State Feedback and State Estimators

"Linear System Theory and Design", Chapter 8

http://zitompul.wordpress.com

2 0 1 4

State Estimator

- In previous section, we have discussed the state feedback, based on the assumption that all state variables are available for feedback.
- Also, we have discussed the output feedback, provided that the output is available for feedback.
- On the purpose of state feedback, practically, the state variables might be not accessible for direct connection.
- The sensing devices or transducers might be not available or very expensive.

- In this case, we need a "**state estimator**" or a "**state observer**". Their output will be the "estimate of the state", provided that the system under consideration is ***observable***.

State Estimator

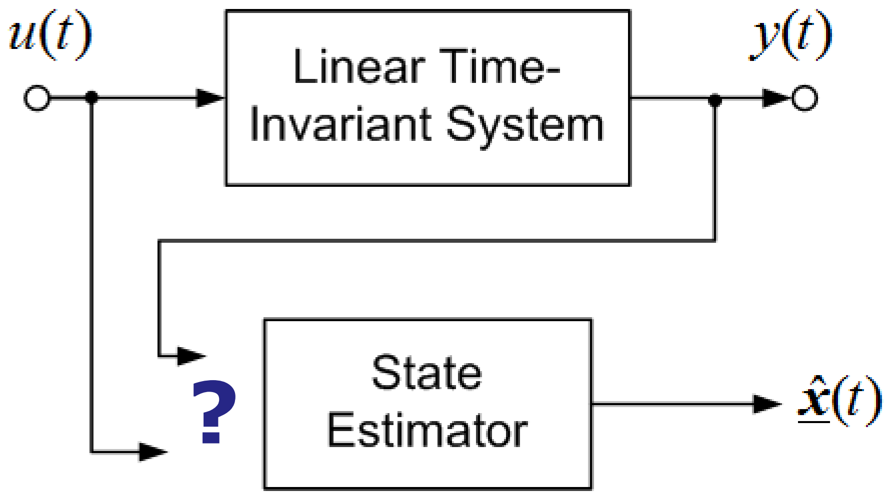
- Consider the *n*-dimensional single-variable state space equations:

$$\dot{\underline{\mathbf{x}}}(t) = \underline{\mathbf{A}}\underline{\mathbf{x}}(t) + \underline{\mathbf{b}}u(t)$$

$$y(t) = \underline{\mathbf{c}}\underline{\mathbf{x}}(t)$$

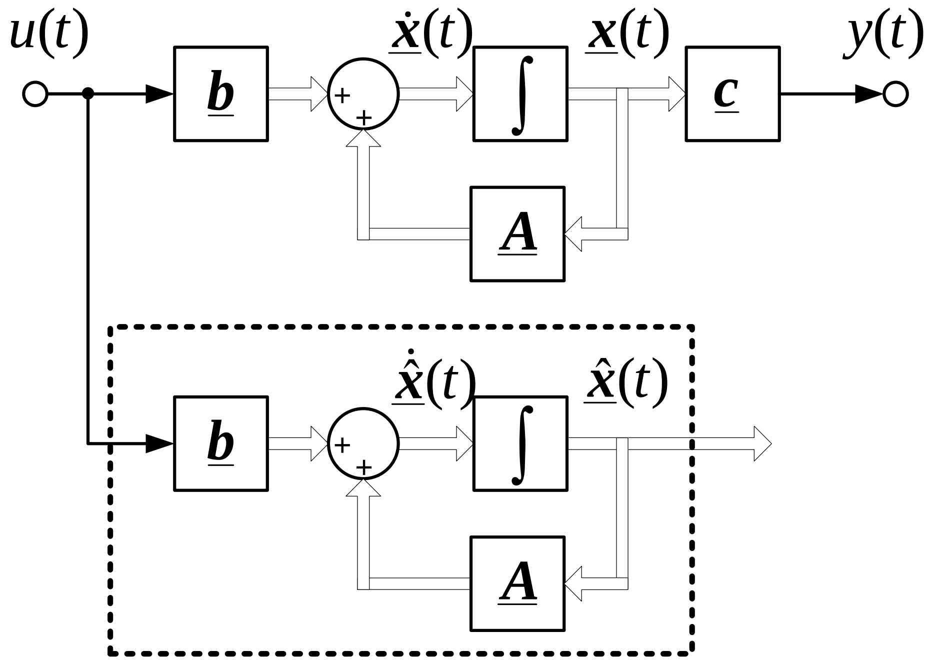
where $\underline{\mathbf{A}}$, $\underline{\mathbf{b}}$, $\underline{\mathbf{c}}$ are given, $u(t)$ and $y(t)$ are available, and the states $\underline{\mathbf{x}}(t)$ are not available.

- **Problem**: How to estimate $\underline{\mathbf{x}}(t)$?



Open-Loop State Estimator

- The block diagram of an open-loop state estimator can be seen below:



- The open-loop state estimator duplicates the original system and deliver:

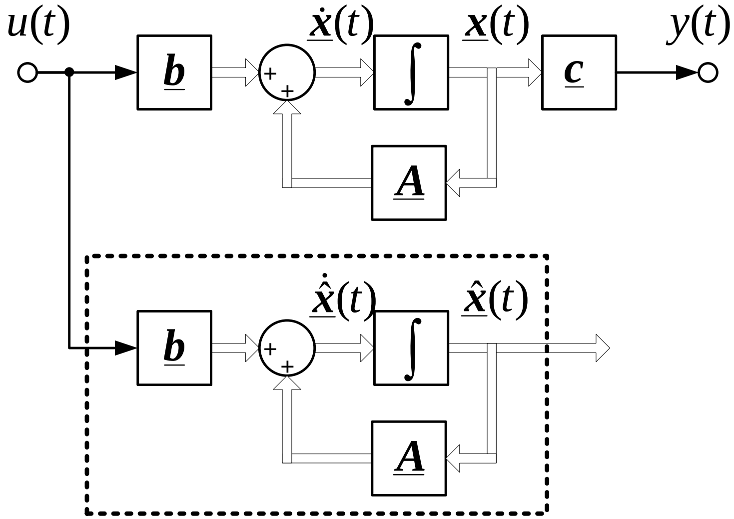
$$\dot{\hat{\underline{x}}}(t) = \underline{A}\hat{\underline{x}}(t) + \underline{b}u(t)$$

