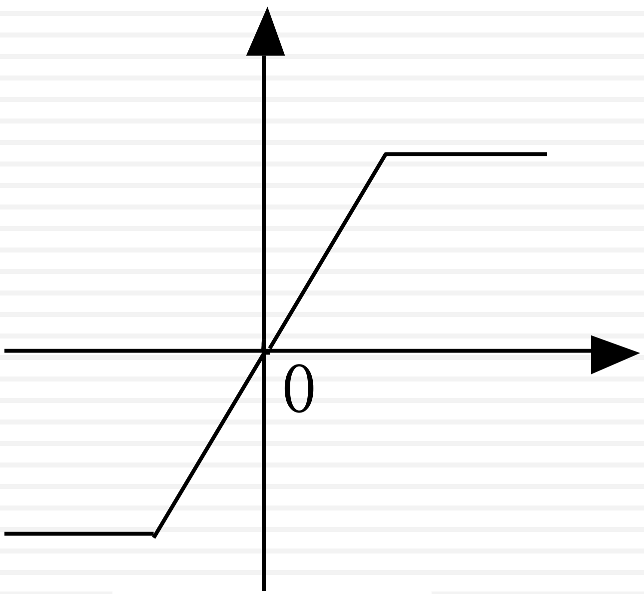
$$\Delta y = k\Delta x$$

This is linearized model of the nonlinear component.

Note: this method is only suitable for systems with weak nonlinearity.



Relay



Saturation

For systems with strong nonlinearity, we cannot use such linearization method.

Example

- Linearize the nonlinear equation $Z=XY$ in the region $5 \leq x \leq 7$, $10 \leq y \leq 12$. Find the error if the linearized equation is used to calculate the value of z when $x=5$, $y=10$

<u>Solution:</u>

Choose $\bar{x}$, $\bar{y}$ as the average values of the given ranges

Then $$\bar{x} = 6, \bar{y} = 11$$

$$\bar{z} = \bar{x}\bar{y} = 66.$$

Example (2)

Expanding the nonlinear equation into a Taylor series about points $x = \bar{x}$, $y = \bar{y}$ and neglecting the higher-order terms

$$z - \bar{z} = a(x - \bar{x}) + b(y - \bar{y})$$

Where

$$a = \left. \frac{\partial(xy)}{\partial x} \right|_{x=\bar{x},\ y=\bar{y}} = \bar{y} = 11$$

$$b = \left. \frac{\partial(xy)}{\partial y} \right|_{x=\bar{x},\ y=\bar{y}} = \bar{x} = 6$$

Example (3)

Hence the linearized equation is

$$z - 66 = 11(x - 6) + 6(y - 11)$$

or

$$z = 11x + 6y - 66$$

When $x=5$, $y=10$, the value of z given by the linearized equation is

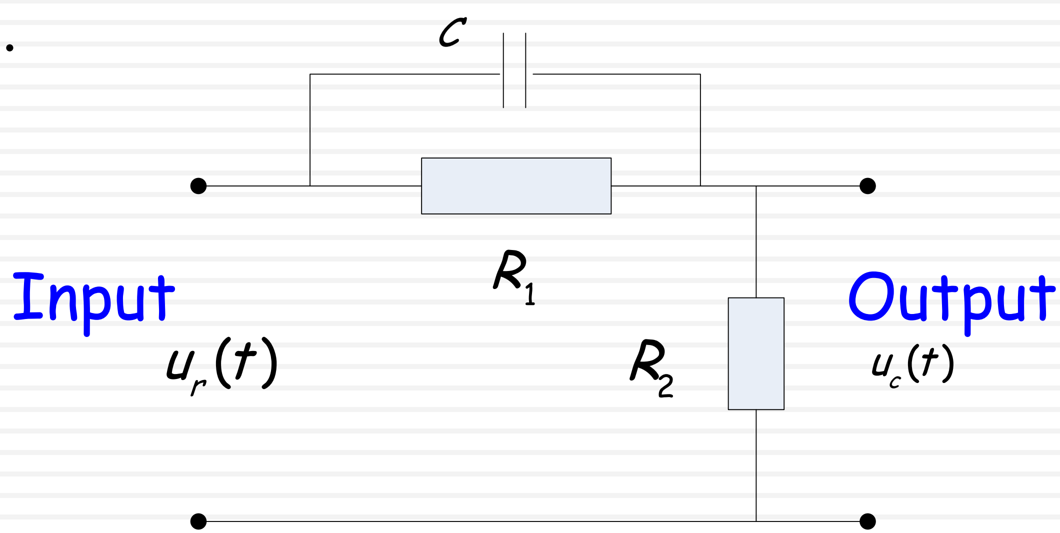
$$z = 11x + 6y - 66 = 55 + 60 - 66 = 49$$

The exact value of z is $z = xy = 50$

The error is thus $50-49=1$ or 2%

Exercise

- E1. Please build the differential equation of the following system.



$$\begin{cases} R_1 i_1 = \dfrac{1}{C}\displaystyle\int i_2 dt \\ i = i_1 + i_2 \\ u_c = R_2 i \\ u_r = R_1 i_1 + u_c \end{cases} \Rightarrow R_1 R_2 C \frac{du_c}{dt} + (R_1 + R_2) u_c = R_1 R_2 C \frac{du_r}{dt} + R_2 u_r$$

2-3 Transfer function

Solving Differential Equations

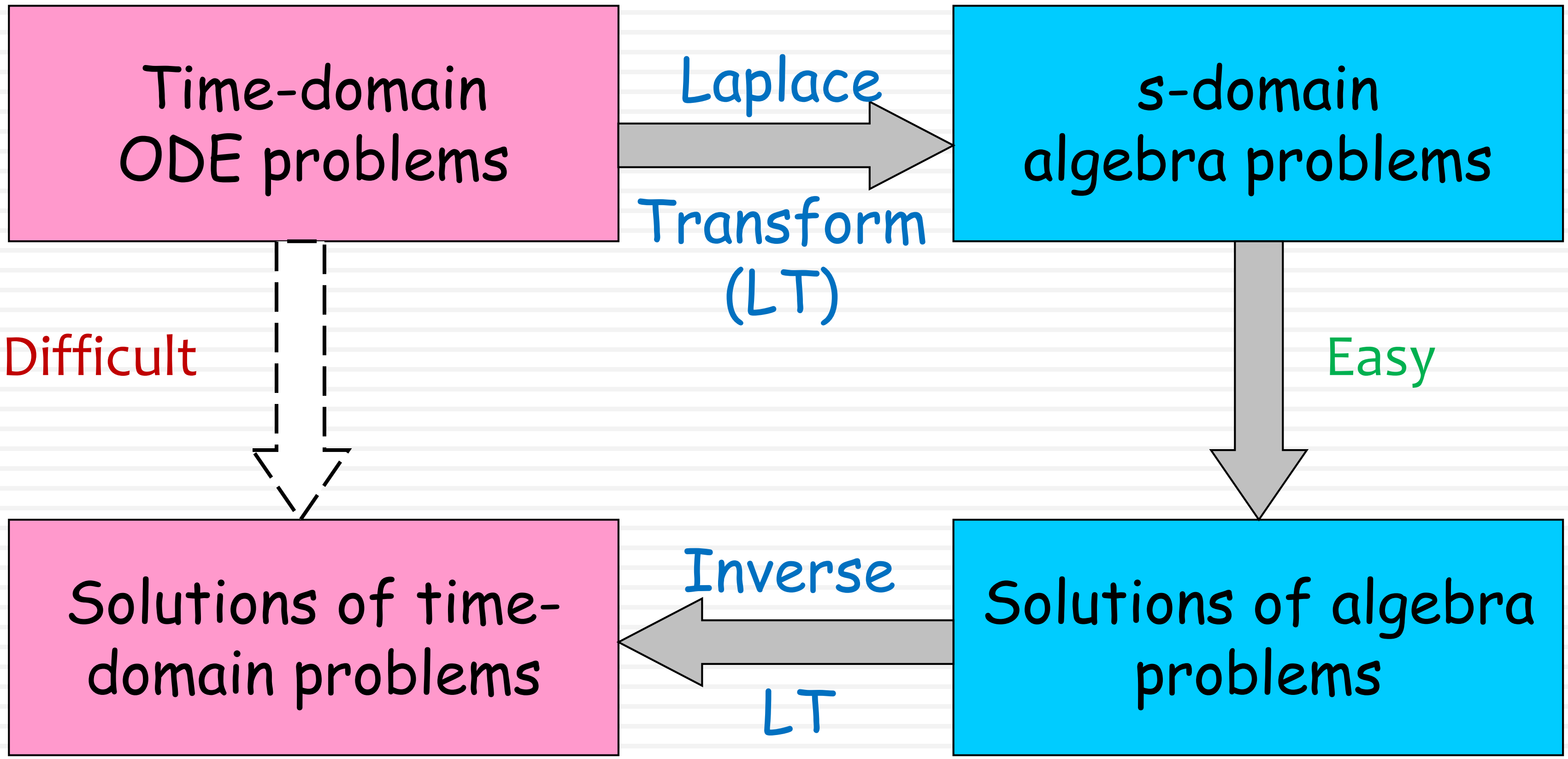
Example

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} = 2t + 1$$

Solving linear differential equations with constant coefficients:

- To find the general solution (involving solving the characteristic equation)
- To find a particular solution of the complete equation (involving constructing the family of a function)

WHY need LAPLACE transform?



Laplace Transform

The Laplace transform of a function *f(t)* is defined as

$$F(s) = \mathcal{L}\left[f(t)\right]$$

$$= \int_0^{\infty} f(t)e^{-st}dt$$

where $s = \sigma + j\omega$ is a complex variable

Examples

Step signal: *f(t)=A*

$$F(s)=\int_0^\infty f(t)e^{-st}dt=\int_0^\infty Ae^{-st}dt=-\frac{A}{s}e^{-st}\Big|_0^\infty=\frac{A}{s}$$

- Exponential signal *f(t)=* e^{-at}

$$F(s)=\int_0^\infty e^{-at}e^{-st}dt=-\frac{1}{s+a}e^{-(a+s)t}\Big|_0^\infty=\frac{1}{s+a}$$

Laplace transform table

f(t)	F(s)	f(t)	F(s)
$\delta(t)$	1	$\sin wt$	$\frac{w}{s^2+w^2}$
1	$\frac{1}{s}$	$\cos wt$	$\frac{s}{s^2+w^2}$
t	$\frac{1}{s^2}$	$e^{-at}\sin wt$	$\frac{w}{(s+a)^2+w^2}$
e^{-at}	$\frac{1}{s+a}$	$e^{-at}\cos wt$	$\frac{s+a}{(s+a)^2+w^2}$

Properties of Laplace Transform

(1) Linearity

$$\mathcal{L}[af_1(t)+bf_2(t)] = a\mathcal{L}[f_1(t)]+b\mathcal{L}[f_2(t)]$$

(2) Differentiation

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s)-f(0)$$

Using Integration By Parts method to prove

where f(0) is the initial value of f(t).

