State Feedback and State Estimators

“Linear System Theory and Design”, Chapter 8

http://zitompul.wordpress.com

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Homework 8

Consider the following linear system given by:

$$\dot{\underline{\boldsymbol{x}}}(t) = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -3 & 4 \\ -1 & 1 & -9 \end{bmatrix} \underline{\boldsymbol{x}}(t) + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \underline{\boldsymbol{x}}(t)$$

(a) Using the transformation to the Observer Form, find the gain vector $\underline{\boldsymbol{l}}$ of the closed-loop state estimator if the desired poles are −3 and −4 ± j2.

(b) Recall again the output feedback. In observer form, its effect on the characteristic equation of the system can be calculated much easier. By calculation, prove that the poles of the system cannot be assigned to any arbitrary location by only setting the value of output feedback j.

Solution of Homework 8

$$a(s) = \det(s\underline{\boldsymbol{I}} - \underline{\boldsymbol{A}})$$
$$= s^3 + \underbrace{13}_{a_2}s^2 + \underbrace{33}_{a_1}s + \underbrace{13}_{a_0}$$

$$\underline{\mathcal{O}} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & -9 \\ 14 & -22 & 93 \end{bmatrix}$$

$$\underline{\mathcal{T}} = \begin{bmatrix} 33 & 13 & 1 \\ 13 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{R}}^{-1} = \begin{bmatrix} 21 & 17 & 9 \\ 11 & 3 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\underline{\boldsymbol{R}} = \begin{bmatrix} -0.0361 & 0.2048 & -0.4940 \\ 0.0843 & -0.1446 & -0.1807 \\ 0.0361 & -0.2048 & 1.4940 \end{bmatrix}$$

Checking the Observer Form:

$$\underline{\hat{\boldsymbol{A}}} = \begin{bmatrix} 0 & 0 & -13 \\ 1 & 0 & -33 \\ 0 & 1 & -13 \end{bmatrix}$$

$$\underline{\hat{\boldsymbol{b}}} = \begin{bmatrix} 46 \\ 13 \\ 0 \end{bmatrix}$$

$$\underline{\hat{\boldsymbol{c}}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Solution of Homework 8

(a) Find the gain vector $\underline{\boldsymbol{l}}$ of the closed-loop state estimator if the desired poles are −3 and −4 ± j2.

The desired characteristic equation of the state observer is:

$$\breve{a}(s) = (s+3)(s+4+j2)(s+4-j2)$$
$$= s^3 + \underbrace{11}_{\breve{a}_2}s^2 + \underbrace{44}_{\breve{a}_1}s + \underbrace{60}_{\breve{a}_0}$$

$$\hat{l}_1 = \breve{a}_0 - a_0 = 60 - 13 = 47$$
$$\hat{l}_2 = \breve{a}_1 - a_1 = 44 - 33 = 11$$
$$\hat{l}_3 = \breve{a}_2 - a_2 = 11 - 13 = -2$$

$$\underline{\hat{\boldsymbol{l}}} = \begin{bmatrix} 47 \\ 11 \\ -2 \end{bmatrix}$$

For the transformed system

$$\underline{\boldsymbol{l}} = \underline{\boldsymbol{R}}\underline{\hat{\boldsymbol{l}}} = \begin{bmatrix} 1.5422 \\ 2.7349 \\ -3.5422 \end{bmatrix}$$

For the original system

Solution of Homework 8

(b) Prove that the poles of the system cannot be placed freely by only setting a single variable *j* of output feedback.

$$\dot{\hat{\underline{\mathbf{x}}}}(t)=\begin{bmatrix}0 & 0 & -13\\ 1 & 0 & -33\\ 0 & 1 & -13\end{bmatrix}\hat{\underline{\mathbf{x}}}(t)+\begin{bmatrix}46\\ 13\\ 0\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}0 & 0 & 1\end{bmatrix}\hat{\underline{\mathbf{x}}}(t)$$

- **Transformation Result, Observer Form**

$$\dot{\underline{\mathbf{x}}}(t)=(\underline{\mathbf{A}}-j\underline{\mathbf{b}}\underline{\mathbf{c}})\underline{\mathbf{x}}(t)+\underline{\mathbf{b}}r(t)$$

$$y(t)=\underline{\mathbf{c}}\underline{\mathbf{x}}(t)$$

- **Output Feedback**

Solution of Homework 8

$$\underline{\dot{\mathbf{x}}}(t) = (\underline{\mathbf{A}} - j\underline{\mathbf{b}}\underline{\mathbf{c}})\underline{\mathbf{x}}(t) + \underline{\mathbf{b}}r(t)$$

$$\underline{\dot{\mathbf{x}}}(t) = \left(\begin{bmatrix} 0 & 0 & -13 \\ 1 & 0 & -33 \\ 0 & 1 & -13 \end{bmatrix} - j \begin{bmatrix} 46 \\ 13 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right) \underline{\mathbf{x}}(t) + \underline{\mathbf{b}}r(t)$$

$$= \left(\begin{bmatrix} 0 & 0 & -13 \\ 1 & 0 & -33 \\ 0 & 1 & -13 \end{bmatrix} - j \begin{bmatrix} 0 & 0 & 46 \\ 0 & 0 & 13 \\ 0 & 0 & 0 \end{bmatrix} \right) \underline{\mathbf{x}}(t) + \underline{\mathbf{b}}r(t)$$

$$= \begin{bmatrix} 0 & 0 & -(13+46j) \\ 1 & 0 & -(33+13j) \\ 0 & 1 & -13 \end{bmatrix} \underline{\mathbf{x}}(t) + \underline{\mathbf{b}}r(t)$$