Open-Loop State Estimator

- Several conclusions can be drawn by comparing both state equations

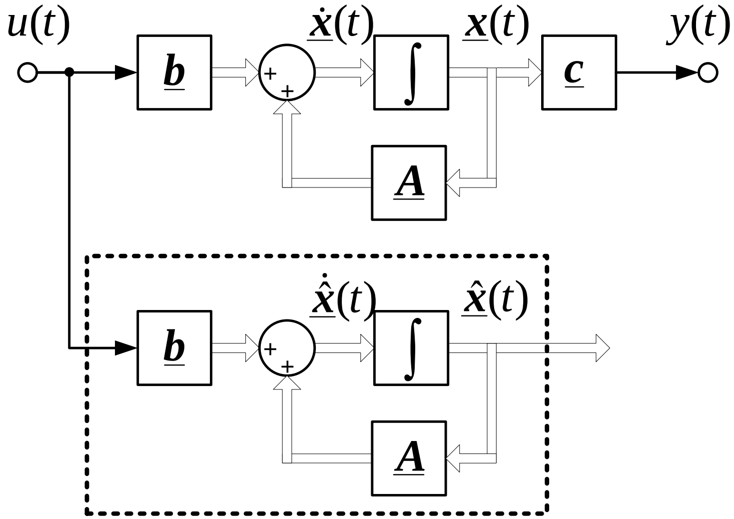
$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{b}u(t)$$
$$\dot{\hat{\underline{x}}}(t) = \underline{A}\hat{\underline{x}}(t) + \underline{b}u(t)$$

- If the initial states of both equations are the same, $\hat{\underline{x}}_0(t) = \underline{x}_0(t)$, then for any $t \geq 0$, $\hat{\underline{x}}(t) = \underline{x}(t)$.
- If the pair $(\underline{A},\underline{c})$ is observable, the initial state can be computed over any time interval $[0,t_0]$, and after setting $\hat{\underline{x}}(t_0)=\underline{x}(t_0)$, then $\hat{\underline{x}}(t)=\underline{x}(t)$ for $t \geq t_0$.



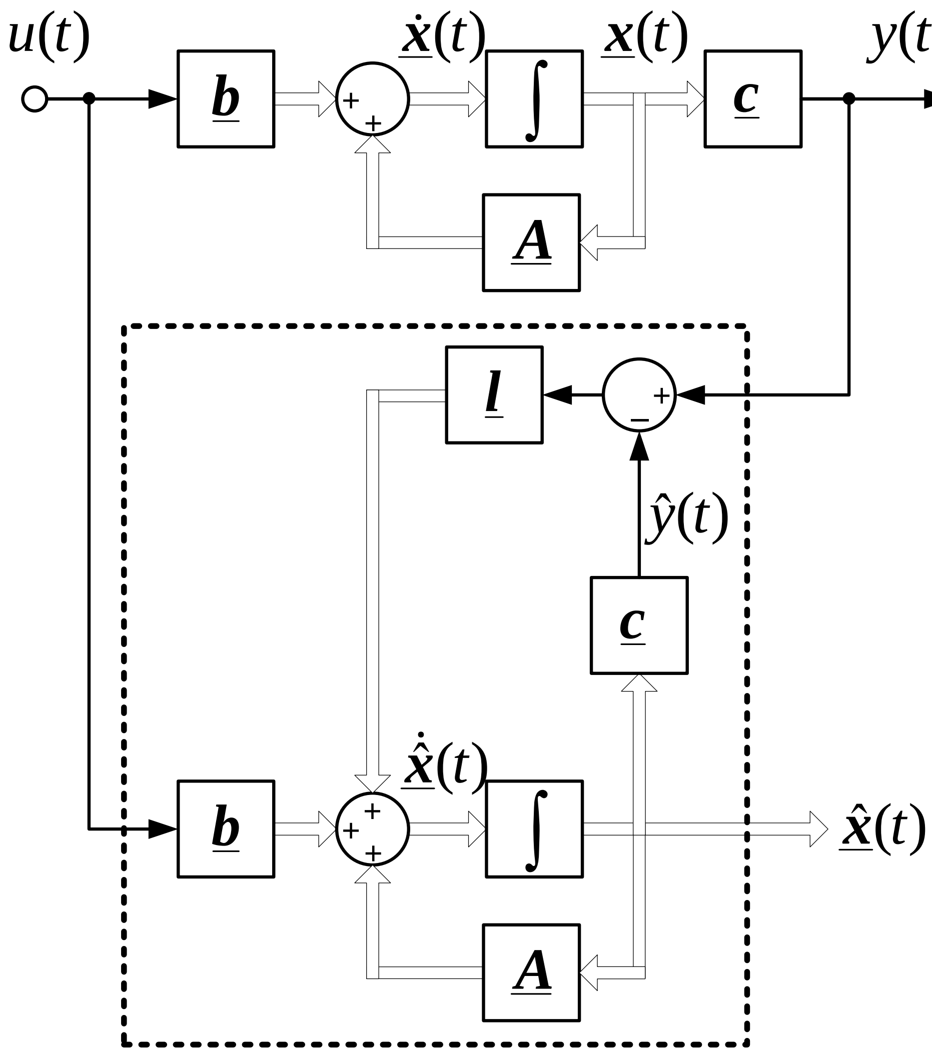
Open-Loop State Estimator

- The disadvantages of open-loop estimator are:
  - Initial state must be computed and appointed each time the estimator is used.
  - If the system is unstable, any small difference between $\underline{\mathbf{x}}(t_0)$ and $\hat{\underline{\mathbf{x}}}(t_0)$ will lead to even bigger difference between $\underline{\mathbf{x}}(t)$ and $\hat{\underline{\mathbf{x}}}(t)$, making $\hat{\underline{\mathbf{x}}}(t)$ unusable.