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s)-s^{n-1}f(0)-s^{n-2}f^{(1)}(0)-\cdots-f^{(n-1)}(0)$$

Properties of Laplace Transform (2)

(3) Integration

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$$

Using Integration By Parts method to prove)

$$\mathcal{L}\left[\int_o^{t_1}\int_o^{t_2}\cdots\int_o^{t_n} f(\tau)d\tau dt_1 dt_2 \cdots dt_{n-1}\right] = \frac{F(s)}{s^n}$$

Properties of Laplace Transform (3)

（4） Final-value Theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

The final-value theorem relates the steady-state behavior of f(t) to the behavior of sF(s) in the neighborhood of s=0

(5) Initial-value Theorem

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

Properties of Laplace Transform (4)

(6)Shifting Theorem:

a. shift in time (real domain)

$$\mathcal{L}[f(t-\tau)] = e^{-\tau \cdot s} F(s)$$

b. shift in complex domain

$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$

(7) Real convolution (Complex multiplication) Theorem

$$\mathcal{L}[\int_0^t f_1(t-\tau) f_2(\tau) d\tau] = F_1(s) \cdot F_2(s)$$

Inverse Laplace transform

Definition : Inverse Laplace transform, denoted by $\mathcal{L}^{-1}[F(s)]$ is given by

$$f(t)=\mathcal{L}^{-1}[F(s)]=\frac{1}{2\pi\cdot j}\int_{C-j\infty}^{C+j\infty}F(s)e^{st}ds(t>0)$$

where C is a real constant。

Note: The inverse Laplace transform operation involving rational functions can be carried out using Laplace transform table and partial-fraction expansion.

Partial-Fraction Expansion method for finding Inverse Laplace Transform

$$F(s)=\frac{N(s)}{D(s)}=\frac{b_0 s^m+b_1 s^{m-1}+\cdots+b_{m-1}s+b_m}{s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n}\,(m<n)$$

If F(s) is broken up into components

$$F(s)=F_1(s)+F_2(s)+\ldots+F_n(s)$$

If the inverse Laplace transforms of components are readily available, then

$$\mathcal{L}^{-1}\left[F(s)\right]=\mathcal{L}^{-1}\left[F_1(s)\right]+\mathcal{L}^{-1}\left[F_2(s)\right]+\ldots+\mathcal{L}^{-1}\left[F_n(s)\right]$$

$$=f_1(t)+f_2(t)+\ldots+f_n(t)$$

Poles and zeros

- Poles
  - A complex number s_0 is said to be a pole of a complex variable function $F(s)$ if $F(s_0)=\infty$.
- Zeros
  - A complex number s_0 is said to be a zero of a complex variable function $F(s)$ if $F(s_0)=0$

Examples:

$$\frac{(s-1)(s+2)}{(s+3)(s+4)}$$ poles: -3, -4; zeros: 1, -2

$$\frac{s+1}{s^2+2s+2}$$ poles: -1+j, -1-j; zeros: -1

Case 1: F(s) has simple real poles

$$F(s)=\frac{N(s)}{D(s)}=\frac{b_0s^m+b_1s^{m-1}+\cdots+b_{m-1}s+b_m}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}$$

Partial-Fraction Expansion

$$=\frac{c_1}{s-p_1}+\frac{c_2}{s-p_2}+\cdots+\frac{c_n}{s-p_n}$$

where $p_i\,(i=1,2,\cdots,n)$ are eigenvalues of $D(s)=0$, and

$$c_i=\left[\frac{N(s)}{D(s)}(s-p_i)\right]\Bigg|_{s=p_i}$$

Inverse LT

$$f(t)=c_1e^{-p_1t}+c_2e^{-p_2t}+\ldots+c_ne^{-p_nt}$$

Parameters p_k give shape and numbers c_k give magnitudes.

Example 1

Partial-Fraction Expansion

$$F(s) = \frac{1}{(s+1)(s-2)(s+3)} = \frac{c_1}{s+1} + \frac{c_2}{s-2} + \frac{c_3}{s+3}$$

$$c_1 = \left[\frac{1}{(s+1)(s-2)(s+3)} \cdot (s+1)\right]\Bigg|_{s=-1} = -\frac{1}{6}$$

$$c_2 = \left[\frac{1}{(s+1)(s-2)(s+3)} \cdot (s-2)\right]\Bigg|_{s=2} = \frac{1}{15}$$

$$c_3 = \left[\frac{1}{(s+1)(s-2)(s+3)} \cdot (s+3)\right]\Bigg|_{s=-3} = \frac{1}{10}$$

$$\therefore F(s) = -\frac{1}{6} \cdot \frac{1}{s+1} + \frac{1}{15} \cdot \frac{1}{s-2} + \frac{1}{10} \cdot \frac{1}{s+3}$$

$$\therefore f(t) = -\frac{1}{6} e^{-t} + \frac{1}{15} e^{2t} + \frac{1}{10} e^{-3t}$$

Case 2: F(s) has simple complex-conjugate poles

Example 2

$$\ddot{y}(t)+4\dot{y}(t)+5y(t)=0,\ y(0)=\dot{y}(0)=1$$

Laplace transform

$$s^2Y(s)-sy(0)-\dot{y}(0)+4sY(s)-4y(0)+5Y(s)=0$$

Applying initial conditions

$$\therefore Y(s)=\frac{s+5}{s^2+4s+5}=\frac{s+5}{(s+2)^2+1}=\frac{s+2+3}{(s+2)^2+1}$$

Partial-Fraction Expansion

$$=\frac{s+2}{(s+2)^2+1}+\frac{3}{(s+2)^2+1}$$

Inverse Laplace transform

$$y(t)=e^{-2t}\cos t+3e^{-2t}\sin t$$

Case 3: F(s) has multiple-order poles

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s-p_1)(s-p_2)\cdots(s-p_{n-r})(s-p_i)^l}$$

$$=\frac{c_1}{s-p_1}+\cdots+\frac{c_{n-l}}{s-p_{n-l}}+\frac{b_l}{(s-p_i)^l}+\frac{b_{l-1}}{(s-p_i)^{l-1}}+\cdots+\frac{b_1}{s-p_i}$$

Simple poles   Multi-order poles

The coefficients $c_1,\cdots,c_{n-l}$ of simple poles can be calculated as Case 1;

The coefficients corresponding to the multi-order poles are determined as

$$b_l=\left[F(s)\cdot(s-p_i)^l\right]\Big|_{s=p_1},\quad b_{l-1}=\left\{\frac{d}{ds}\left[F(s)\cdot(s-p_i)^l\right]\right\}\Big|_{s=pi},\cdots,$$

$$b_{l-m}=\frac{1}{m!}\left\{\frac{d^m}{ds}\left[\frac{N(s)}{D(s)}(s-p_i)^l\right]\right\}\Big|_{s=p_1},\quad b_1=\frac{1}{(l-1)!}\left\{\frac{d^{l-1}}{ds}\left[\frac{N(s)}{D(s)}(s-p_i)^l\right]\right\}\Big|_{s=p_i}$$

Example 3

Solve the following differential equation

$$y^{(3)} + 3\ddot{y} + 3\dot{y} + y = 1, y(0) = \dot{y}(0) = \ddot{y}(0) = 0$$

Laplace transform:

$$s^3Y(s) - s^2y(0) - s\dot{y}(0) - \ddot{y}(0) + 3s^2Y(s) - 3sy(0) - 3\dot{y}(0)$$

$$+ 3sY(s) - 3y(0) + Y(s) = \frac{1}{s}$$

Applying initial conditions:

$$Y(s) = \frac{1}{s(s+1)^3}$$

s= -1 is a *3*-order pole

Partial-Fraction Expansion

$$Y(s) = \frac{c_1}{s} + \frac{b_3}{(s+1)^3} + \frac{b_2}{(s+1)^2} + \frac{b_1}{s+1}$$

Determining coefficients:

$$c_1 = \frac{1}{s(s+1)^3} s \bigg|_{s=0} = 1$$

$$b_3 = [\frac{1}{s(s+1)^3}(s+1)^3]_{s=-1} = -1 \qquad b_1 = \frac{1}{2!}(2s^{-3})\bigg|_{s=-1} = -1$$

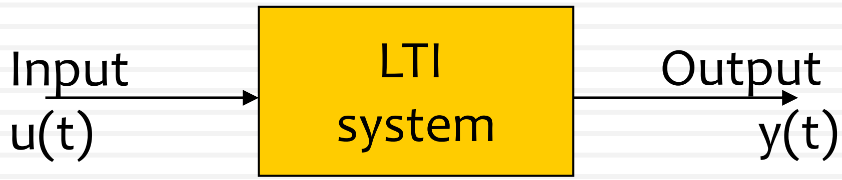
$$b_2 = \left\{ \frac{d}{ds}[\frac{1}{s(s+1)^3}(s+1)^3] \right\}_{s=-1} = [\frac{d}{ds}(\frac{1}{s})]_{s=-1} = (-s^{-2})\big|_{s=-1} = -1$$

$$\therefore Y(s) = \frac{1}{s} - \frac{1}{(s+1)^3} - \frac{1}{(s+1)^2} - \frac{1}{s+1}$$

Inverse Laplace transform:

$$y(t) = 1 - \frac{1}{2}t^2 e^{-t} - te^{-t} - e^{-t}$$

Transfer function



Consider a linear system described by differential equation

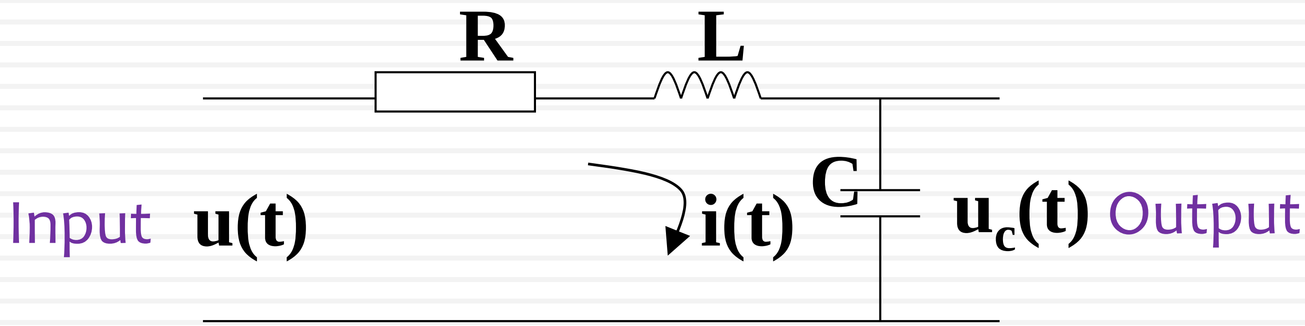
$$y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\cdots+a_0 y(t)= b_m u^{(m)}(t)+b_{m-1}u^{(m-1)}(t)+\cdots+bu^{(1)}(t)+b_0 u(t)$$