- **Only a_0 and a_1 can be adjusted, both dependent to each other**
- **The poles of the system cannot be placed in any wished position**

$$\Rightarrow \underline{\underline{\breve{a}(s) = s^3 + 13s^2 + (33+13j)s + (13+46j)}}$$

Reference Input in State Feedback

- The state feedback has been proven to be able to place the poles of closed-loop system in arbitrary locations, and therefore can be used to design the **transient response** of a system.
- However, the **steady-state response** is still neglected until now, and the system will almost surely have a nonzero error to a step input.

$$u(t) = r(t) - \underline{\mathbf{k}}\,\underline{\mathbf{x}}(t)$$

Reference Value

- Now, two ways to incorporate the tracing of reference input while using state feedback will be introduced:
  - Pre-scaling/ Pre-amplifying
  - Integral Control

Tracing of Reference Input: Pre-Scaling

- If the desired value of the states and the required process input to reach them are $\underline{\boldsymbol{x}}_{\mathrm{r}}(t)$ and $u_{\mathrm{r}}(t)$, then the new feedback formula should be:

$$u(t) = u_{\mathrm{r}}(t) - \underline{\boldsymbol{k}}\left(\underline{\boldsymbol{x}}(t) - \underline{\boldsymbol{x}}_{\mathrm{r}}(t)\right)$$

If $\underline{\boldsymbol{x}}(t) \rightarrow \underline{\boldsymbol{x}}_{\mathrm{r}}(t)$, then $u(t) \rightarrow u_{\mathrm{r}}(t)$

- Consider again the n-dimensional state-space equations:

$$\dot{\underline{\boldsymbol{x}}}(t) = \underline{\boldsymbol{A}}\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{b}}u(t)$$
$$y(t) = \underline{\boldsymbol{c}}\underline{\boldsymbol{x}}(t)$$

- In <u>steady-state condition</u>, these equations reduce to:

$$\underline{\boldsymbol{0}} = \underline{\boldsymbol{A}}\underline{\boldsymbol{x}}_{\mathrm{ss}}(t) + \underline{\boldsymbol{b}}u_{\mathrm{ss}}(t)$$
$$y_{\mathrm{ss}}(t) = \underline{\boldsymbol{c}}\underline{\boldsymbol{x}}_{\mathrm{ss}}(t)$$

Tracing of Reference Input: Pre-Scaling

- Now, comparing the values from the above equations and the desired values, we obtain:

$$y_{\mathrm{ss}}(t) = r(t)$$

$$\underline{\boldsymbol{x}}_{\mathrm{ss}}(t) = \underline{\boldsymbol{x}}_{\mathrm{r}}(t)$$

$$u_{\mathrm{ss}}(t) = u_{\mathrm{r}}(t)$$

- Let us now define:

$$\underline{\boldsymbol{x}}_{\mathrm{ss}}(t) = \underline{\boldsymbol{N}}r(t)$$

$$u_{\mathrm{ss}}(t) = Mr(t)$$

• **How? Why?**

- The equations in <u>steady-state condition</u> can now be written as:

$$\begin{bmatrix} \underline{\boldsymbol{0}} \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{\boldsymbol{A}} & \underline{\boldsymbol{b}} \\ \underline{\boldsymbol{c}} & 0 \end{bmatrix} \begin{bmatrix} \underline{\boldsymbol{N}} \\ M \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \underline{\boldsymbol{N}} \\ M \end{bmatrix} = \begin{bmatrix} \underline{\boldsymbol{A}} & \underline{\boldsymbol{b}} \\ \underline{\boldsymbol{c}} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \underline{\boldsymbol{0}} \\ 1 \end{bmatrix}$$