Closed-Loop State Estimator

- The block diagram of a closed-loop state estimator can be seen below:



- In closed-loop estimator, $y(t) = \mathbf{c}\underline{\mathbf{x}}(t)$ is compared with $\hat{y}(t) = \mathbf{c}\underline{\hat{\mathbf{x}}}(t)$.
- Their difference, after multiplied by the matrix $\underline{\mathbf{l}}$, is used as a correcting term in the calculation of $\underline{\hat{\mathbf{x}}}(t)$.
- If $\underline{\mathbf{l}}$ is properly assigned, the difference will drive $\underline{\hat{\mathbf{x}}}(t)$ to $\underline{\mathbf{x}}(t)$.

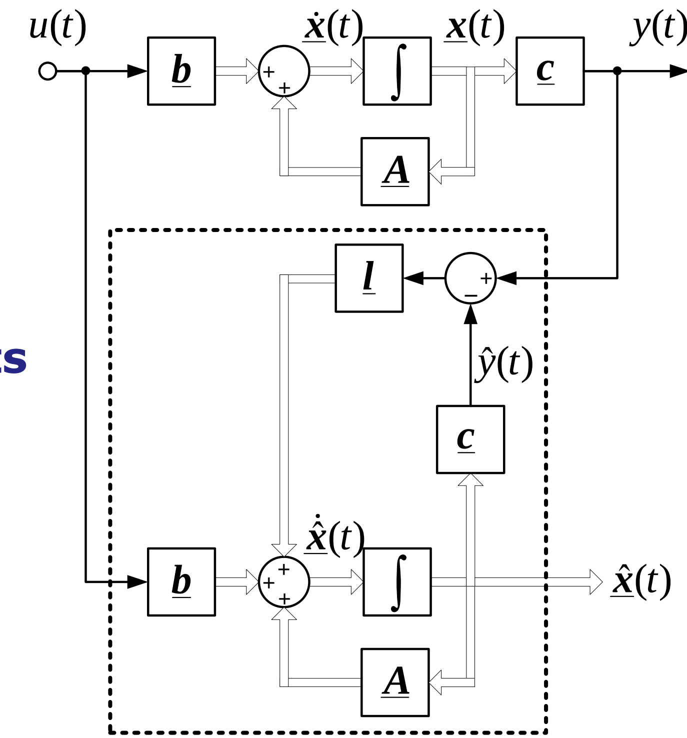
Closed-Loop State Estimator

- Following the previous figure,

$$\dot{\hat{\underline{x}}}(t)=\underline{A}\hat{\underline{x}}(t)+\underline{b}u(t)+\underline{l}\left(y(t)-\underline{c}\,\hat{\underline{x}}(t)\right)$$

$$\dot{\hat{\underline{x}}}(t)=\left(\underline{A}-\underline{l}\underline{c}\right)\hat{\underline{x}}(t)+\underline{b}u(t)+\underline{l}y(t)$$

• "New" inputs



- We now define

$$\underline{e}(t)=\underline{x}(t)-\hat{\underline{x}}(t)$$

$$\dot{\underline{e}}(t)=\dot{\underline{x}}(t)-\dot{\hat{\underline{x}}}(t)$$

$$=\left\{\underline{A}\underline{x}(t)+\underline{b}u(t)\right\}-\left\{\left(\underline{A}-\underline{l}\underline{c}\right)\hat{\underline{x}}(t)+\underline{b}u(t)+\underline{l}y(t)\right\}$$

$$=\left(\underline{A}-\underline{l}\underline{c}\right)\underline{x}(t)-\left(\underline{A}-\underline{l}\underline{c}\right)\hat{\underline{x}}(t)$$

$$=\left(\underline{A}-\underline{l}\underline{c}\right)\left(\underline{x}(t)-\hat{\underline{x}}(t)\right)$$

$$\boxed{\dot{\underline{e}}(t)=\left(\underline{A}-\underline{l}\underline{c}\right)\underline{e}(t)}$$

Closed-Loop State Estimator

$$\underline{\dot{e}}(t) = \left(\underline{A} - \underline{l}\,\underline{c}\right)\underline{e}(t) \quad \Longrightarrow \quad \underline{e}(t) = e^{(\underline{A}-\underline{l}\,\underline{c})t}\,\underline{e}(0)$$

- **The time domain solution of estimation error $\underline{e}(t)$ depends on $\underline{e}(0)$, but not $u(t)$**

Canonical Transformation $\Downarrow$

$$\underline{\dot{\mathcal{E}}}(t) = \underline{\Lambda}\,\underline{\mathcal{E}}(t) \quad \Longrightarrow \quad \underline{\mathcal{E}}(t) = e^{\underline{\Lambda}t}\,\underline{\mathcal{E}}(0)$$

$$\underline{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \qquad e^{\underline{\Lambda}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix}$$

Closed-Loop State Estimator

- From the fact that $\underline{\mathcal{E}}(t) = e^{\underline{\mathbf{\Lambda}}t}\underline{\mathcal{E}}(0)$, we can conclude that:
  - If all eigenvalues of $\underline{\mathbf{\Lambda}} = (\underline{\mathbf{A}}-\underline{\mathbf{lc}})$ are negative, then the estimation error $\underline{\mathbf{e}}(t)$ will approach zero as t increases.
  - There is no need to calculated the initial states each time the closed-loop estimator will be used.
  - After a certain time, the estimation error $\underline{\mathbf{e}}(t)$ will approach zero and the state estimates $\hat{\underline{\mathbf{x}}}(t)$ will be equal to the system's state $\underline{\mathbf{x}}(t)$.

Example 1: State Estimators

Consider a linear time-invariant system with the following state equations:

$$\dot{\underline{\boldsymbol{x}}}(t) = \begin{bmatrix} 0 & 4 \\ -1 & -2 \end{bmatrix} \underline{\boldsymbol{x}}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{\boldsymbol{x}}(t)$$

Design a state estimator with eigenvalues of –10 and –10.