Assume all initial conditions are zero, we get the transfer function(TF) of the system as

$$TF=G(s)=\left.\frac{\mathcal{L}\left[output\ \ y(t)\right]}{\mathcal{L}\left[input\ u(t)\right]}\right|_{zero\ \ initial\ \ condition}$$

$$=\frac{Y(s)}{U(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}$$

Example 1. Find the transfer function of the RLC



Solution:

1) Writing the differential equation of the system according to physical law:

$$LC\ddot{u}_C(t)+RC\dot{u}_C(t)+u_C(t)=u(t)$$

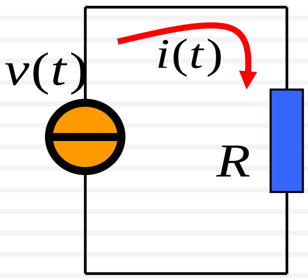
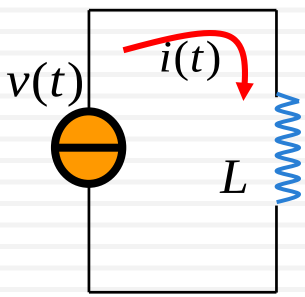
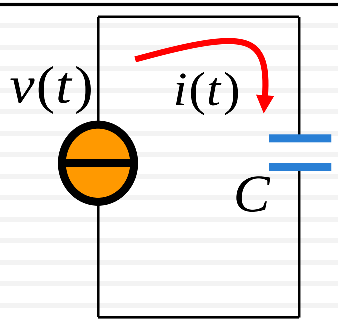
2) Assuming all initial conditions are zero and applying Laplace transform

$$LCs^2U_c(s)+RCsU_c(s)+U_c(s)=U(s)$$

3) Calculating the transfer function $G(s)$ as

$$G(s)=\frac{U_c(s)}{U(s)}=\frac{1}{LCs^2+RCs+1}$$

Transfer function of typical components

Component	ODE	TF
	$v(t) = Ri(t)$	$G(s) = \dfrac{V(s)}{I(s)} = R$
	$v(t) = L\dfrac{di(t)}{dt}$	$G(s) = \dfrac{V(s)}{I(s)} = sL$
	$v(t) = \dfrac{1}{C}\int_0^t i(\tau)d\tau$	$G(s) = \dfrac{V(s)}{I(s)} = \dfrac{1}{sC}$

Properties of transfer function

- The transfer function is defined only for a linear time-invariant system, not for nonlinear system.
- All initial conditions of the system are set to zero.
- The transfer function is independent of the input of the system.
- The transfer function $G(s)$ is the Laplace transform of the unit impulse response $g(t)$.

How poles and zeros relate to system response

- Why we strive to obtain TF models?
- Why control engineers prefer to use TF model?
- How to use TF model to analyze and design control systems?

- we start from the relationship between the locations of zeros and poles of TF and the output responses of a system

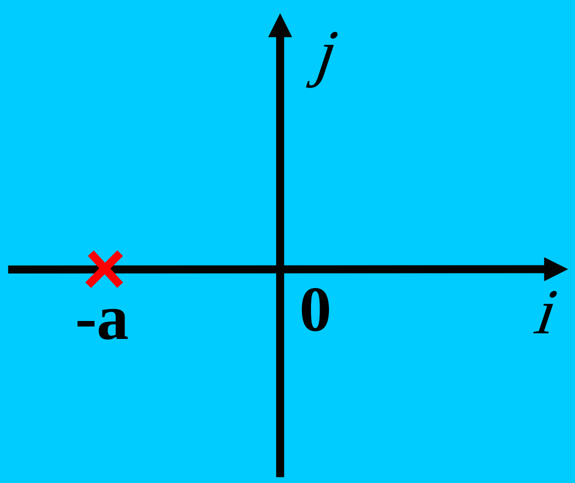
Transfer function

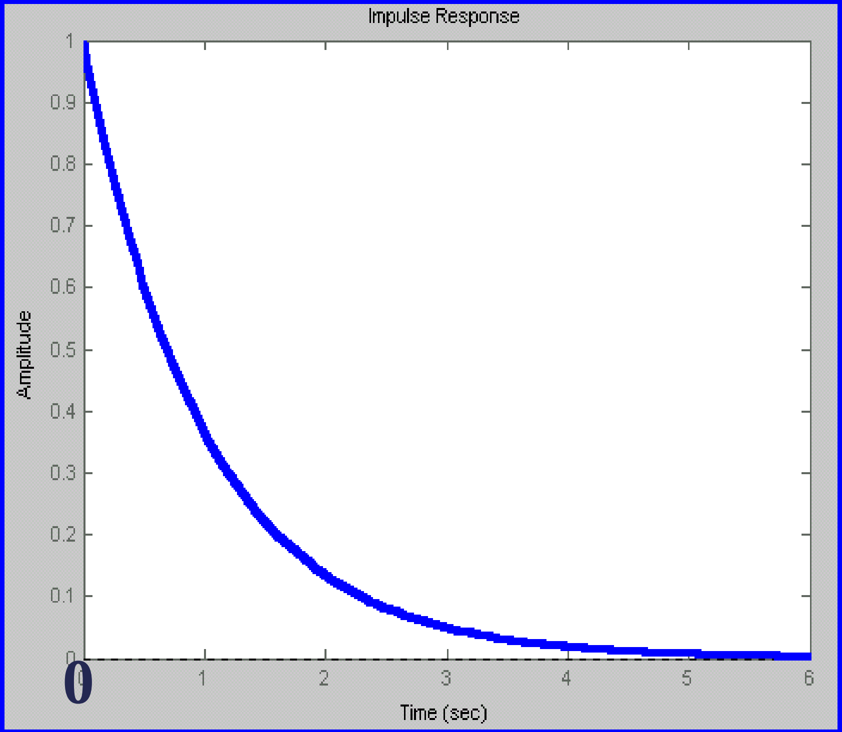
$$X(s) = \frac{A}{s+a}$$

Time-domain impulse response

$$x(t) = Ae^{-at}$$

Position of poles and zeros





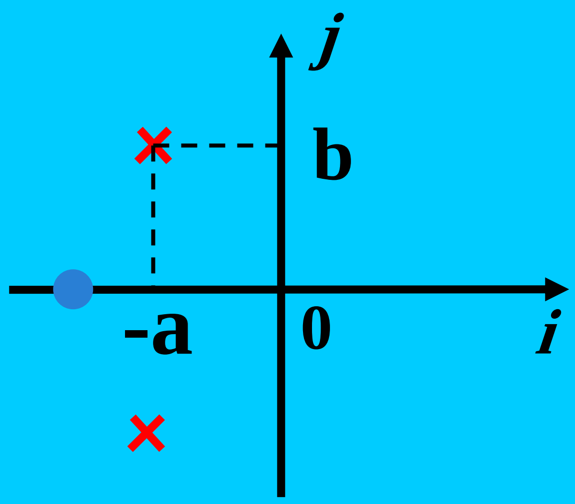
Transfer function

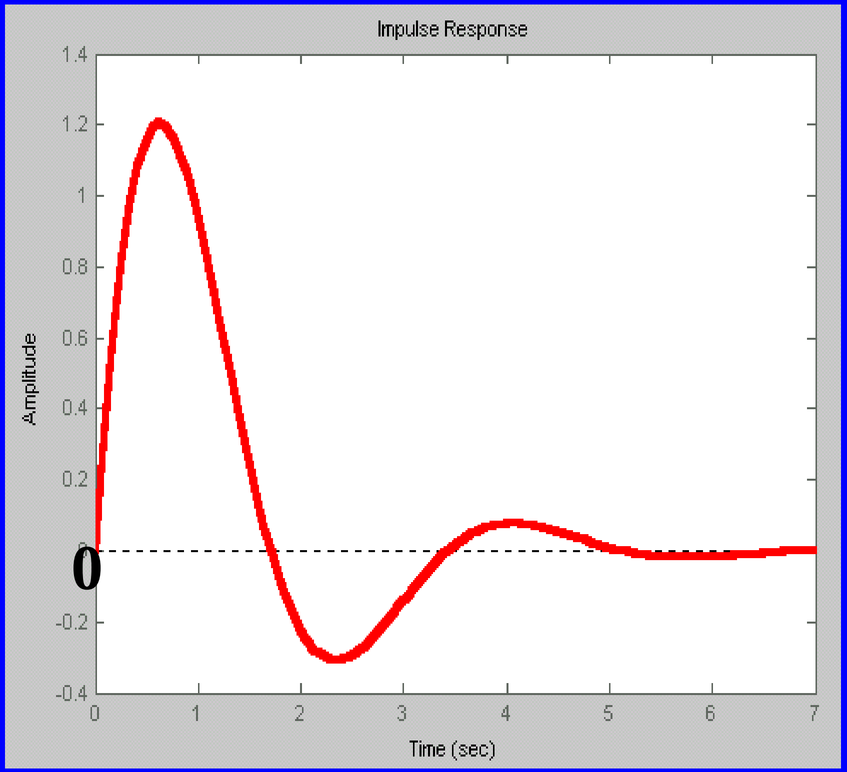
$$X(s) = \frac{A_1 s + B_1}{(s+a)^2 + b^2}$$

