State Feedback and State Estimators

“Linear System Theory and Design”, Chapter 8

http://zitompul.wordpress.com

2 0 1 4

Homework 5

A state-space equation of a third-order system is given as:

$$\dot{\underline{\tilde{\boldsymbol{x}}}}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \underline{\tilde{\boldsymbol{x}}}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 6 & -6 & 1 \end{bmatrix} \underline{\tilde{\boldsymbol{x}}}(t)$$

a. Perform a step-by-step transformation of the given model to Frobenius Form.

b. Calculate the required feedback gain $\underline{\boldsymbol{k}}$ so that the system may have two conjugate poles at $-2\pm j1$ and -4.

Solution of Homework 5

a. Transformation to Frobenius Form.

- Find the characteristic equation

$$a(s)=\det(s\underline{\boldsymbol{I}}-\underline{\boldsymbol{A}})=\det\left(\begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+2 & 0 \\ 0 & 0 & s+3 \end{bmatrix}\right)$$

$$=(s+1)(s+2)(s+3)$$

$$=s^3+6s^2+11s+6$$

- Calculate $\underline{\mathcal{C}}$, $\underline{\mathcal{T}}$

$$\underline{\mathcal{C}}=\left[\underline{\boldsymbol{b}} \;\vdots\; \underline{\boldsymbol{A}}\underline{\boldsymbol{b}} \;\vdots\; \underline{\boldsymbol{A}}^2\underline{\boldsymbol{b}}\right]=\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix}$$

$$\underline{\mathcal{T}}=\begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution of Homework 5

- Calculate $\underline{\boldsymbol{Q}}$

$$\underline{\boldsymbol{Q}} = \underline{\mathcal{C}}\,\underline{\mathcal{T}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix} \begin{bmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 5 & 1 \\ 3 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\underline{\boldsymbol{Q}}^{-1} = \begin{bmatrix} 0.5 & -1 & 0.5 \\ -0.5 & 2 & -1.5 \\ 0.5 & -4 & 4.5 \end{bmatrix}$$

- Perform transformation

$$\underline{\hat{\boldsymbol{A}}} = \underline{\boldsymbol{Q}}^{-1}\underline{\boldsymbol{A}}\,\underline{\boldsymbol{Q}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$\underline{\hat{\boldsymbol{b}}} = \underline{\boldsymbol{Q}}^{-1}\underline{\boldsymbol{b}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- **Transformation accomplished**

Solution of Homework 5

b. Finding $\underline{\boldsymbol{k}}$.

- Calculate $\hat{\underline{\boldsymbol{k}}}$

$$\breve{a}(s) = (s+2-j)\cdot(s+2+j)\cdot(s+4)$$
$$\breve{a}(s) = s^3 + 8s^2 + 21s + 20$$
$$a(s) = s^3 + 6s^2 + 11s + 6$$

$$\left.\begin{aligned} \hat{k}_1 &= \breve{a}_0 - a_0 = 14 \\ \hat{k}_2 &= \breve{a}_1 - a_1 = 10 \\ \hat{k}_3 &= \breve{a}_2 - a_2 = 2 \end{aligned}\right\} \hat{\underline{\boldsymbol{k}}} = \begin{bmatrix} 14 & 10 & 2 \end{bmatrix}$$

- Calculate $\underline{\boldsymbol{k}}$

$$\underline{\boldsymbol{k}} = \hat{\underline{\boldsymbol{k}}}\underline{\boldsymbol{Q}}^{-1}$$
$$= \begin{bmatrix} 14 & 10 & 2 \end{bmatrix} \begin{bmatrix} 0.5 & -1 & 0.5 \\ -0.5 & 2 & -1.5 \\ 0.5 & -4 & 4.5 \end{bmatrix}$$
$$= \underline{\underline{\begin{bmatrix} 3 & -2 & 1 \end{bmatrix}}}$$

Solution of Homework 5

c. Direct calculation of $\underline{\boldsymbol{k}}$ without transformation.

$$a(s) = \det(s\underline{\boldsymbol{I}} - (\underline{\boldsymbol{A}} - \underline{\boldsymbol{b}}\underline{\boldsymbol{k}}))$$

$$= \det\left(s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \right) \right)$$

$$= \det\left(s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -(1+k_1) & -k_2 & -k_3 \\ -k_1 & -(2+k_2) & -k_3 \\ -k_1 & -k_2 & -(3+k_3) \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} s+(1+k_1) & k_2 & k_3 \\ k_1 & s+(2+k_2) & k_3 \\ k_1 & k_2 & s+(3+k_3) \end{bmatrix} \right)$$