Example 1: State Estimators

$$\underline{\mathcal{O}} = \begin{bmatrix} \underline{c} \\ \cdots \\ \underline{c}\,\underline{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

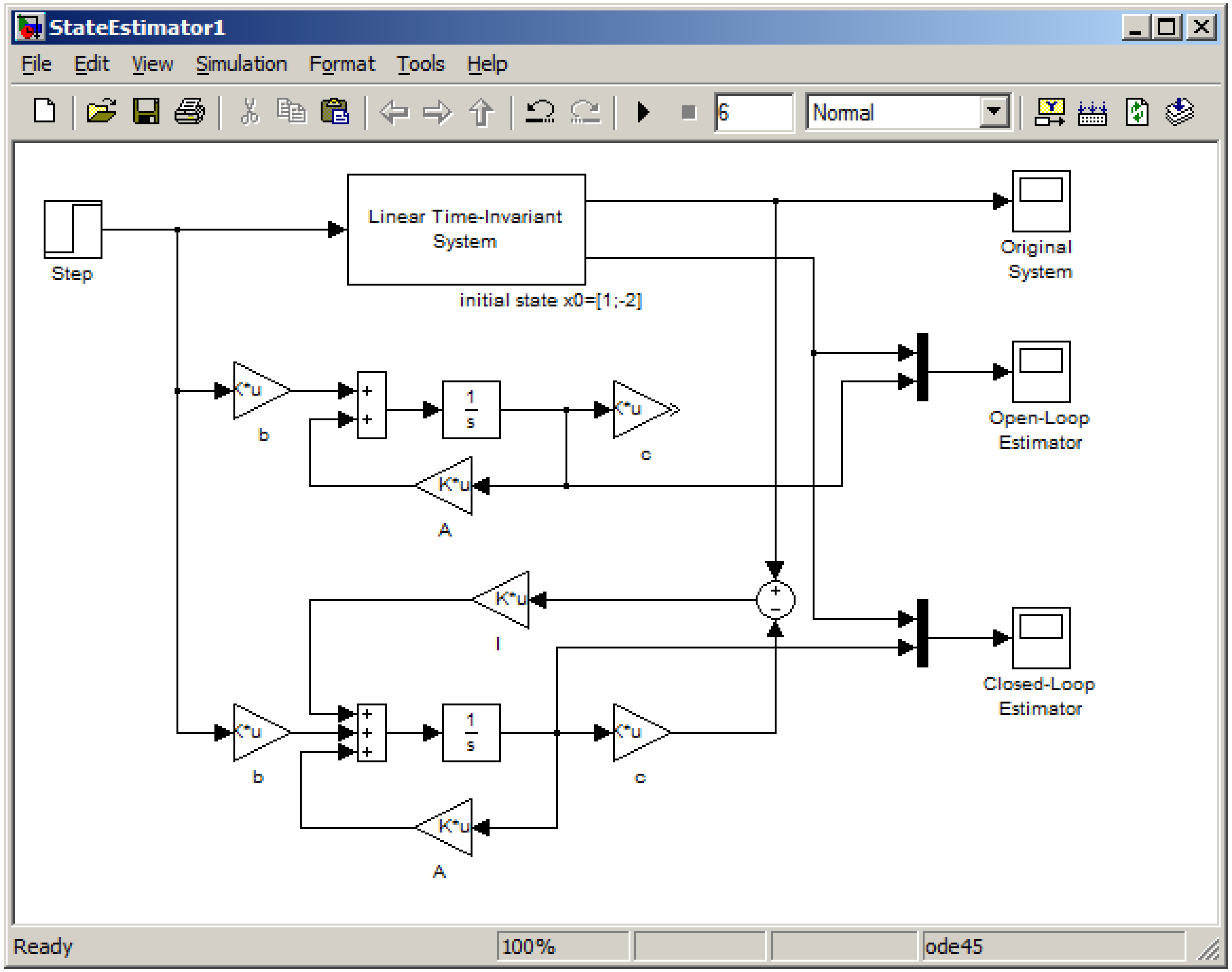
- **Observable → a state estimator can be designed**

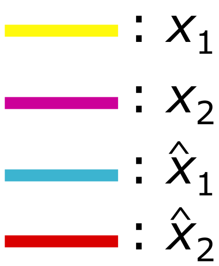
$$\begin{aligned} \alpha(s) &= \det(s\underline{I} - \underline{A} + \underline{l}\,\underline{c}) \\ &= \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \\ &= \det\left(\begin{bmatrix} s + l_1 & -4 \\ l_2 + 1 & s + 2 \end{bmatrix} \right) \\ &= s^2 + (l_1 + 2)s + (2l_1 + 4l_2 + 4) \\ &\equiv (s+10)(s+10) \end{aligned}$$

$$l_1 = 18, \quad l_2 = 15$$

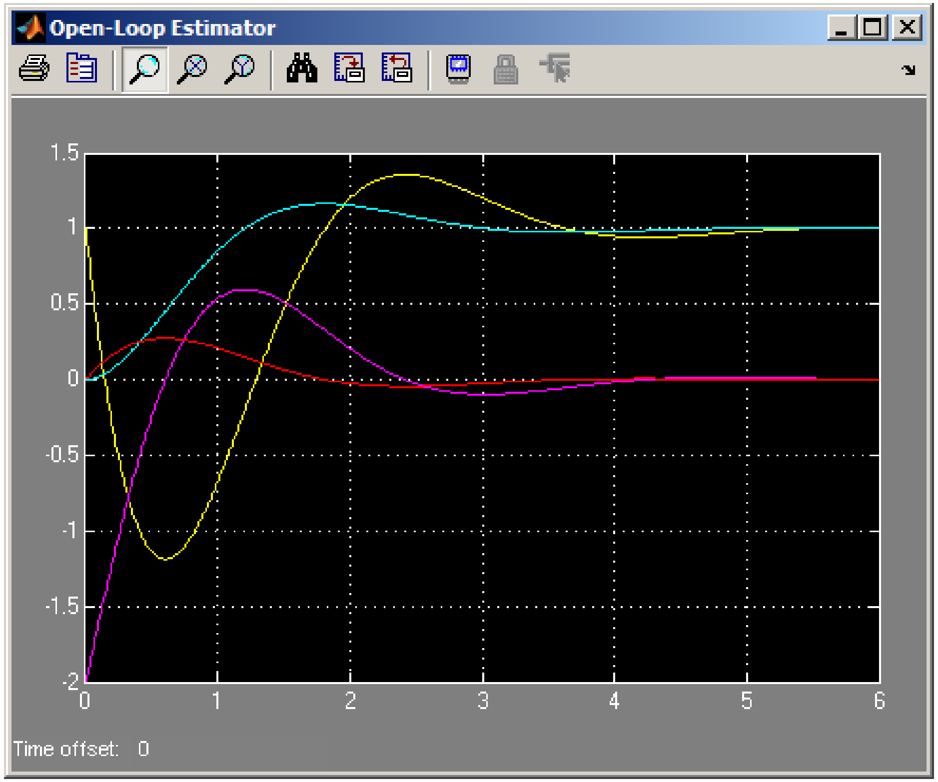
$$\underline{l} = \begin{bmatrix} 18 \\ 15 \end{bmatrix}$$