Time-domain impulse response

$$x(t) = Ae^{-at}\sin(bt + \phi)$$

Position of poles and zeros





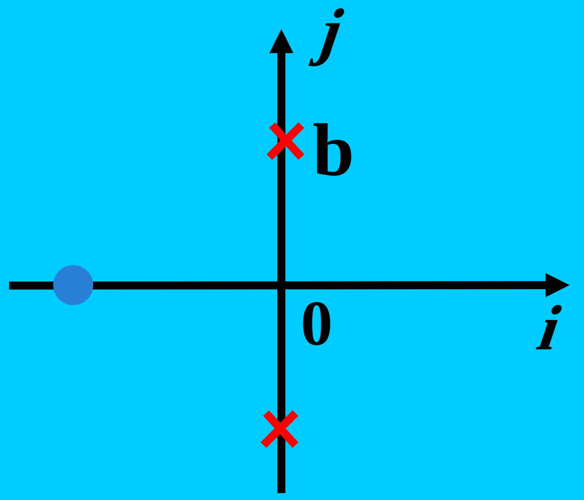
Transfer function

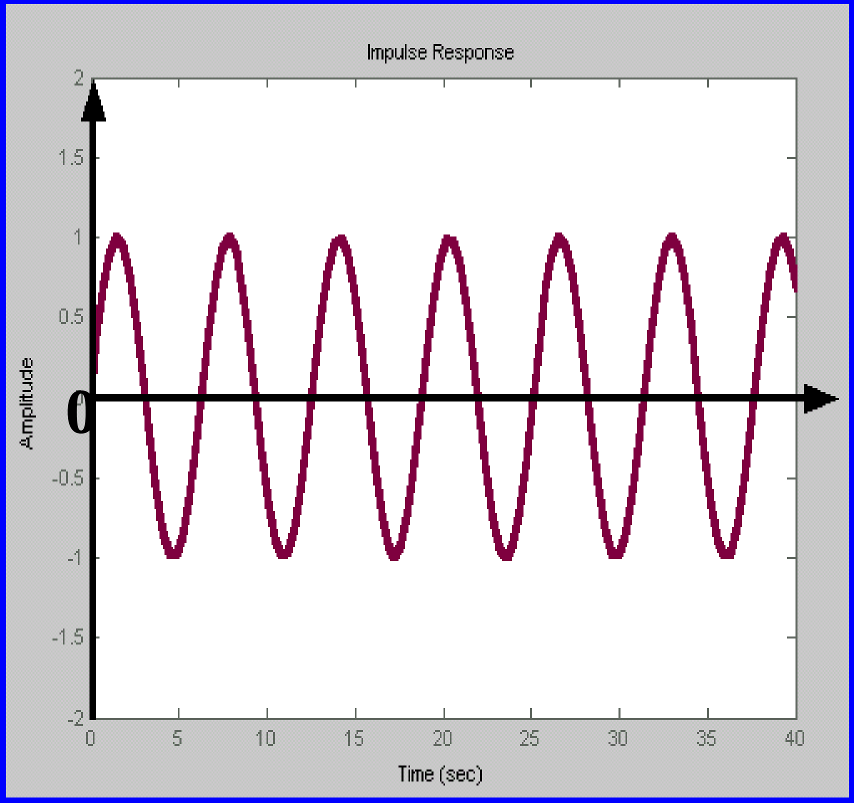
$$X(s) = \frac{A_1 s + B_1}{s^2 + b^2}$$

Time-domain impulse response

$$x(t) = A\sin(bt + \phi)$$

Position of poles and zeros





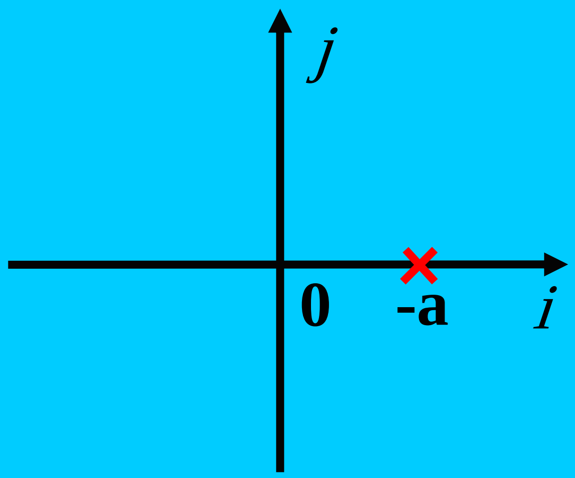
Transfer function

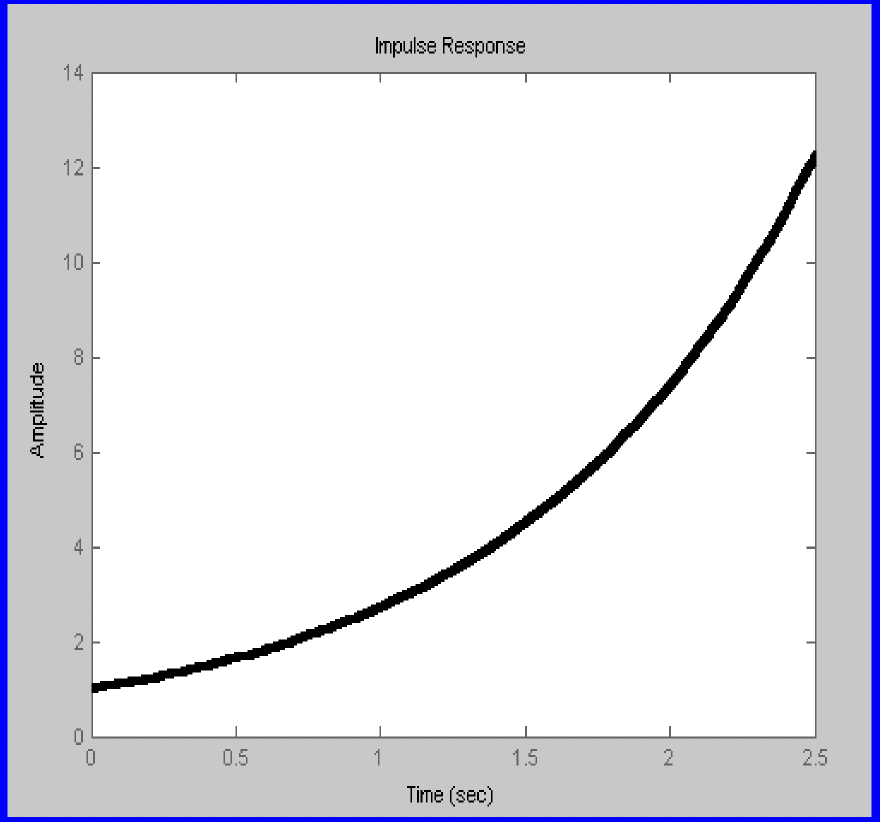
$$X(s) = \frac{A}{s-a}$$

Time-domain impulse response

$$x(t) = Ae^{at}$$

Position of poles and zeros





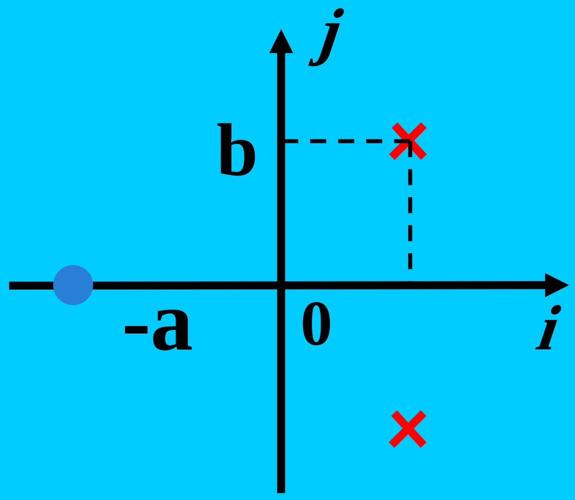
Transfer function:

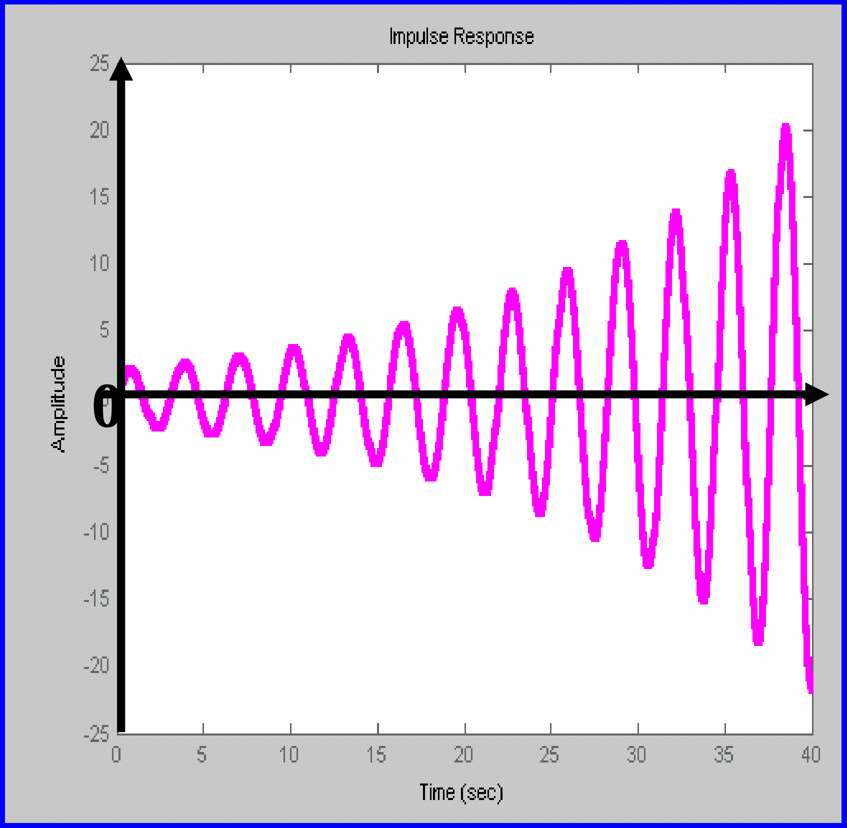
$$X(s) = \frac{A_1 s + B_1}{(s-a)^2 + b^2}$$