Tracing of Reference Input: Pre-Scaling

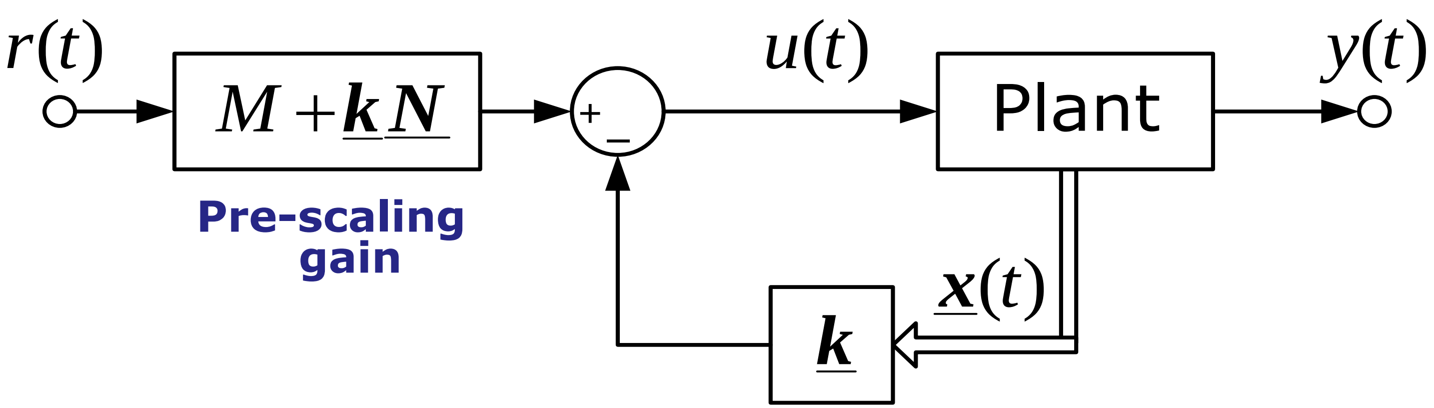
- After finding $\underline{\boldsymbol{N}}$ and M, the required input to the system, $u(t)$, that guarantees zero steady-state error to a step input can be calculated as:

$$u(t) = u_{\mathrm{r}}(t) - \underline{\boldsymbol{k}}\left(\underline{\boldsymbol{x}}(t) - \underline{\boldsymbol{x}}_{\mathrm{r}}(t)\right)$$

$$u(t) = Mr(t) - \underline{\boldsymbol{k}}\left(\underline{\boldsymbol{x}}(t) - \underline{\boldsymbol{N}}r(t)\right)$$

$$= \underbrace{\left(M + \underline{\boldsymbol{k}}\,\underline{\boldsymbol{N}}\right)}_{E}r(t) - \underline{\boldsymbol{k}}\,\underline{\boldsymbol{x}}(t)$$

• **New scalar gain for *r*(*t*)**



Example 1: Pre-Scaling

Referring again to the state-space equation that has been used before,

$$\underline{\dot{\boldsymbol{x}}}(t) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \underline{\boldsymbol{x}}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{\boldsymbol{x}}(t)$$

For the desired eigenvalues of –1 and –2, it is already calculated that the required feedback gain is $\underline{\boldsymbol{k}}$ = [4 1].

Now, it is desired that the output $y(t)$ should follow $r(t)$ = 1.5(t). Calculate the gain E for the reference value $r(t)$

Example 1: Pre-Scaling

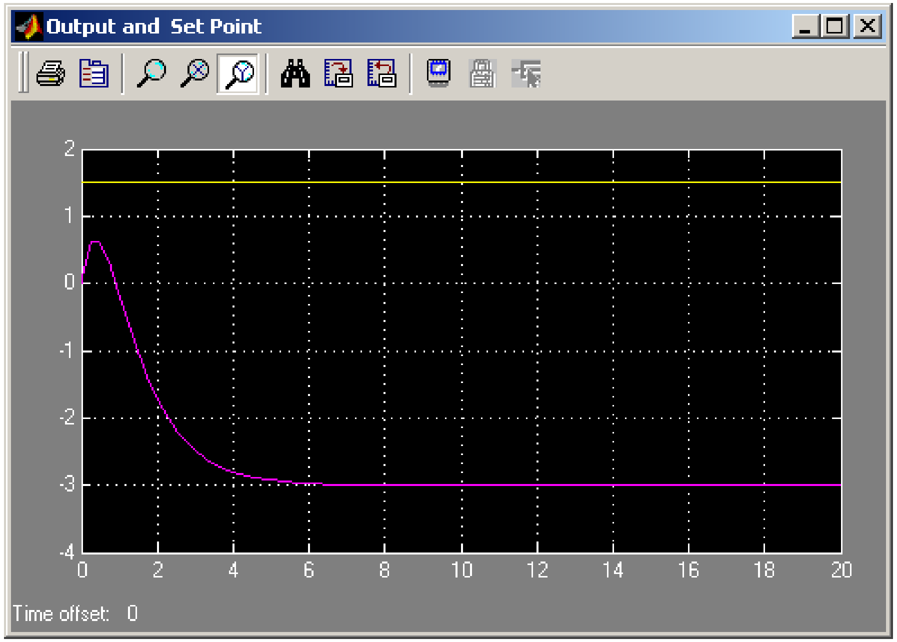
$$\begin{bmatrix} \underline{\boldsymbol{N}} \\ M \end{bmatrix} = \begin{bmatrix} \underline{\boldsymbol{A}} & \underline{\boldsymbol{b}} \\ \underline{\boldsymbol{c}} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \underline{\boldsymbol{0}} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \underline{\boldsymbol{N}} \\ M \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 1.25 \\ -0.75 \end{bmatrix}$$