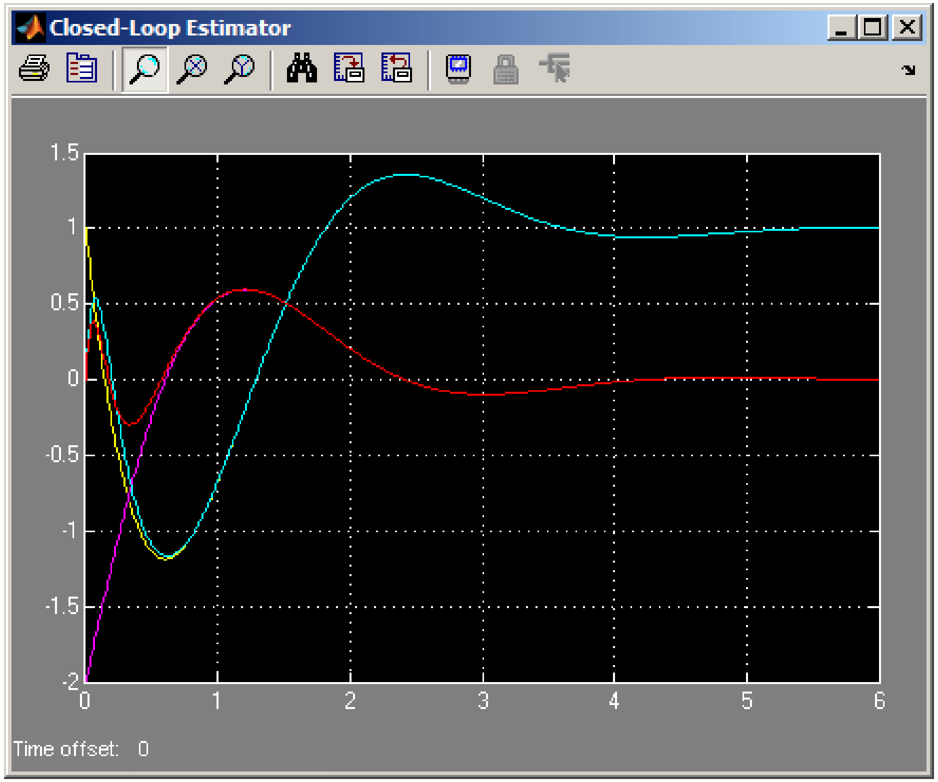
Example 1: State Estimators





Example 1: State Estimators





(yellow) : x_1

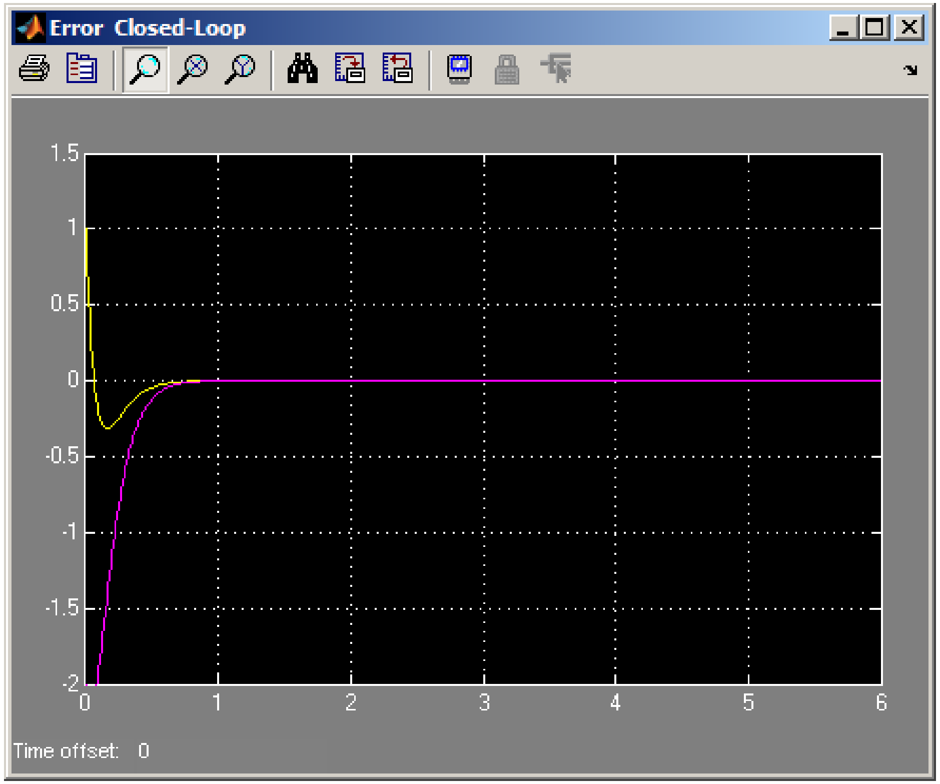
(magenta) : x_2

(cyan) : $\hat{x}_1$

(red) : $\hat{x}_2$

- **Conclusion: ...**

Example 1: State Estimators



: $x_1 - \hat{x}_1$

: $x_2 - \hat{x}_2$

- **Conclusion: ...**

Example 2: State Estimators

A system is given in state space form as below:

$$\dot{\underline{\mathbf{x}}}(t) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \underline{\mathbf{x}}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{\mathbf{x}}(t)$$

(a) Find a state feedback gain $\underline{\boldsymbol{k}}$, so that the closed-loop system has –1 and –2 as its eigenvalues.

(b) Design a closed-loop state estimator for the system, with eigenvalues $-2 \pm j2$.