- **Complicated to be done**

Output Feedback

- Consider the *n*-dimensional ***controllable*** single-variable state space equations:

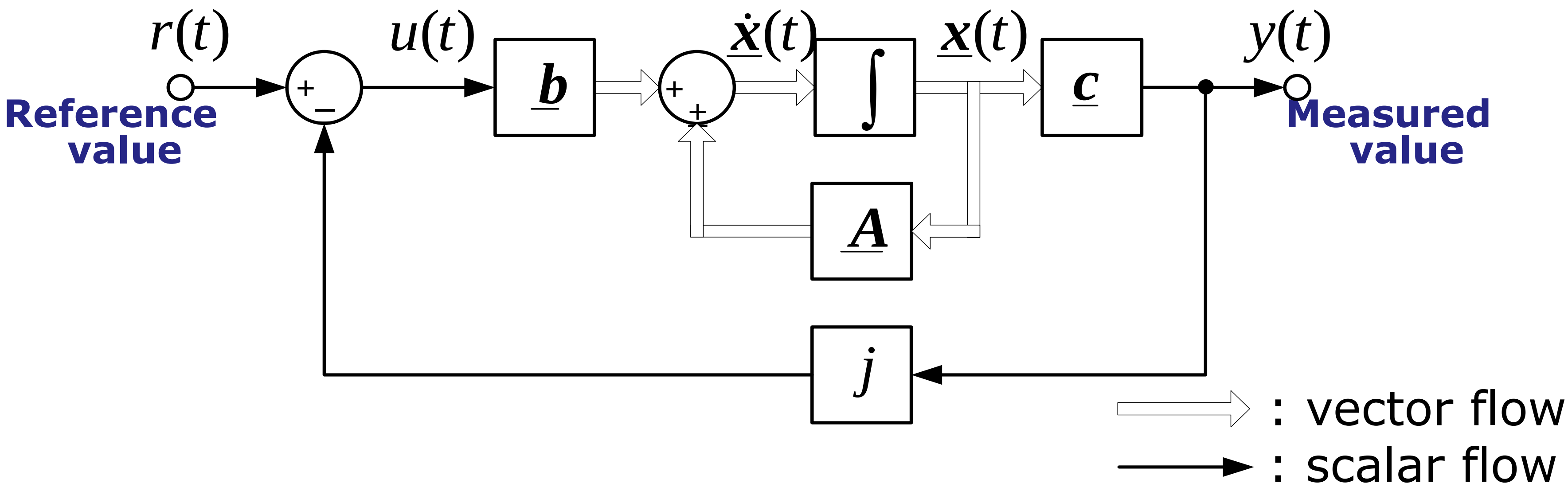
$$\dot{\underline{\mathbf{x}}}(t) = \underline{\mathbf{A}}\underline{\mathbf{x}}(t) + \underline{\mathbf{b}}u(t)$$

$$y(t) = \underline{\mathbf{c}}\underline{\mathbf{x}}(t)$$

- **Main idea**: Using measurement of output variable $y(t)$, determine an input $u(t)=f(r(t),y(t))$ such that the dynamic properties of the system can be changed to fulfill a certain criteria.

- In contrast to state feedback, output feedback has less degree of freedom in the controller parameter.
- However, the output feedback method is superior to the state feedback method from the practical point of view, because the output $y(t)$ is known and measurable.
- On the contrary, it is almost always difficult, if not impossible, to measure the entire state vector $x(t)$ due to practical limitations. (This can be encompassed by using state observer which will be discussed later.)

Output Feedback



- The output $y(t)$ is fed back through a feedback gain j.
- The input $u(t)$ is given by:

$$u(t) = r(t) - jy(t)$$

$$= r(t) - j\underline{\boldsymbol{c}}\,\underline{\boldsymbol{x}}(t)$$

- By inspection the state feedback gain $\underline{\boldsymbol{k}}$ and output feedback gain j are actually related via the following equation:

$$\underline{\boldsymbol{k}} \Leftrightarrow j\underline{\boldsymbol{c}}$$

Output Feedback

- Substituting $u(t)$ to the original state space equations,

$$\dot{\underline{\boldsymbol{x}}}(t) = (\underline{\boldsymbol{A}} - j\underline{\boldsymbol{b}}\underline{\boldsymbol{c}})\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{b}}r(t)$$

$$y(t) = \underline{\boldsymbol{c}}\underline{\boldsymbol{x}}(t)$$

- The roots of the characteristic equation can now be repositioned through

$$a(s) = \det(s\underline{\boldsymbol{I}} - (\underline{\boldsymbol{A}} - j\underline{\boldsymbol{b}}\underline{\boldsymbol{c}}))$$

- The output feedback has less degree of freedom in placing the poles, thus it may happen that the location to place the poles is limited.