Time-domain dynamic response

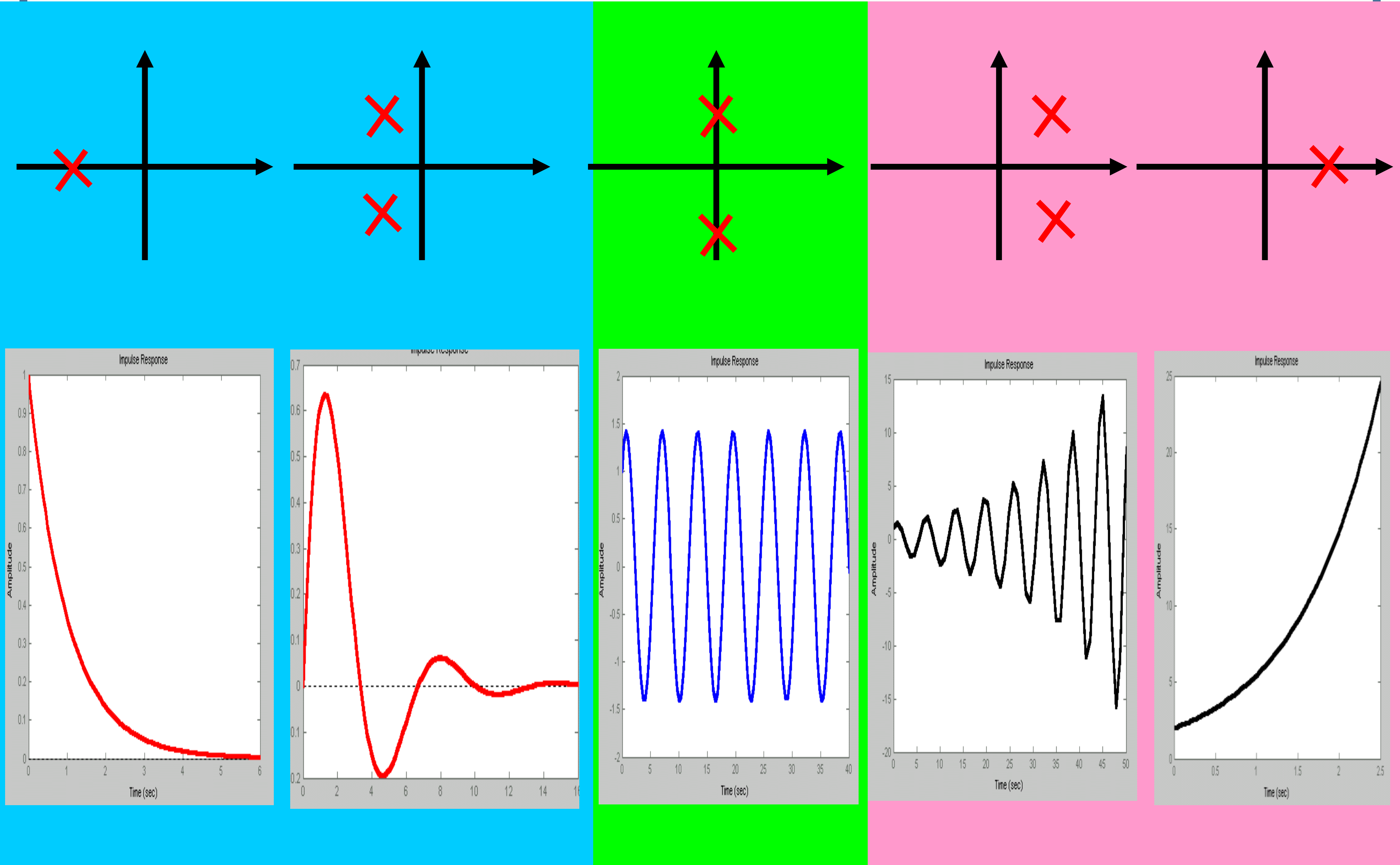
$$x(t) = Ae^{at}\sin(bt+\phi)$$

Position of poles and zeros





Summary of pole position & system dynamics



Characteristic equation

-obtained by setting the denominator polynomial of the transfer function to zero

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

Note: stability of linear single-input, single-output systems is completely governed by the roots of the characteristic equation.

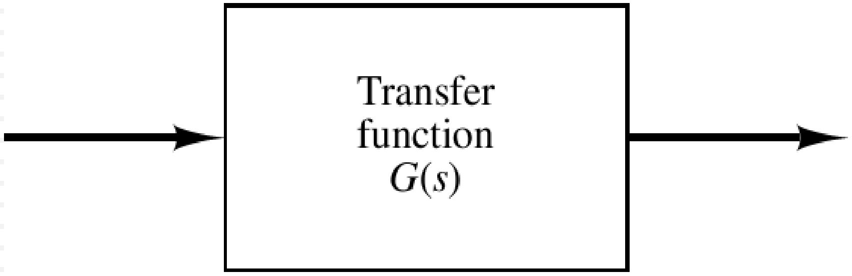
2-4 Block diagram and Signal-flow graph (SFG)

Block Diagrams

- In a block diagram all system variables are linked to each other through functional blocks
- The transfer functions of the components are usually entered in the corresponding blocks
- Blocks are connected by arrows to indicate the direction of the flow of signals

Note: The dimension of the output signal from the block is the dimension of the input signal multiplied by the dimension of the transfer function in the block

Element of a block diagram



Block Diagrams(2)

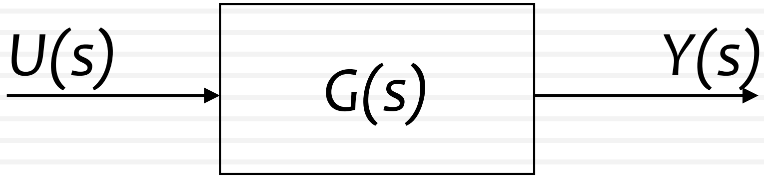
- The advantage of the block diagram representation is the simplicity of forming the overall block diagram for the entire system by connecting the blocks of the components according to the signal flow
- A block diagram contains information concerning dynamic behavior, but it does not include any information on the physical construction of the system
- A number of different block diagrams can be drawn for a system

Block Diagram Representation

The transfer function relationship

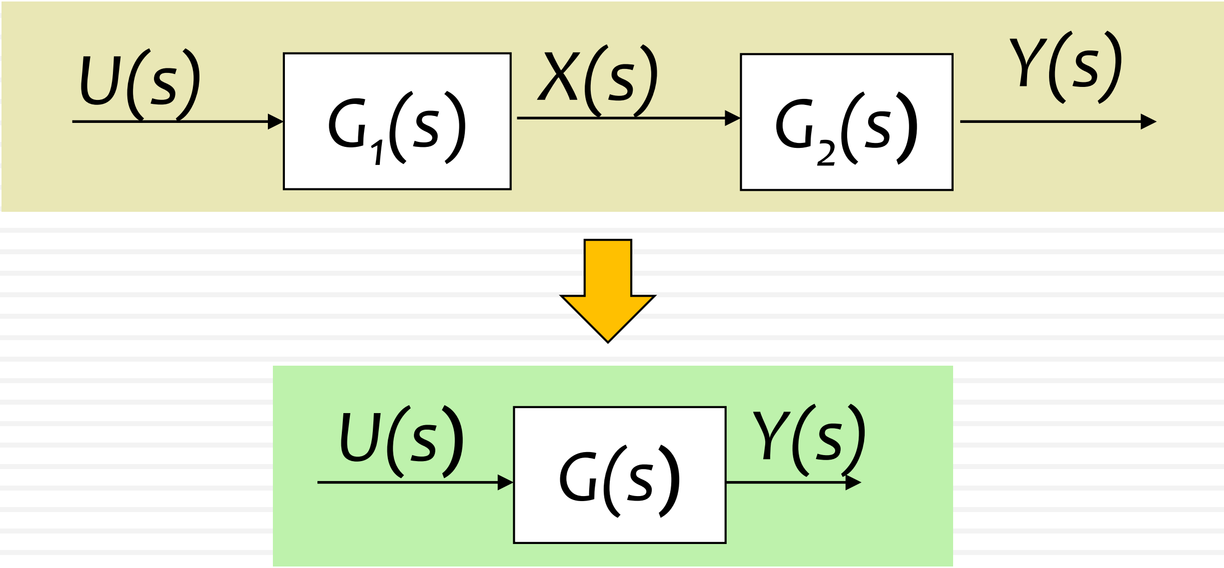
$$Y(s) = G(s)U(s)$$

can be graphically denoted through a block diagram.



Equivalent Transform of Block Diagram

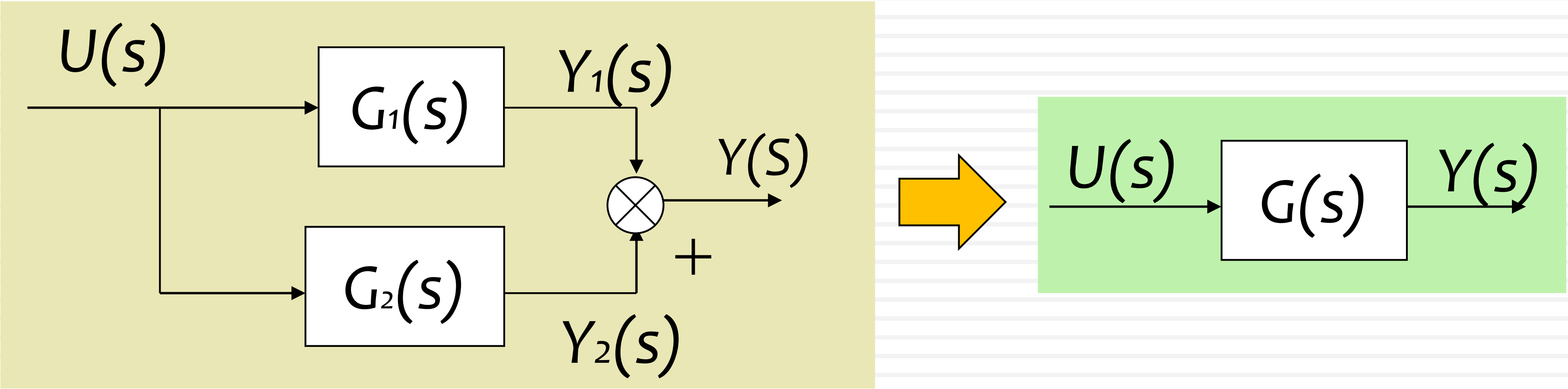
1. Connection in series



$$G(s) = ?$$

$$G(s) = \frac{Y(s)}{U(s)} = G_1(s) \cdot G_2(s)$$

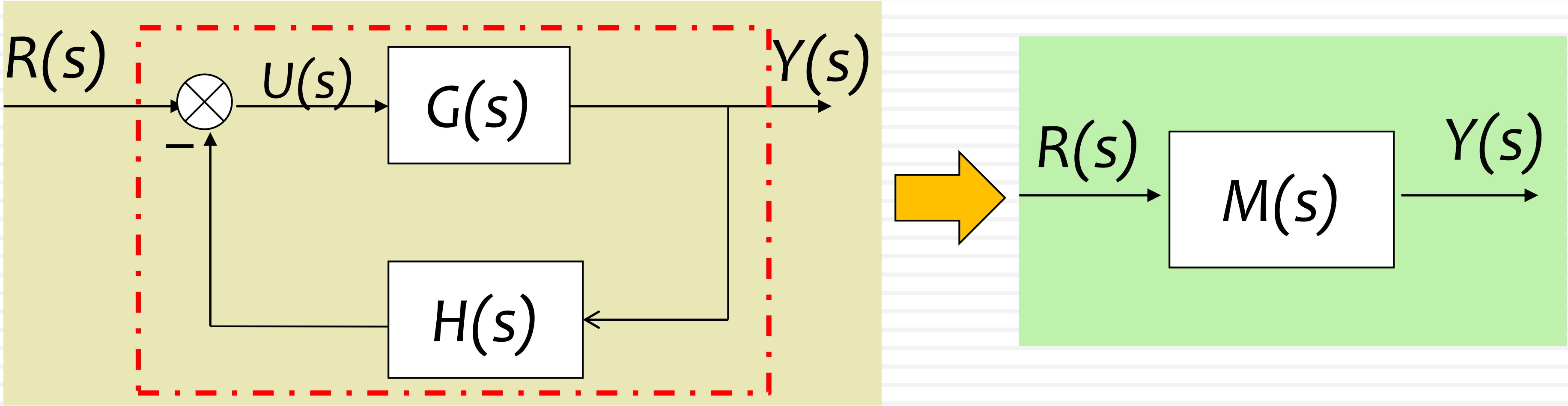
2. Connection in parallel



$$G(s) = ?$$

$$G(s) = \frac{Y(s)}{U(s)} = G_1(s) + G_2(s)$$

3. Negative feedback



$$\begin{cases} Y(s) = U(s)G(s) \\ U(s) = R(s) - Y(s)H(s) \end{cases} \Rightarrow Y(s) = \left[R(s) - Y(s)H(s)\right]G(s)$$