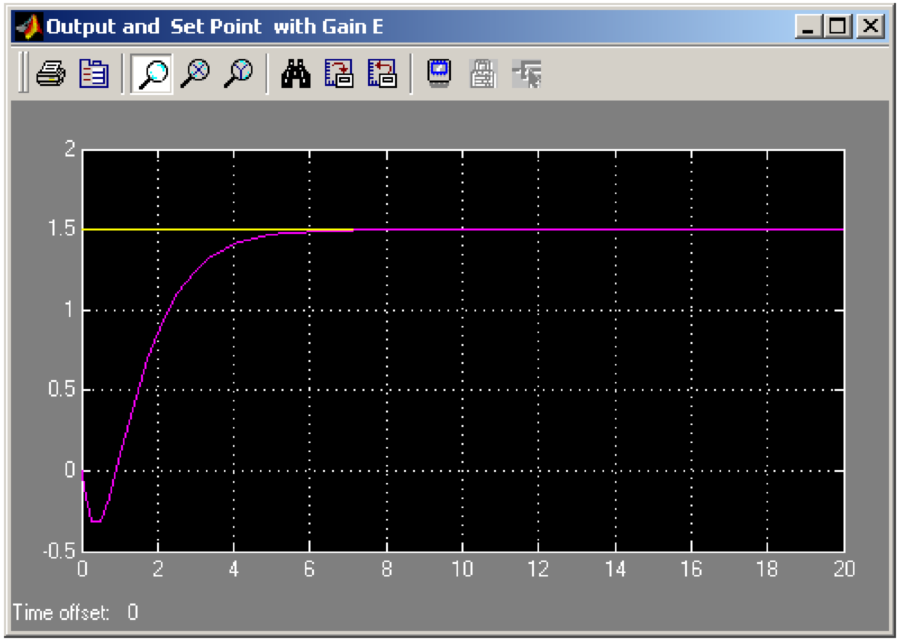
$$\Rightarrow \underline{\boldsymbol{N}} = \begin{bmatrix} -0.25 \\ 1.25 \end{bmatrix}, \quad M = -0.75$$

$$E = \left(M + \underline{\boldsymbol{k}}\,\underline{\boldsymbol{N}}\right) = \left(-0.75 + \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} -0.25 \\ 1.25 \end{bmatrix}\right) = \underline{\underline{-0.5}}$$

Example 1: Pre-Scaling



Step Response Without Reference Gain *E*



Step Response With Reference Gain *E*

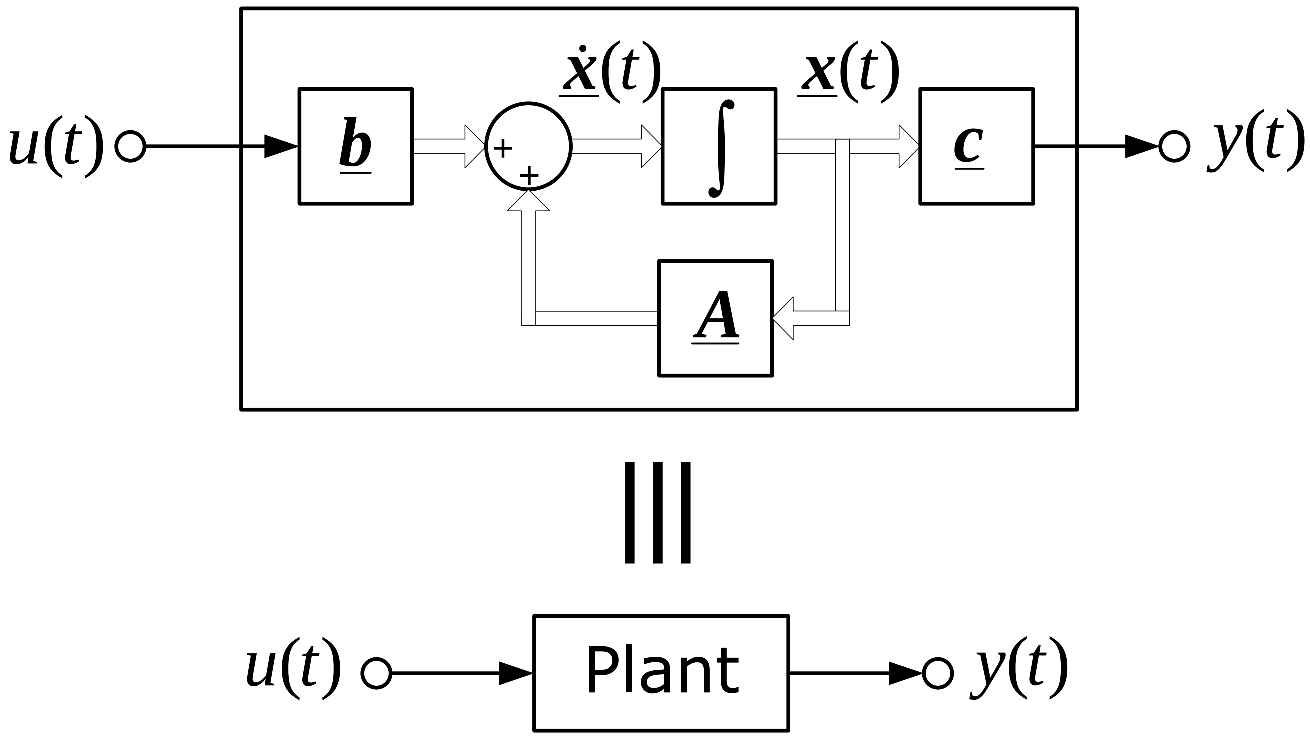
- **The previous steady-state value of the system is $y(\infty) = -3$, see left scope.**
- **The reference gain ($E = -0.5$) invert $y(\infty)$ to the desired value of $r(t) = 1.5(t)$, see right scope.**