Example 2: State Estimators

(a) Find a state feedback gain $\underline{k}$, so that the closed-loop system has –1 and –2 as its eigenvalues.

$$a(s) = \det(s\underline{I} - \underline{A} + \underline{b}\underline{k})$$
$$= \det\left(\begin{bmatrix} s-2+k_1 & -1+k_2 \\ 1+2k_1 & s-1+2k_2 \end{bmatrix}\right)$$
$$= s^2 + (k_1 + 2k_2 - 3)s + (k_1 - 5k_2 + 3)$$
$$\equiv (s+1)(s+2)$$

$$k_1 = 4 \qquad k_2 = 1$$

$$\underline{k} = \begin{bmatrix} 4 & 1 \end{bmatrix}$$

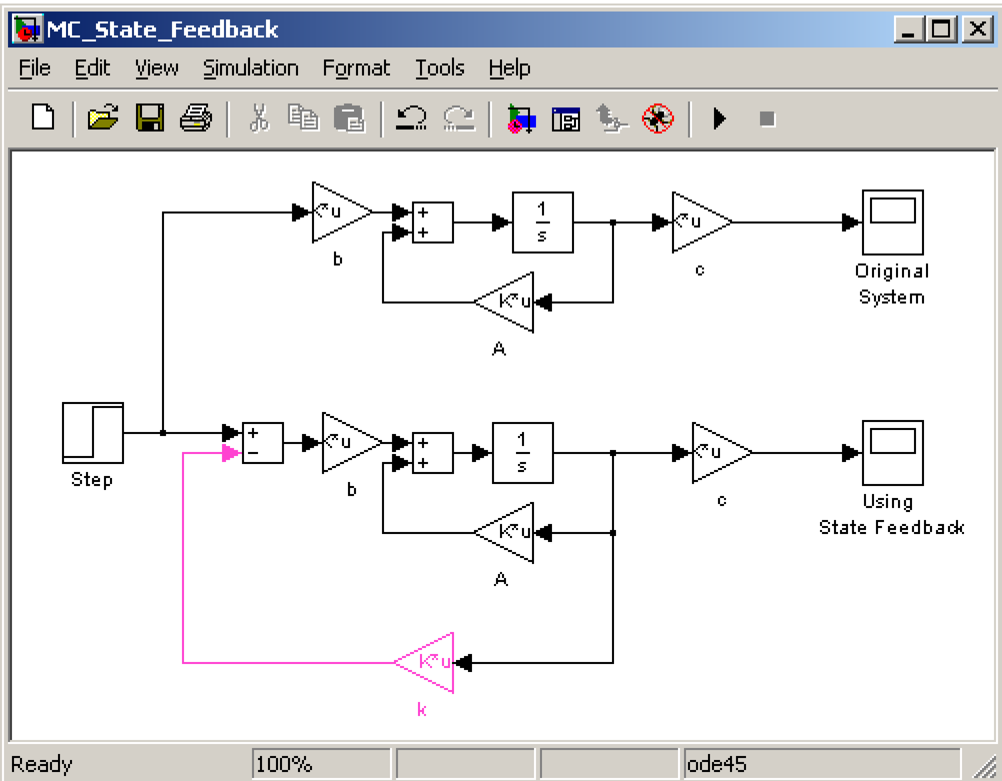
(b) Design a closed-loop state estimator for the system, with eigenvalues –2 ± j2.

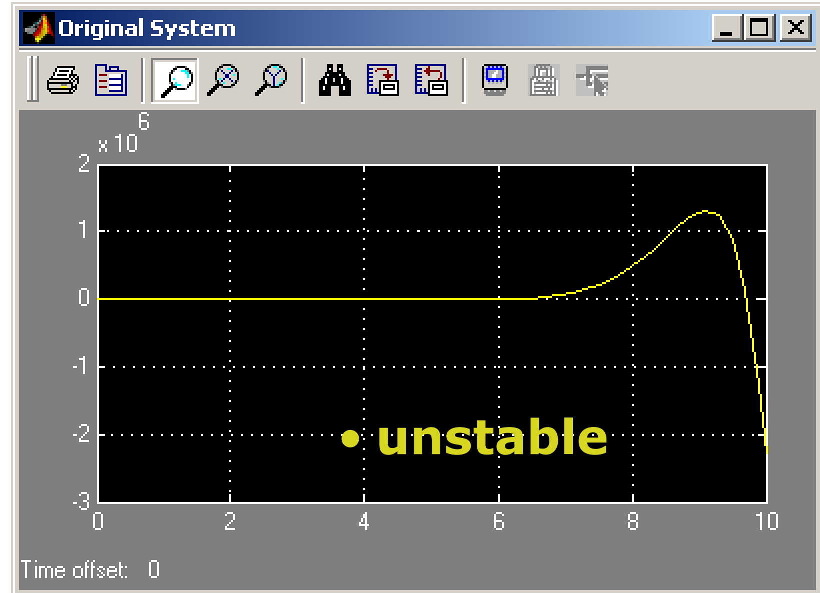
$$\alpha(s) = \det(s\underline{I} - \underline{A} + \underline{l}\underline{c})$$
$$= \det\left(\begin{bmatrix} s-2+l_1 & -1+l_1 \\ 1+l_2 & s-1+l_2 \end{bmatrix}\right)$$
$$= s^2 + (l_1 + l_2 - 3)s + (-2l_1 - l_2 + 3)$$
$$\equiv (s+2+j2)(s+2-j2)$$