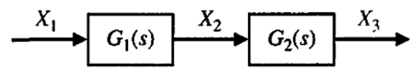
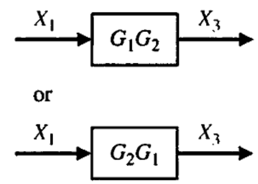
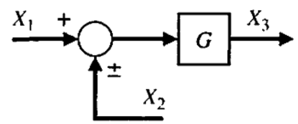
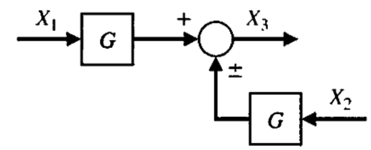
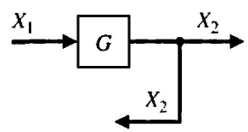
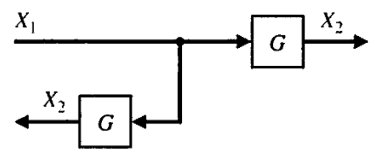
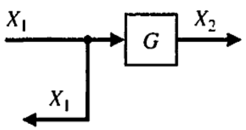
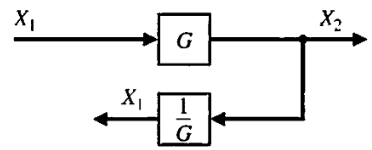
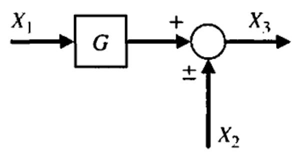
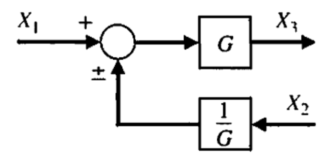
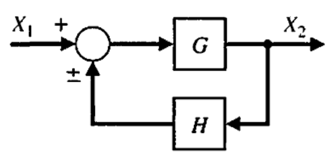
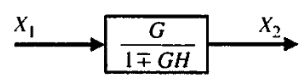
Transfer function of a negative feedback system:

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\text{gain of the forward path}}{1 + \text{gain of the loop}}$$

Block Diagram Reduction

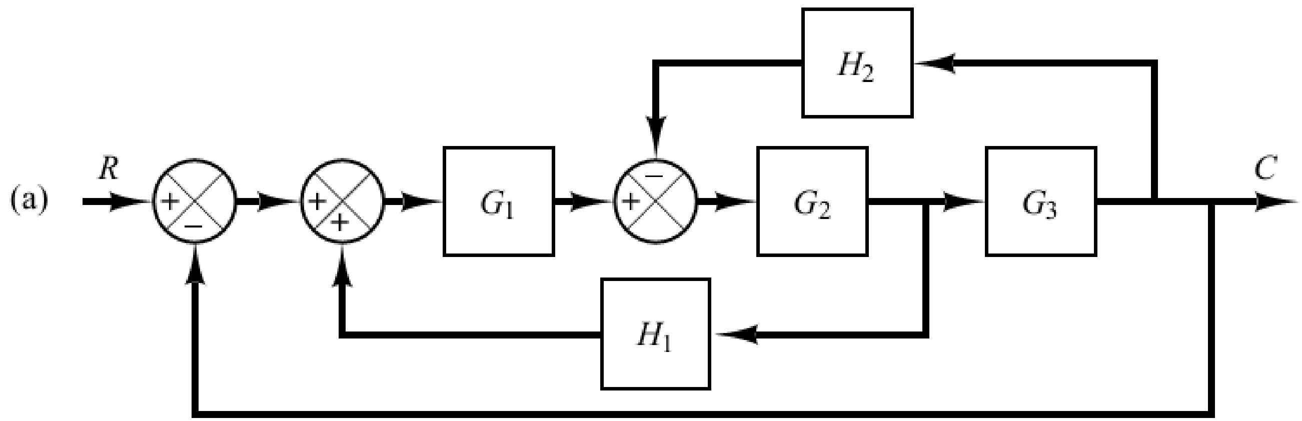
- Any number of cascaded blocks can be replaced by a single block, the transfer function of which is simply the product of the individual transfer functions
- Blocks can be connected in series only if the output of one block is not affected by the next following block (no feedback)
- A complicated block diagram involving many feedback loops can be simplified by a step-by-step rearrangement
- Simplification of the block diagram by rearrangements considerably reduces the labor needed for subsequent mathematical analysis

Block Diagram Transformations

Transformation	Original Diagram	Equivalent Diagram
1. Combining blocks in cascade		
2. Moving a summing point behind a block		
3. Moving a pickoff point ahead of a block		
4. Moving a pickoff point behind a block		
5. Moving a summing point ahead of a block		
6. Eliminating a feedback loop		

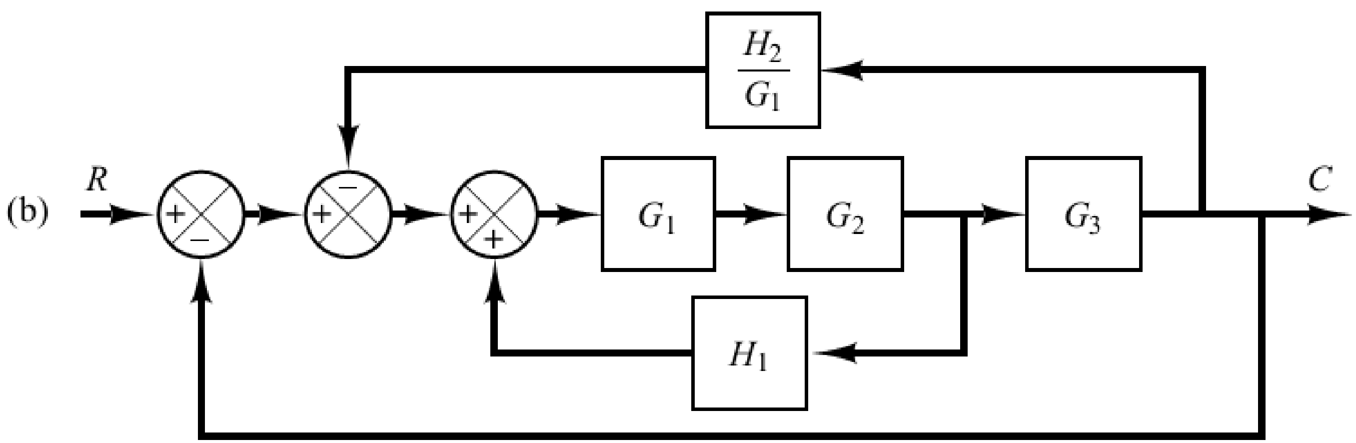
Example

- Simplify this diagram



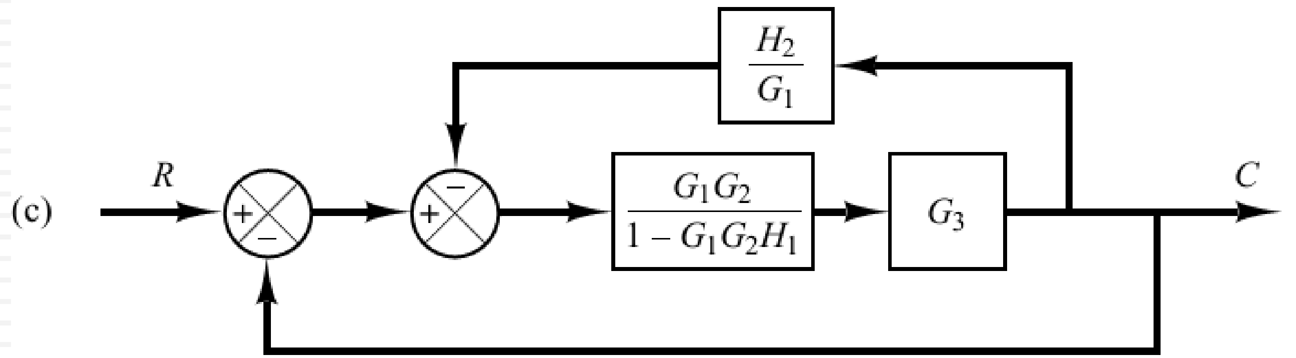
Solution:

- By moving the summing point of the negative feedback loop containing H_2 outside the positive feedback loop containing H_1, we obtain

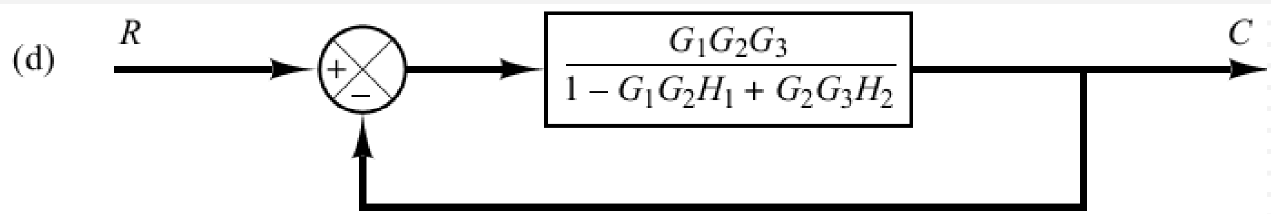


Example (2)