Tracing of Reference Input: Integral Control

- The integral control is included by augmenting the state vector $\underline{\boldsymbol{x}}(t)$ with the desired dynamics, such that the states of the system is increased, but still with the same form of:

$$\dot{\underline{\mathbf{x}}}(t) = \underline{\mathbf{A}}\underline{\mathbf{x}}(t) + \underline{\mathbf{b}}u(t)$$

$$y(t) = \underline{\mathbf{c}}\underline{\mathbf{x}}(t)$$



Tracing of Reference Input: Integral Control

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$$\dot{\underline{\mathbf{x}}}(t) = \underline{\mathbf{A}}\underline{\mathbf{x}}(t) + \underline{\mathbf{b}}u(t)$$

$$y(t) = \underline{\mathbf{c}}\underline{\mathbf{x}}(t)$$

- The feedback is set to contain the integral of the error, $e = r - y$, as well as the state of the system, $x(t)$.
- We add the existing state with an extra integral state x_{int}, given by the following equation:

$$\dot{x}_{\mathrm{int}}(t) = r(t) - \underline{\mathbf{c}}\underline{\mathbf{x}}(t) = e(t)$$

This implies that

$$x_{\mathrm{int}}(t) = \int_0^{\tau} e(t)dt$$

Tracing of Reference Input: Integral Control

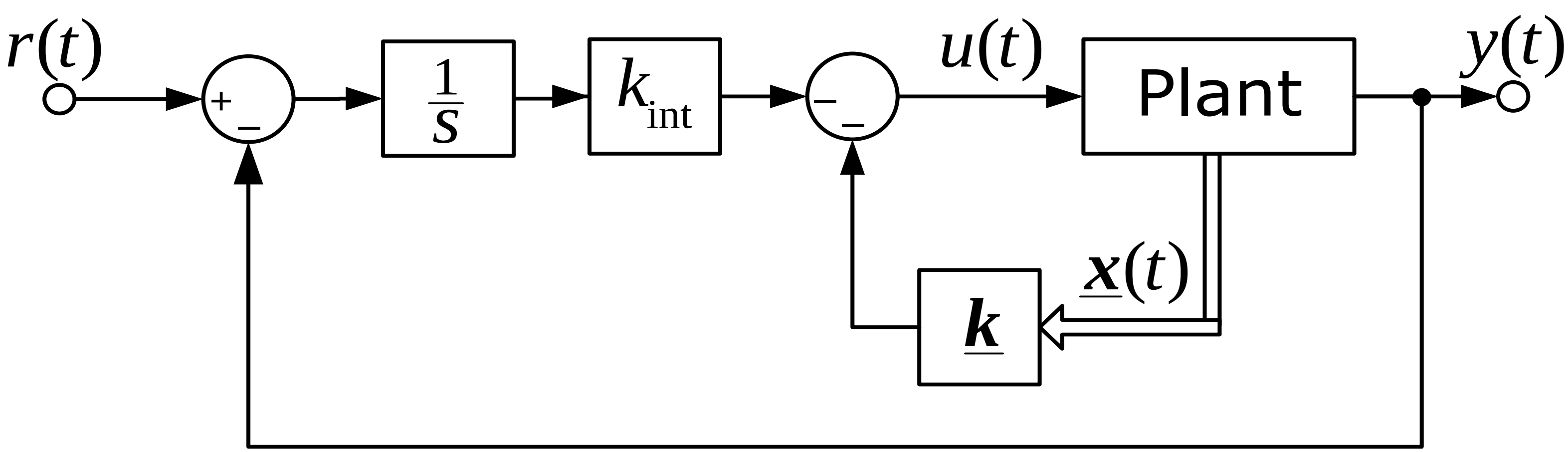
- The augmented state-space equations become