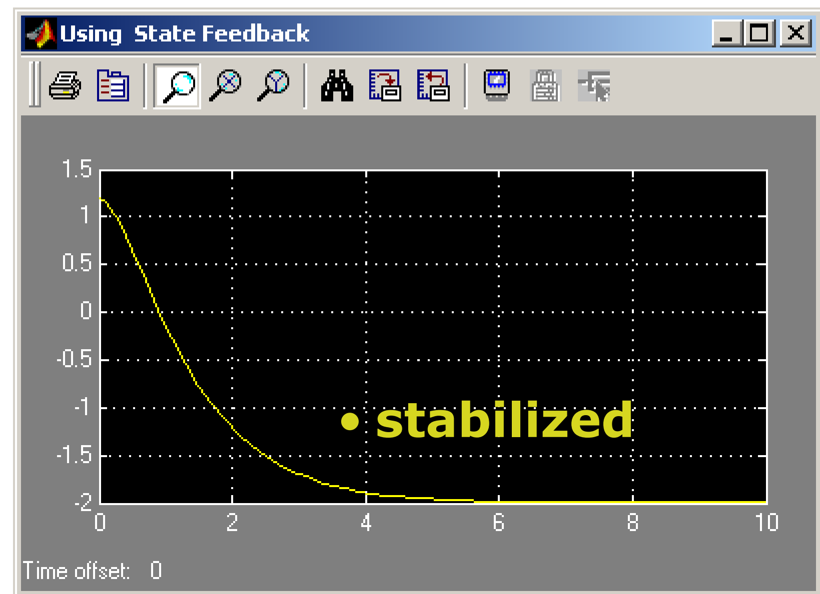
$$l_1 = -12 \qquad l_2 = 19$$

$$\underline{l} = \begin{bmatrix} -12 \\ 19 \end{bmatrix}$$

Example 2: State Estimators

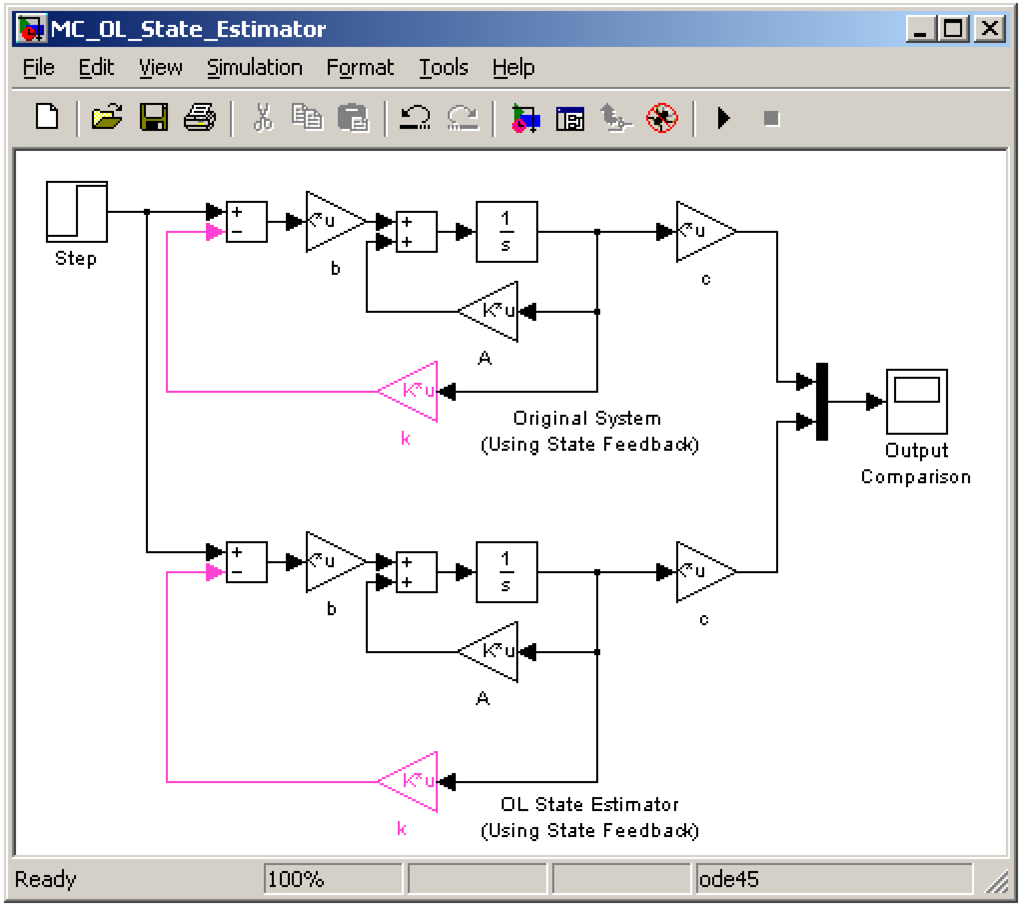


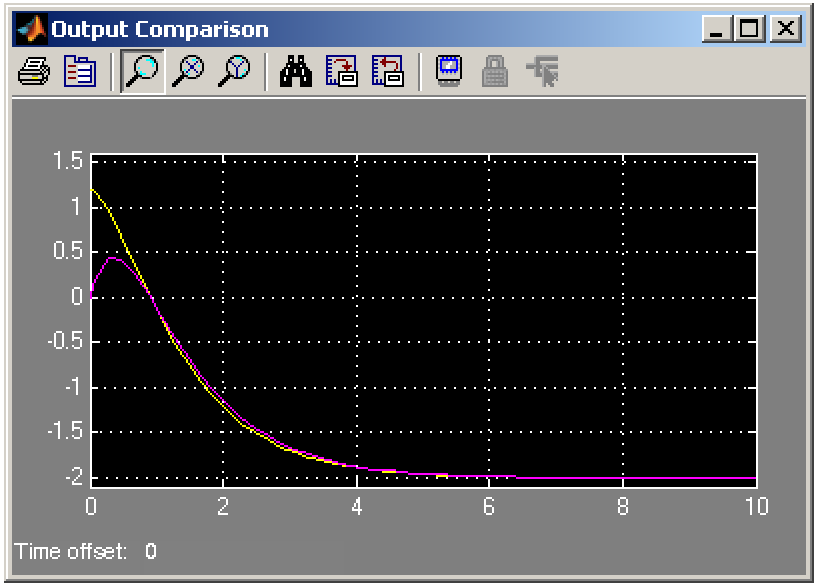




$$\underline{x}_0 = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$$

Example 2: State Estimators

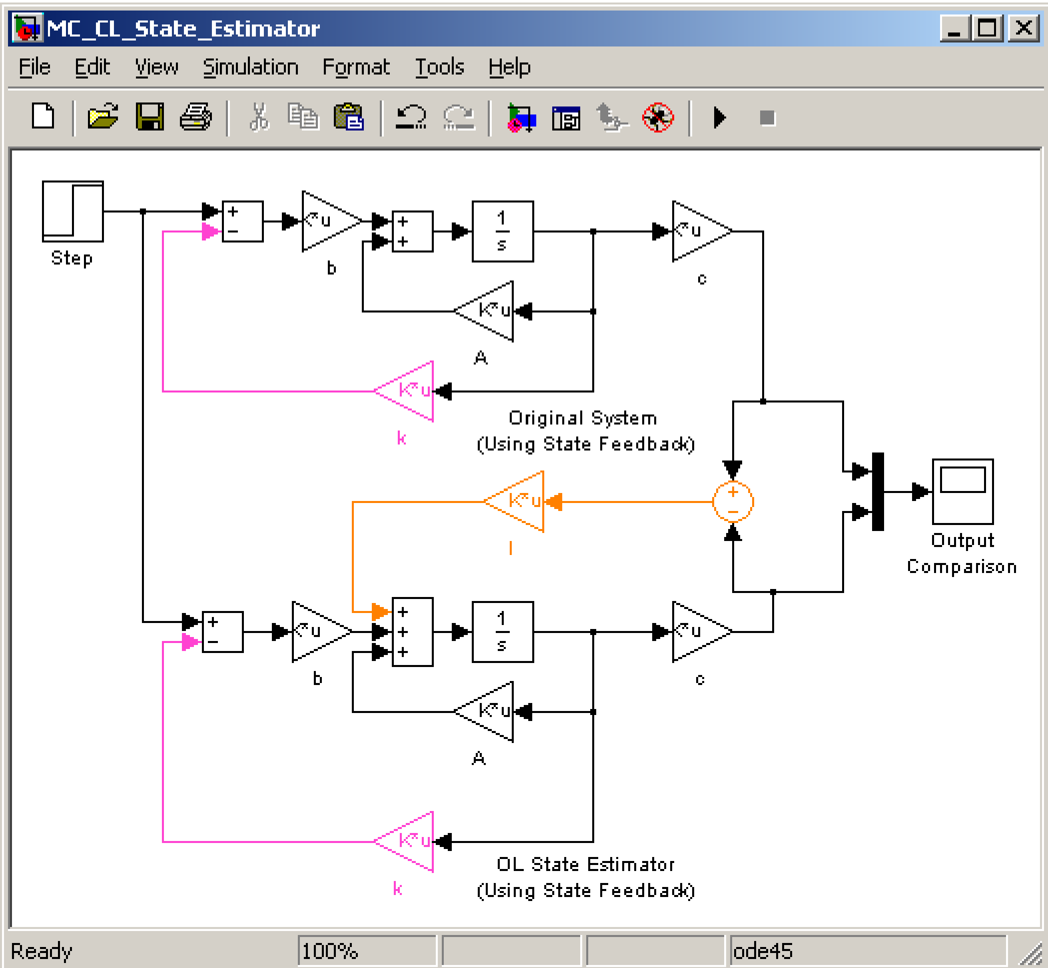


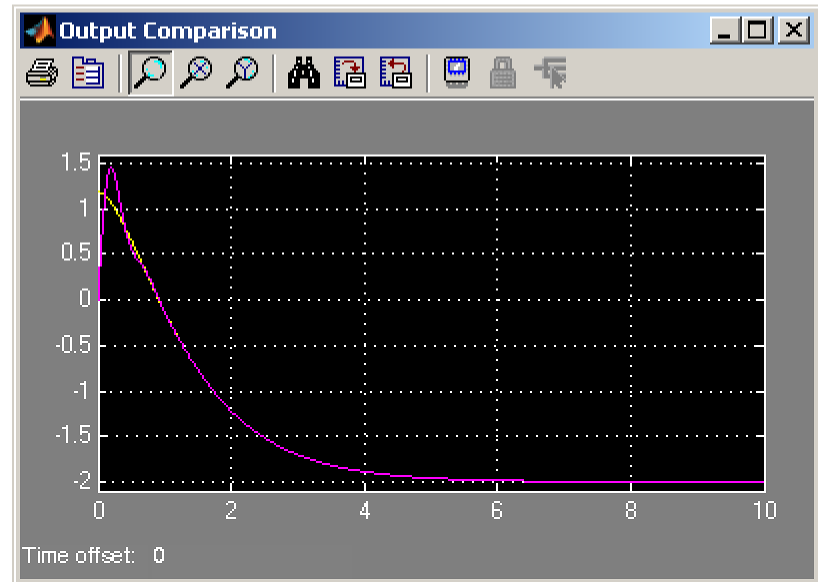


$$\underline{x}_0 = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}, \quad \hat{\underline{x}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- **If $\hat{\underline{x}}_0 \neq \underline{x}_0$, then for a certain amount of time there will be estimation error $e = \underline{x} - \hat{\underline{x}}$, which in the end will decay to zero**
- **For unstable system, the estimation error will increase, $e \rightarrow \infty$, no decay**

Example 2: State Estimators





$$\underline{\boldsymbol{x}}_0 = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}, \quad \underline{\hat{\boldsymbol{x}}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- **Although $\underline{\hat{x}}_0 \neq \underline{x}_0$, in <u>shorter time</u> the estimation error decays to zero**
- **Furthermore, in case a system is unstable, the estimation error *e* will also decay**

Homework 7: State Estimators

(a) For the same system as discussed in Example 2, design another closed-loop state estimator, with eigenvalues at –3 and –4.

(b) Compare the performance of the estimator in the previous slides and the one you have designed. Do simulation using Matlab Simulink.

(c) Give some explanations of the comparison results.

Homework 7A: State Estimators

(a) For the same system as discussed in Example 2, design another closed-loop state estimator, with eigenvalues at $-0.5 \pm j1$. This means, the eigenvalues of the estimator is to the right of those of the system, which is -1 and -2.

(b) Compare the performance of the estimator in the previous slides and the one you have designed. Do simulation using Matlab Simulink.

(c) Give some explanations of the comparison results.

- Deadline: Wednesday, 5 November 2014.
- For Johnson, Rayhan, Kristiantho, and Anthony.

Homework 7B: State Estimators

Consider a second order system.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u$$

where $u = 3 + 0.5\ \sin(0.75t)$ is the input and x is the output, $\zeta = 1$, $\omega = 1$ rad/s. The initial states are $x(0) = 2$, $\dot{x}(0) = 1$.

(a) Design an observer with poles at –0.25 and –0.5.

(b) Perform simulation in Matlab Simulink and determine the time required by observer states to catch up with the actual system states.