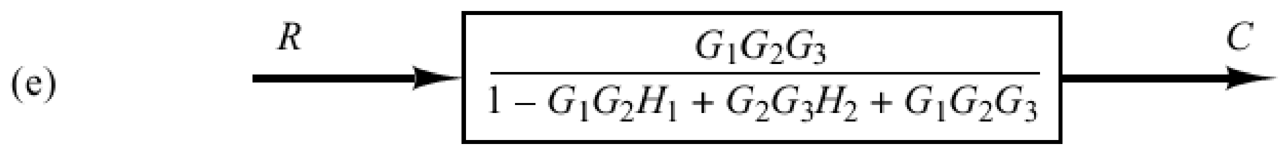
- Eliminating the positive feedback loop



- The elimination of the loop containing H_2/G_1 gives



- Finally, eliminating the feedback loop results in



Example (3)

- Notice that the numerator of the closed-loop transfer function *C*(*s*)/*R*(*s*) is the product of the transfer functions of the feed-forward path.
- The denominator of *C*(*s*)/*R*(*s*) is equal to

$$1 + \sum (\text{product of the transfer functions around each loop})$$

$$= 1 + \left(-G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3\right)$$

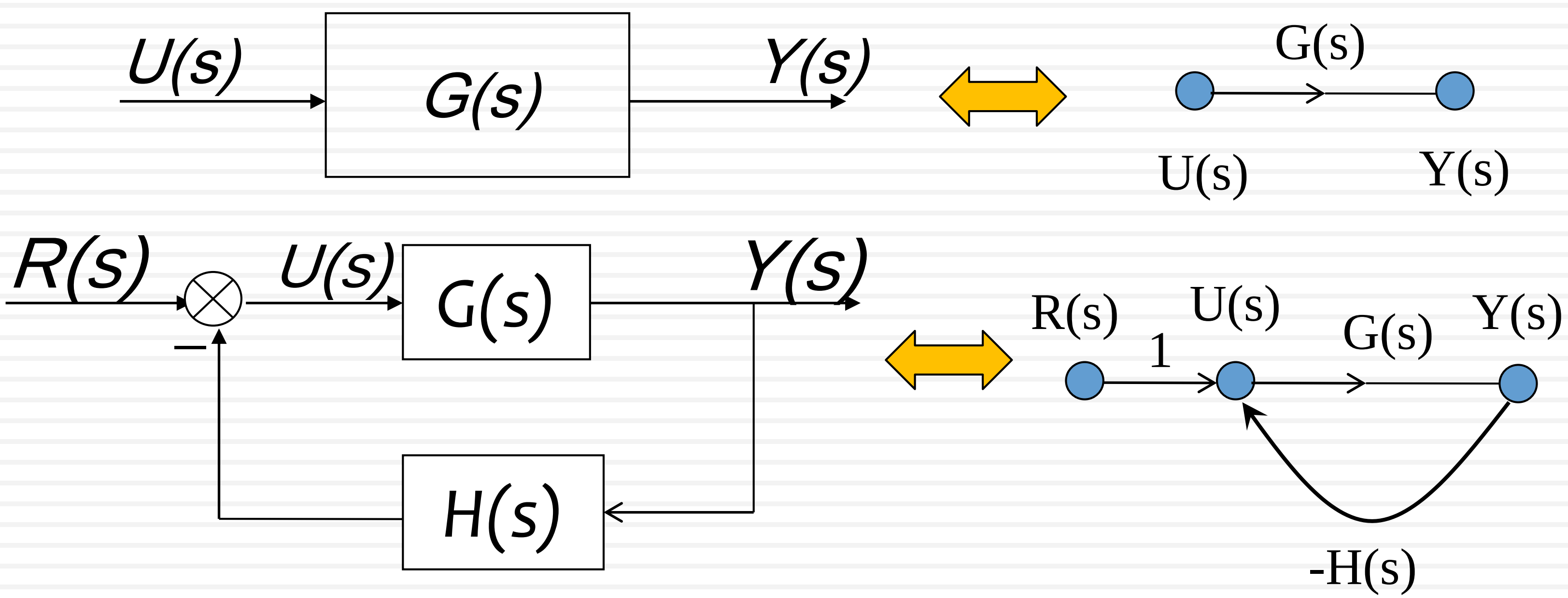
$$= 1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3$$

Signal Flow Graph

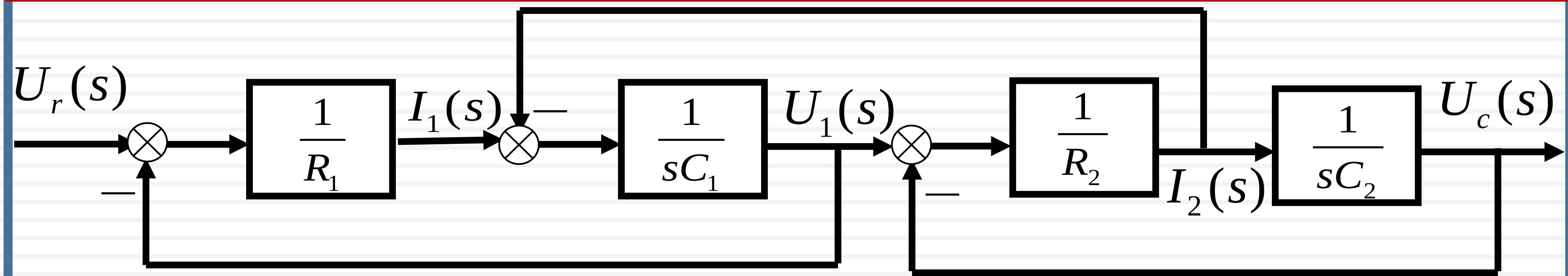
Signal Flow Graph (SFG)

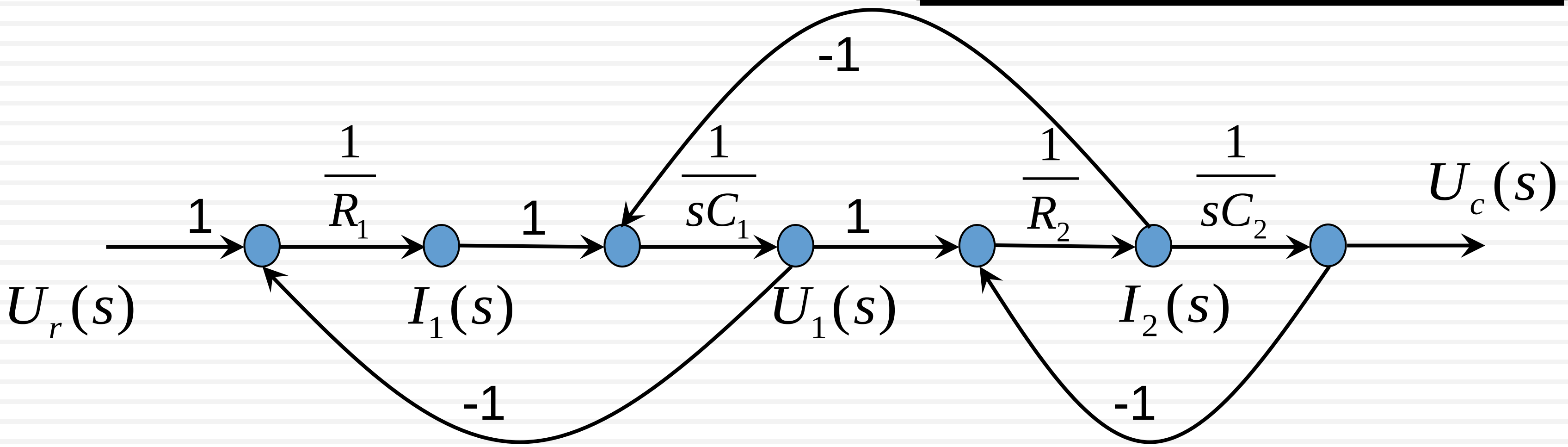
SFG was introduced by S.J. Mason for the cause-and-effect representation of linear systems.

1. Each signal is represented by a node.
2. Each transfer function is represented by a branch.



Block Diagram and Signal-Flow Graph





- Note: A signal flow graph and a block diagram contain exactly the same information (there is no advantage to one over the other; there is only personal preference)

Mason's Rule

$$M(s)=\frac{Y(s)}{U(s)}=\frac{1}{\Delta}\sum_{k=1}^{N}M_k\Delta_k$$

N = Total number of forward paths between output $Y(s)$ and input $U(s)$

M_k = Path gain of the k^{th} forward path

Δ = 1 − ∑(all individual loop gains)

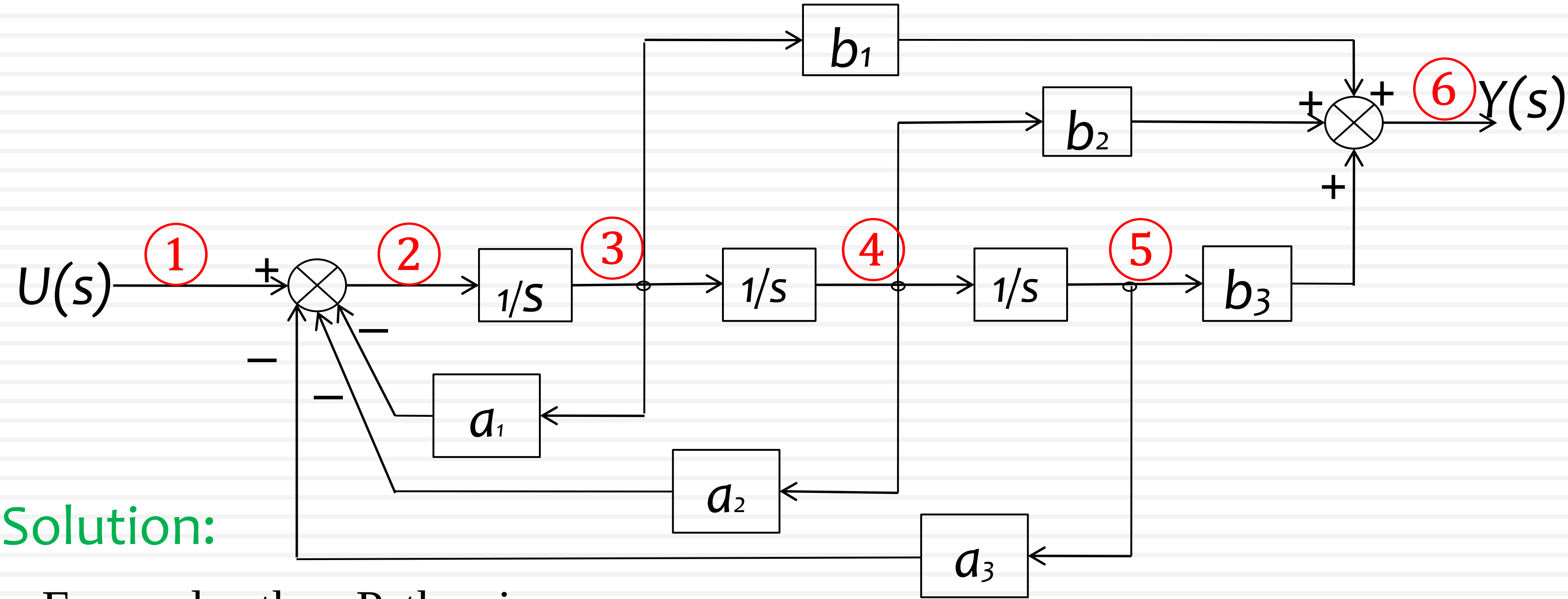
+∑(gain products of all possible two loops that do not touch)