$$\begin{bmatrix} \dot{x}_{\text{int}} \\ \dot{\underline{\boldsymbol{x}}} \end{bmatrix} = \begin{bmatrix} 0 & -\underline{\boldsymbol{c}} \\ \boldsymbol{0} & \underline{\boldsymbol{A}} \end{bmatrix} \begin{bmatrix} x_{\text{int}} \\ \underline{\boldsymbol{x}} \end{bmatrix} + \begin{bmatrix} 0 \\ \underline{\boldsymbol{b}} \end{bmatrix} u + \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix} r$$

$$y = \begin{bmatrix} 0 & \underline{\boldsymbol{c}} \end{bmatrix} \begin{bmatrix} x_{\text{int}} \\ \underline{\boldsymbol{x}} \end{bmatrix}$$

with the feedback law –to incorporate the feedback $\underline{\boldsymbol{k}}$ gain and integrator gain k_{int}– is chosen as:

$$u(t) = -\underline{\boldsymbol{k}}\,\underline{\boldsymbol{x}}(t) - k_{\text{int}} x_{\text{int}}(t) = -\begin{bmatrix} k_{\text{int}} & \underline{\boldsymbol{k}} \end{bmatrix} \begin{bmatrix} x_{\text{int}} \\ \underline{\boldsymbol{x}} \end{bmatrix}$$



Tracing of Reference Input: Integral Control

- Substituting $u(t)$ to the augmented state-space equations,

$$\begin{bmatrix} \dot{x}_{\text{int}} \\ \dot{\underline{\boldsymbol{x}}} \end{bmatrix} = \left(\begin{bmatrix} 0 & -\underline{\boldsymbol{c}} \\ \boldsymbol{0} & \underline{\boldsymbol{A}} \end{bmatrix} - \begin{bmatrix} 0 \\ \underline{\boldsymbol{b}} \end{bmatrix} \begin{bmatrix} k_{\text{int}} & \underline{\boldsymbol{k}} \end{bmatrix} \right) \begin{bmatrix} x_{\text{int}} \\ \underline{\boldsymbol{x}} \end{bmatrix} + \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix} r$$

$$\begin{bmatrix} \dot{x}_{\text{int}} \\ \dot{\underline{\boldsymbol{x}}} \end{bmatrix} = \begin{bmatrix} 0 & -\underline{\boldsymbol{c}} \\ -k_{\text{int}}\underline{\boldsymbol{b}} & \underline{\boldsymbol{A}} - \underline{\boldsymbol{b}}\underline{\boldsymbol{k}} \end{bmatrix} \begin{bmatrix} x_{\text{int}} \\ \underline{\boldsymbol{x}} \end{bmatrix} + \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix} r$$

$$y = \begin{bmatrix} 0 & \underline{\boldsymbol{c}} \end{bmatrix} \begin{bmatrix} x_{\text{int}} \\ \underline{\boldsymbol{x}} \end{bmatrix}$$

- The characteristic equation of the augmented system is now given as

$$a(s) = \det \left(s\underline{\boldsymbol{I}} - \begin{bmatrix} 0 & -\underline{\boldsymbol{c}} \\ -k_{\text{int}}\underline{\boldsymbol{b}} & \underline{\boldsymbol{A}} - \underline{\boldsymbol{b}}\underline{\boldsymbol{k}} \end{bmatrix} \right)$$

with the possibility to place the poles by means of $\underline{\boldsymbol{k}}$ and k_{int}.

Example 2: Integral Control

The scheme should now be implemented on the state-space equations that has been used before,

$$\dot{\underline{x}}(t) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x}(t)$$

with the desired eigenvalues of −1 and −2, and $r(t)$ = 1.5(t).

The integrator increases the order of the system **by one** to become a third-order system. The third eigenvalues is assumed to be–3.

The augmented state-space equations is given by:

$$\begin{bmatrix} \dot{x}_{\text{int}} \\ \dot{\underline{x}} \end{bmatrix} = \begin{bmatrix} 0 & -\underline{c} \\ -k_{\text{int}}\underline{b} & \underline{A} - \underline{b}\underline{k} \end{bmatrix} \begin{bmatrix} x_{\text{int}} \\ \underline{x} \end{bmatrix} + \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} r$$

$$\begin{bmatrix} \dot{x}_{\text{int}} \\ \dot{\underline{x}} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ -k_{\text{int}} & 2-k_1 & 1-k_2 \\ -2k_{\text{int}} & -1-2k_1 & 1-2k_2 \end{bmatrix} \begin{bmatrix} x_{\text{int}} \\ \underline{x} \end{bmatrix} + \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} r$$