Example: Output Feedback

Let us redo the previous example with the following state equations:

$$\underline{\dot{x}}(t) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x}(t)$$

The characteristic equation of the given system is:

$$a(s) = \det(s\underline{I} - \underline{A})$$

$$= (s-4)(s+2)$$

- **Unstable eigenvalues or unstable pole**

Introducing the output feedback,

$$u(t) = r(t) - jy(t) = r(t) - j\underline{b}\underline{c}\underline{x}(t)$$

The state space is now:

$$\underline{\dot{x}}(t) = \begin{bmatrix} 1 & 3 \\ 3-j & 1-j \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

Example: Output Feedback

The characteristic equation becomes:

$$a(s) = \det\left(\begin{bmatrix} s-1 & -3 \\ -(3-j) & s-(1-j) \end{bmatrix}\right)$$

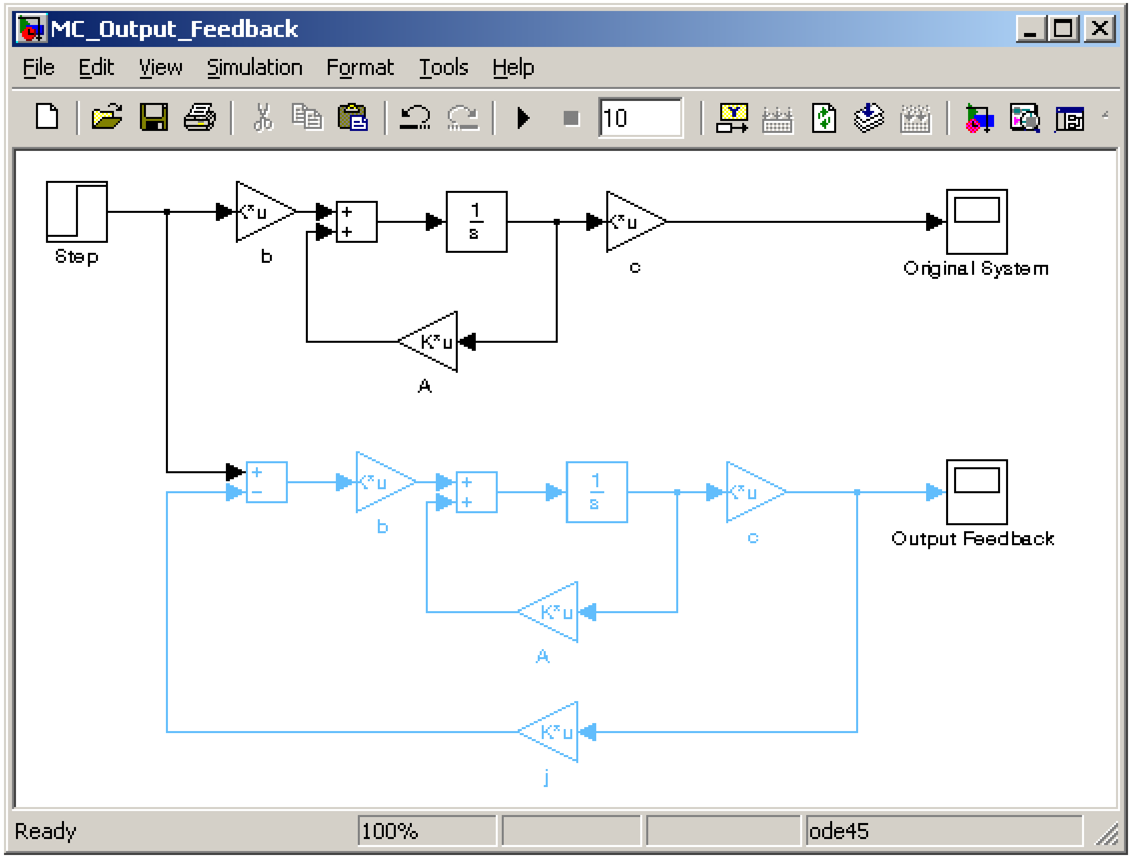
$$= (s-1)(s-(1-j)) - (3-j)(3)$$

$$= s^2 + (j-2)s + (2j-8)$$

- **The roots of the new characteristic equation can be moved to a new stable position.**
- **But, the value of the stable poles cannot be freely chosen.**