−∑(gain products of all possible three loops that do not touch)

$+\cdots$

Δ_k = Value of Δ for that part of the block diagram that does not touch the k^{th} forward path

Example 1 Find the transfer function for the following block diagram



Solution:

Forward path	Path gain
1236	$M_1 = 1\left(\frac{1}{s}\right)(b_1)(1)$
12346	$M_2 = 1\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)(b_2)(1)$
123456	$M_3 = 1\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)(b_3)(1)$

and the determinates are

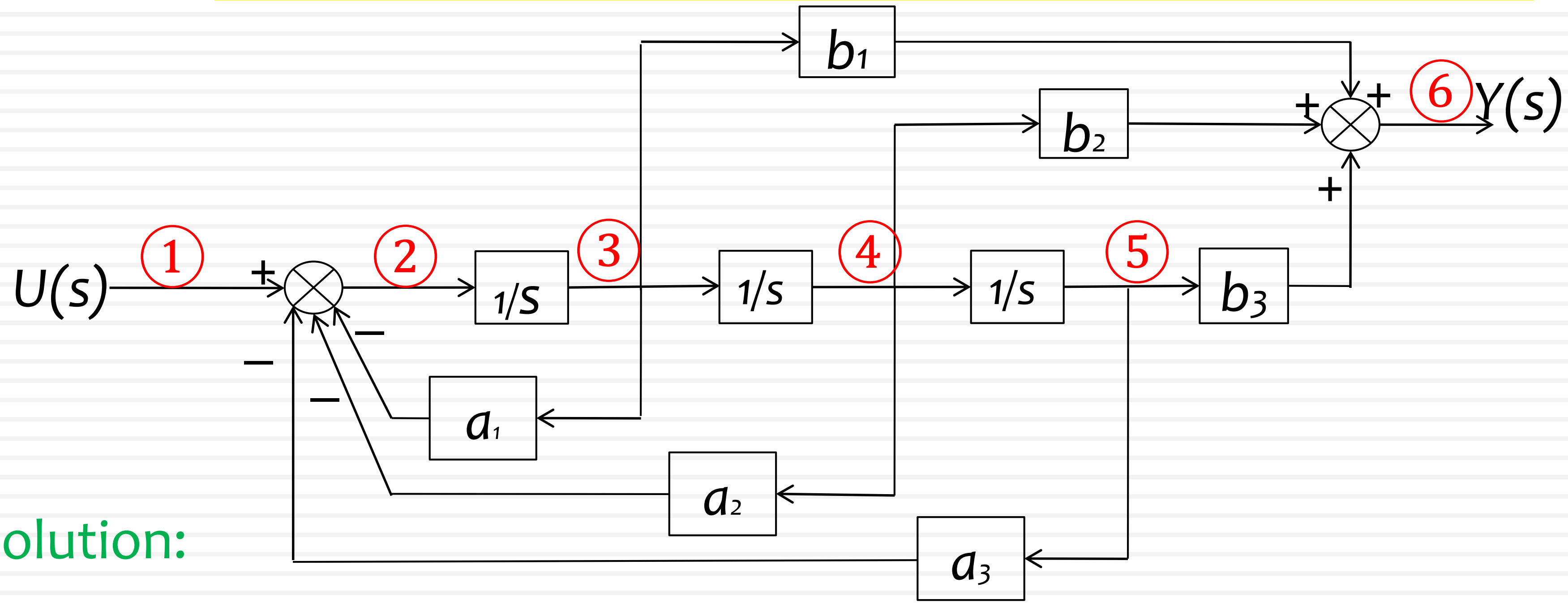
$$\Delta = 1 - \left(-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3}\right) + 0$$

$$\Delta_1 = 1 - 0$$

$$\Delta_2 = 1 - 0$$

$$\Delta_3 = 1 - 0$$

Example 1 **Find the transfer function for the following block diagram**



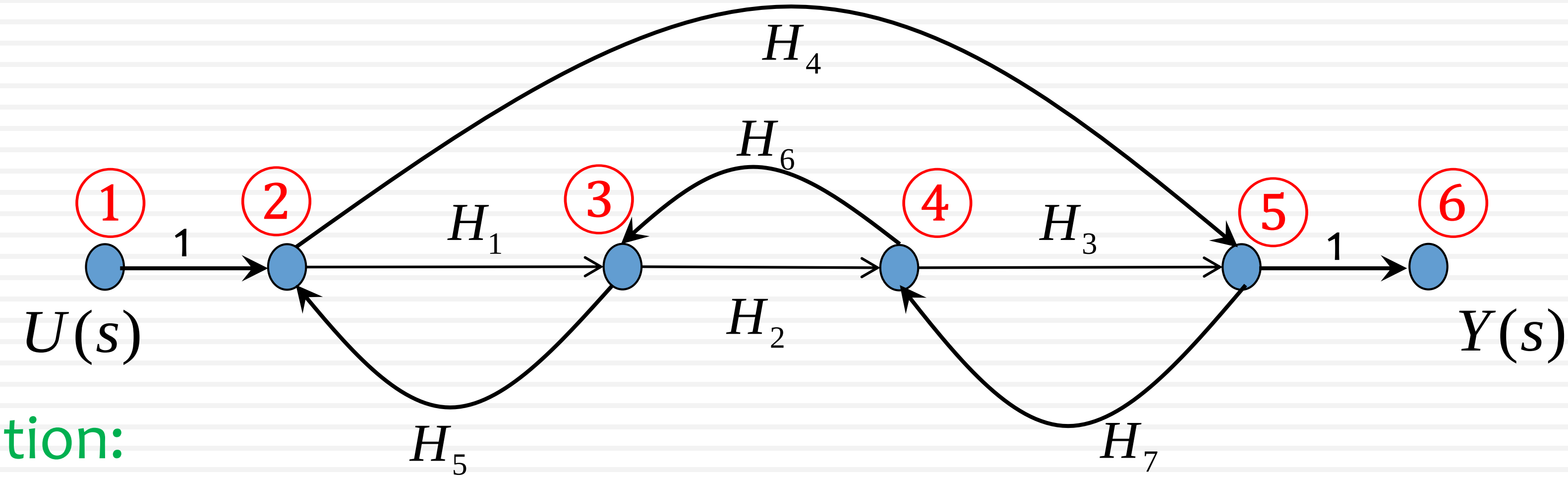
Solution:

Applying Mason's rule, we find the transfer function to be

$$M(s) = \frac{Y(s)}{U(s)} = \sum_{k=1}^{N} \frac{M_k \Delta_k}{\Delta}$$

$$= \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

Example 2 Find the transfer function for the following SFG



Solution:

Forward path	Path gain
123456	$M_1 = H_1 H_2 H_3$
1256	$M_2 = H_4$

Loop path	Path gain
232	$l_1 = H_1 H_5$
343	$l_2 = H_2 H_6$
454	$l_3 = H_3 H_7$
25432	$l_4 = H_4 H_7 H_6 H_5$

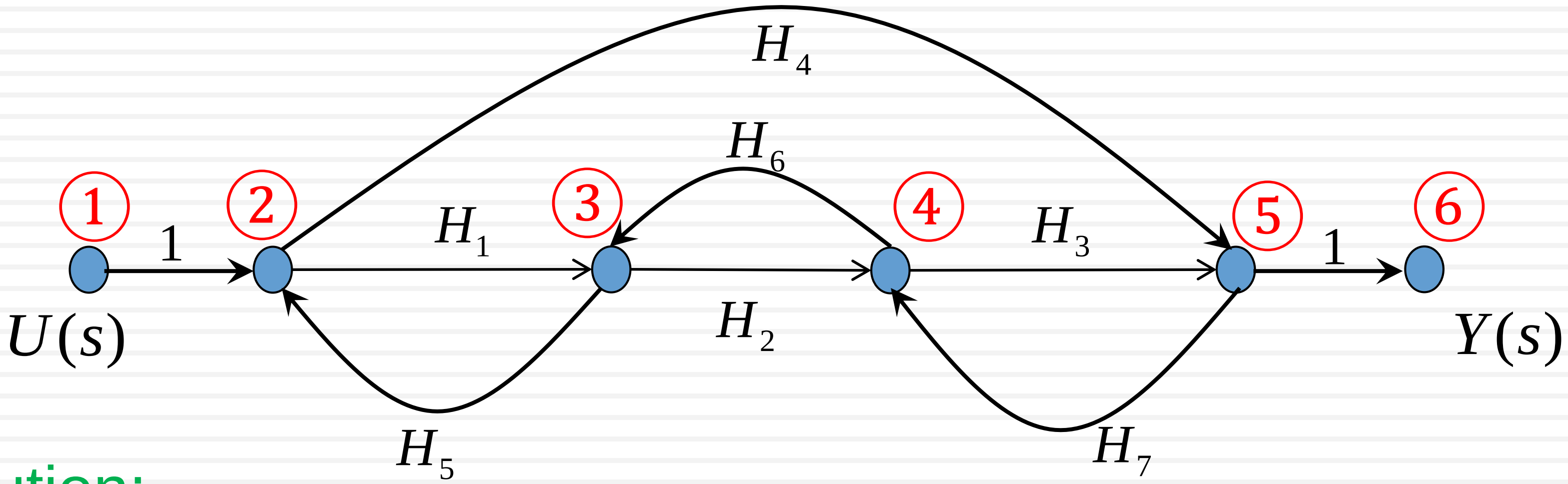
and the determinates are

$$\Delta = 1 - \left(l_1 + l_2 + l_3 + l_4\right) + (l_1 l_3)$$

$$\Delta_1 = 1 - 0$$

$$\Delta_2 = 1 - H_2 H_6$$

Example 2 Find the transfer function for the following SFG



Solution:

Applying Mason's rule, we find the transfer function to be

$$M(s)=\frac{Y(s)}{U(s)}=\sum_{k=1}^{N}\frac{M_k\Delta_k}{\Delta}$$

$$=\frac{H_1H_2H_3+H_4-H_4H_2H_6}{1-H_1H_5-H_2H_6-H_3H_7-H_4H_7H_6H_5+H_1H_5H_3H}$$