Stability, Observability, and Controllability

“Linear System Theory and Design”, Chapter 5,6

http://zitompul.wordpress.com

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Homework 3: Transfer Function to State Space

- Find the state-space realizations of the following transfer function in *Frobenius Form*, *Observer Form*, and *Canonical Form*.

$$G(s)=\frac{Y(s)}{U(s)}=\frac{s+2}{s^3+8s^2+19s+12}$$

- Hint: Learn the following functions in Matlab and use the to solve this problem: **roots**, **residue**, **conv**.

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- ***Frobenius Form***

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -12 & -19 & -8\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}2 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+0u(t)$$

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- ***Observer Form***

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}0 & 0 & -12\\ 1 & 0 & -19\\ 0 & 1 & -8\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+\begin{bmatrix}2\\ 1\\ 0\end{bmatrix}u(t)$$

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- Using Matlab function, [R,P,K] = residue(NUM,DEN),

$$\frac{s+2}{s^3+8s^2+19s+12}=\frac{-2/3}{s+4}+\frac{1/2}{s+3}+\frac{1/6}{s+1}$$

$$=\frac{r_1}{s-\lambda_1}+\frac{r_2}{s-\lambda_2}+\frac{r_3}{s-\lambda_3}$$

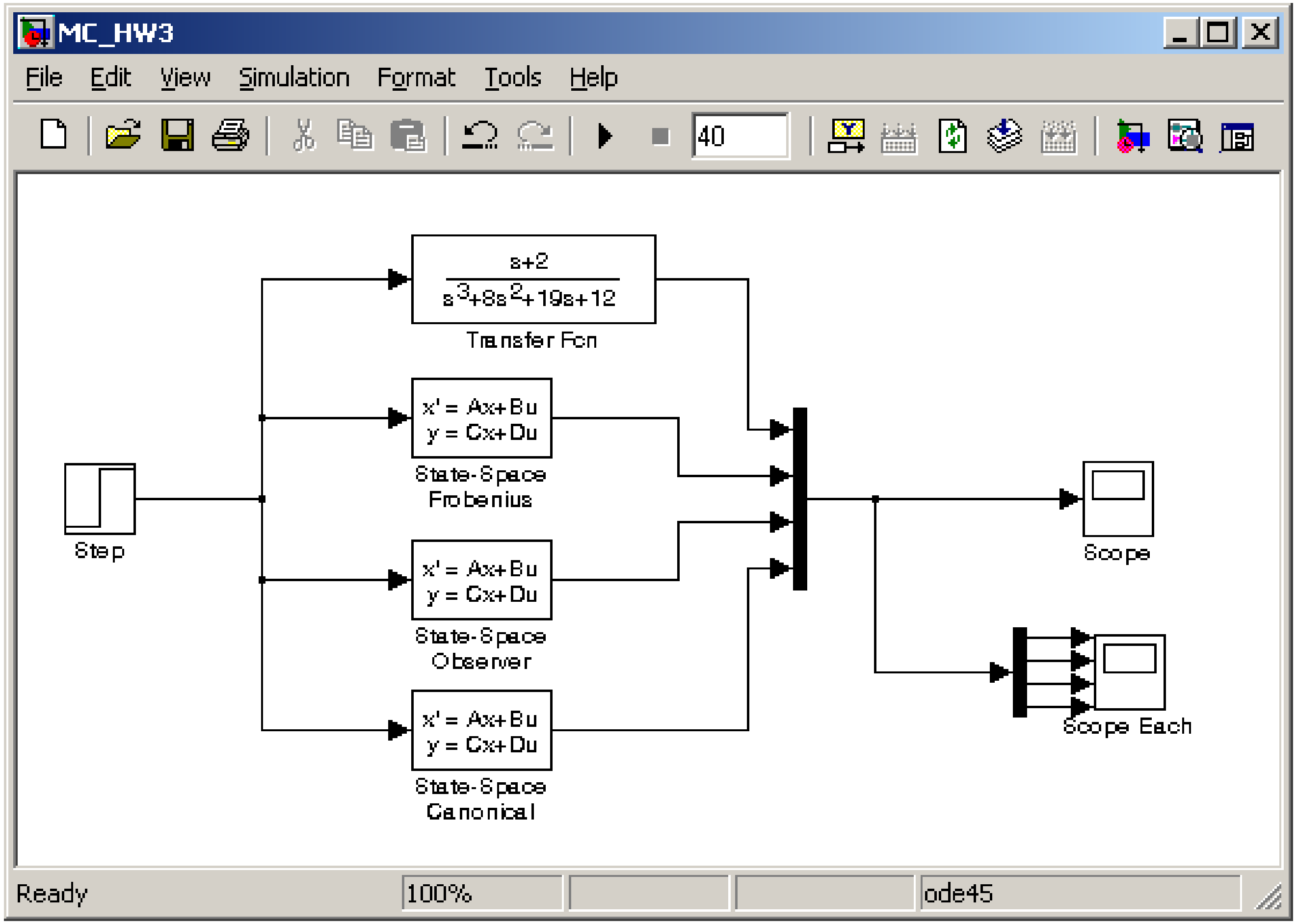
Homework 3: Transfer Function to State Space

- ***Canonical Form***

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -2/3 & 1/2 & 1/6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + 0u(t)$$

Homework 3: Transfer Function to State Space



Canonical Form

- The state space equations in case all poles are distinct:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + r_0 u(t)$$

Canonical Form, Distinct Poles

- **The resulting matrix $\underline{A}$ is a diagonal matrix.**
- **The ODEs are decoupled, each of them can be solved independently.**