Example 2: Integral Control

$$a(s)=\det\left(s\underline{\boldsymbol{I}}-\begin{bmatrix}0 & -1 & -1\\ -k_{\text{int}} & 2-k_1 & 1-k_2\\ -2k_{\text{int}} & -1-2k_1 & 1-2k_2\end{bmatrix}\right)$$

$$a(s)=\det\left(\begin{bmatrix}s & 1 & 1\\ k_{\text{int}} & s-(2-k_1) & -(1-k_2)\\ 2k_{\text{int}} & 1+2k_1 & s-(1-2k_2)\end{bmatrix}\right)$$

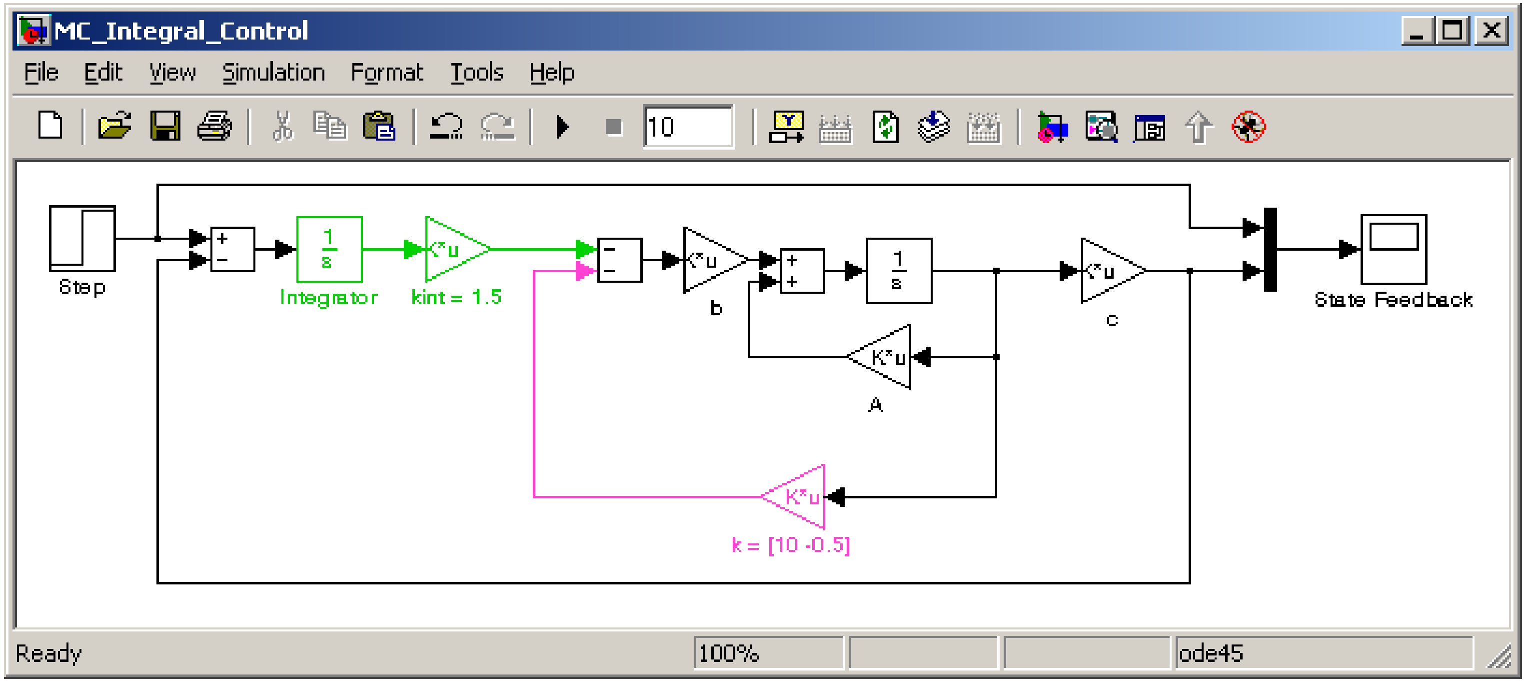
$$=s^3+(k_1+2k_2-3)s^2+(k_1-5k_2-3k_{\text{int}}+3)s+4k_{\text{int}}$$