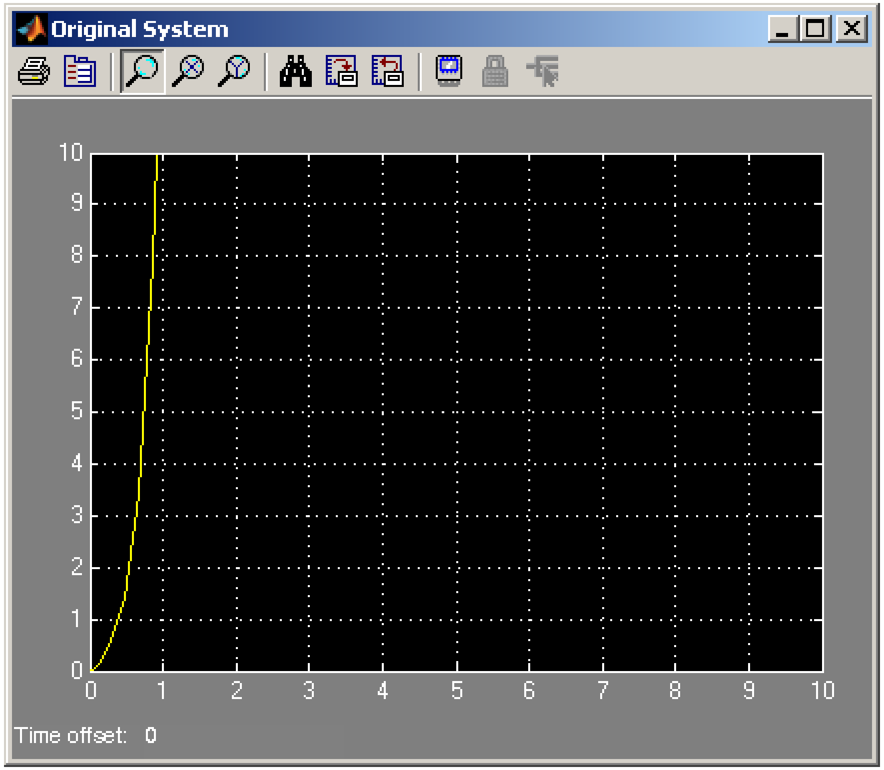
Example: Output Feedback

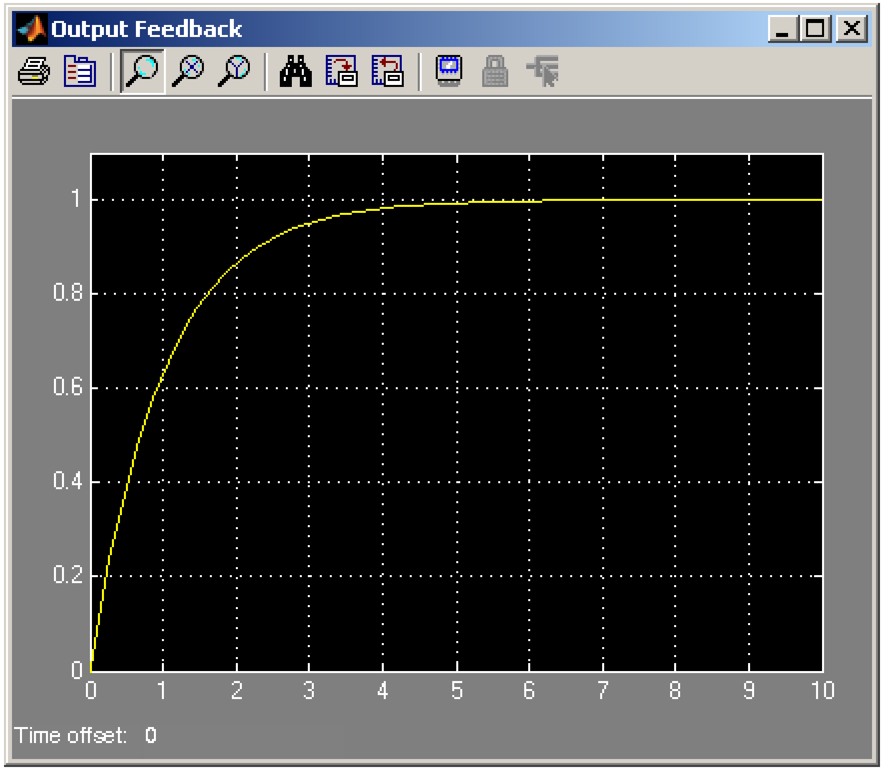
Unit step input



- **Matlab Simulink realization**

Example: Output Feedback



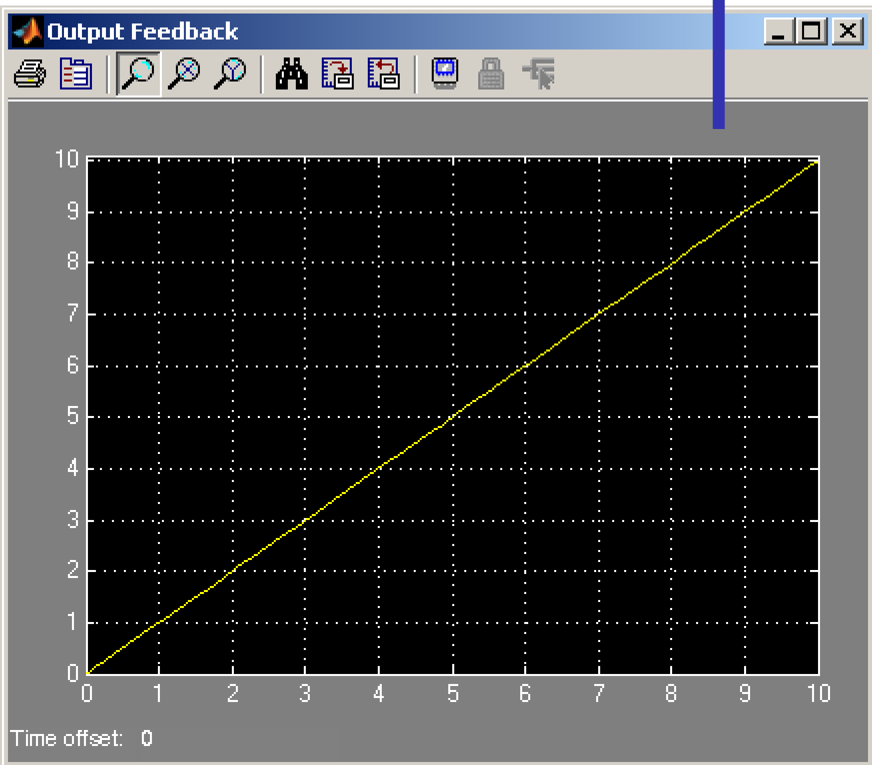


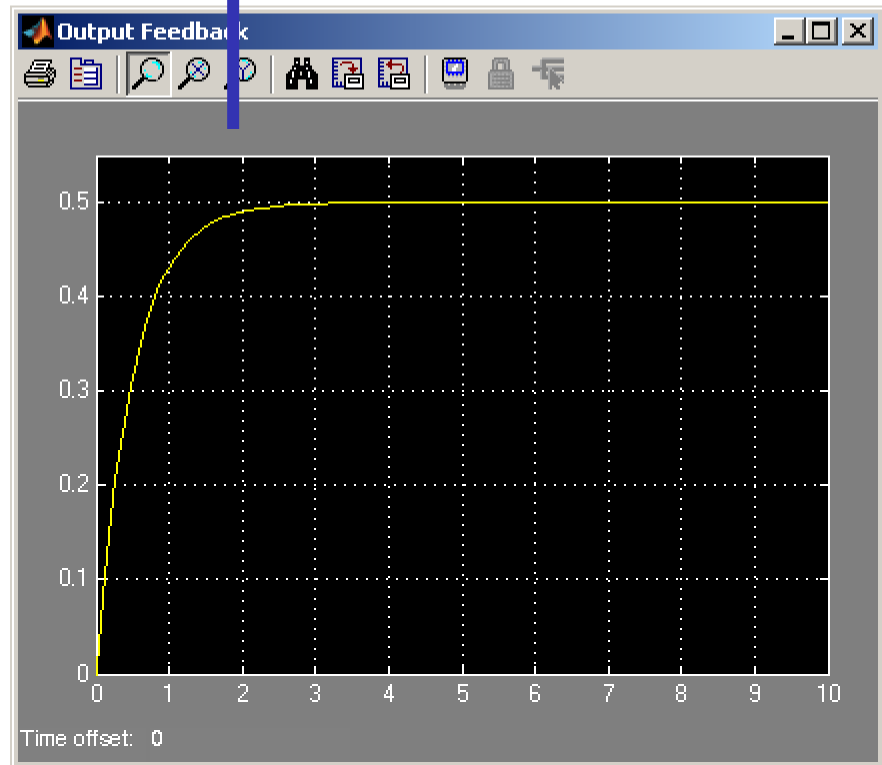
- **Output of original system**
- **Output of system with output feedback, *j* = 5**

Example: Output Feedback

One pole at the origin, Integrator property

Steady-state error





- **Output of system with output feedback, j = 4**

- **Output of system with output feedback, j = 6**