$$\equiv s^3+6s^2+11s+6$$

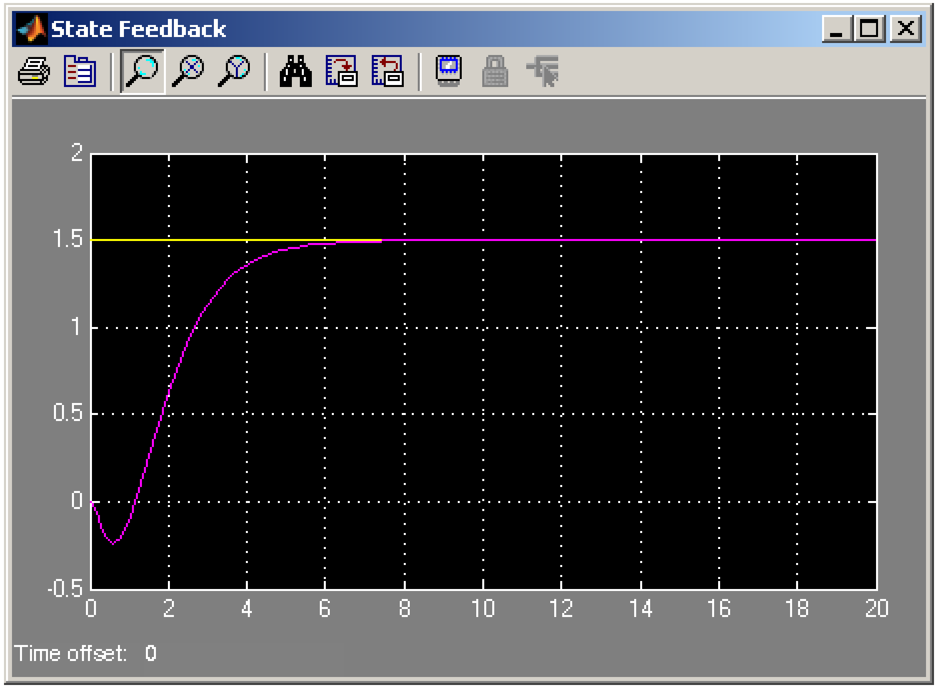
$$\equiv (s+1)(s+2)(s+3)$$

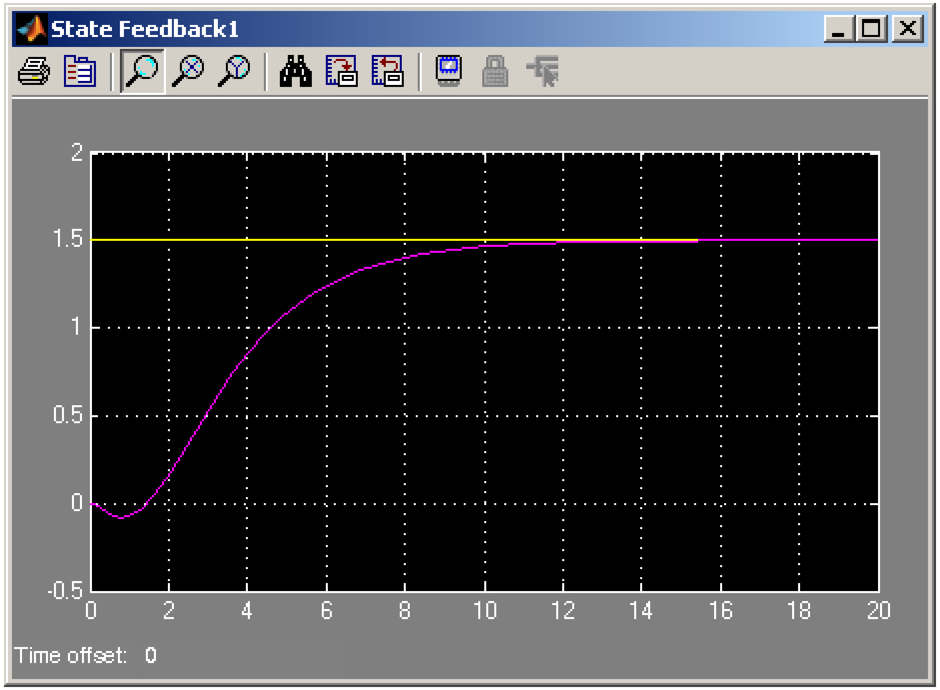
$$\therefore k_1=10, k_2=-0.5, k_{\text{int}}=1.5$$

Example 2: Integral Control



Example 2: Integral Control





Third pole at *s* = –3

$$k_1 = 10, k_2 = -0.5, k_{\text{int}} = 1.5$$

Third pole at *s* = –0.5

$$k_1 = 5, k_2 = 0.75, k_{\text{int}} = 0.25$$

- **What conclusion can be taken?**

Homework 9

Refer to the last example.

(a) Calculate the transfer function $G(s)$ of the system.

(b) Calculate the steady-state value of the system to a unit step input, using the Final Value Theorem of Laplace Transform.

(c) Determine the gain K so that the steady-state response of $K \cdot G(s)$ has zero error to a unit step input.

(d) Find out the relation between the transfer function gain K and the reference gain E.

Homework 9A

It is desired that the following linear system has zero steady state error to a unit step input. Find the solution by using:

$$\dot{\underline{x}}(t) = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -3 & 4 \\ -1 & 1 & -9 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \underline{x}(t)$$

(a) **Pre-scaling method**, by calculating the gain E.

(b) **Integral control method**, by calculating the gain $[k_{\text{int}}\ \boldsymbol{k}]$.
Hint: Assume the additional pole to be –1 and do not move the original poles of the system.

(c) Implement the original system, the system at (a) and the system at (b) in one Matlab Simulink file and compare the outputs.
Hint: For the matrix calculations, you may use Matlab. Write down or print the result on your homework papers.