Math Preliminaries

- Let $\underline{\boldsymbol{M}}$ be an $m\times n$ matrix, then:

$$\underline{\boldsymbol{M}}\,\underline{\boldsymbol{x}} = \underline{\mathbf{0}},\ \text{for } \underline{\boldsymbol{x}} \neq \underline{\mathbf{0}} \quad \Rightarrow \quad \operatorname{rank}(M) < n$$

$$\underline{\boldsymbol{y}}^{\mathrm{T}}\underline{\boldsymbol{M}} = \underline{\mathbf{0}},\ \text{for } \underline{\boldsymbol{y}}^{\mathrm{T}} \neq \underline{\mathbf{0}} \quad \Rightarrow \quad \operatorname{rank}(M) < m$$

- If $\underline{\boldsymbol{M}}$ is $n\times n$ matrix, then:

$$\underline{\boldsymbol{M}} \text{ is nonsingular} \quad \Leftrightarrow \quad \operatorname{rank}(M) = n$$

- For a nonsingular matrix $\underline{\boldsymbol{M}}$,

$$\underline{\boldsymbol{M}}\,\underline{\boldsymbol{x}} = \underline{\mathbf{0}} \quad \Leftrightarrow \quad \underline{\boldsymbol{x}} = \underline{\mathbf{0}}$$

$$\underline{\boldsymbol{x}}^{\mathrm{T}}\underline{\boldsymbol{M}} = \underline{\mathbf{0}} \quad \Leftrightarrow \quad \underline{\boldsymbol{x}} = \underline{\mathbf{0}}$$

Math Preliminaries

- A symmetric $n \times n$ matrix $\underline{\boldsymbol{P}}$ is *positive semidefinite* if:

  $$\underline{\boldsymbol{x}}^{\mathrm{T}}\,\underline{\boldsymbol{P}}\,\underline{\boldsymbol{x}} \geq \underline{\mathbf{0}}, \quad \text{for all } \underline{\boldsymbol{x}} \neq \underline{\mathbf{0}}$$

  It is positive definite if:

  $$\underline{\boldsymbol{x}}^{\mathrm{T}}\,\underline{\boldsymbol{P}}\,\underline{\boldsymbol{x}} > \underline{\mathbf{0}}, \quad \text{for all } \underline{\boldsymbol{x}} \neq \underline{\mathbf{0}}$$

- A matrix $\underline{\boldsymbol{P}}$ is *positive definite*
  <u>if and only if</u>
  all eigenvalues of $\underline{\boldsymbol{P}}$ are positive.
- A symmetric positive semidefinite matrix is positive definite
  <u>if and only if</u>
  it is nonsingular.

Stability

- There are several ways to define the stability of a system. One of them is "BIBO (Bounded Input Bounded Output) Stability".
- A system is said to be **BIBO stable** if every bounded input excites a bounded output also.
- Bounded input means, there exists a constant u_{m} such that

$$|u(t)| < u_{\mathrm{m}} < \infty, \quad \text{for all } t \geq 0$$

- Thus, a SISO system, described by a transfer function $G(s)$ is said to be BIBO stable **if and only if** every pole of $G(s)$ has a negative real part.
- Other way stated, a SISO system $G(s)$ is stable if every pole of $G(s)$ lies on the left half plane of s.

Stability

- A state space in the form of:

$$\underline{\dot{\boldsymbol{x}}}(t) = \underline{\boldsymbol{A}}\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{B}}\underline{\boldsymbol{u}}(t)$$

$$\underline{\boldsymbol{y}}(t) = \underline{\boldsymbol{C}}\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{D}}\underline{\boldsymbol{u}}(t)$$

is said to be marginally stable if for $\underline{\boldsymbol{u}}(t)=\underline{\mathbf{0}}$, every finite initial state $\underline{\boldsymbol{x}}_0$ will excite a bounded response.

- The state space is said to be asymptotically stable if for $\underline{\boldsymbol{u}}(t)=\underline{\mathbf{0}}$, every finite initial state $\underline{\boldsymbol{x}}_0$ will excite a bounded response and it approaches $\underline{\mathbf{0}}$ as $t\to\infty$.

Controllability

- Consider the n-dimensional state equations with r inputs:

$$\dot{\underline{\boldsymbol{x}}}(t) = \underline{\boldsymbol{A}}\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{B}}\underline{\boldsymbol{u}}(t)$$

- The state equations above are said to be "controllable" if for any initial state $\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$ and any final state $\underline{\boldsymbol{x}}(t_1) = \underline{\boldsymbol{x}}_1$, there exists an input that transfers $\underline{\boldsymbol{x}}_0$ to $\underline{\boldsymbol{x}}_1$ in a finite time.
- Otherwise, the state equations are said to be "uncontrollable".

Controllability Matrix

- The controllability of state equations can be checked using the [$n \times nr$] controllability matrix:

$$\underline{\mathcal{C}} = \left[\underline{\boldsymbol{B}} \;\vdots\; \underline{\boldsymbol{A}}\underline{\boldsymbol{B}} \;\vdots\; \underline{\boldsymbol{A}}^2\underline{\boldsymbol{B}} \;\vdots\; \cdots \;\vdots\; \underline{\boldsymbol{A}}^{n-1}\underline{\boldsymbol{B}} \right]$$

- A state space described by the pair ($\underline{\boldsymbol{A}}$,$\underline{\boldsymbol{B}}$) is controllable if the column rank of $\underline{\mathcal{C}}$ = n, or equivalently, if matrix $\underline{\mathcal{C}}$ has n linearly independent columns.

Example 1

Investigate the controllability of

$$\underline{\boldsymbol{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \underline{\boldsymbol{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{\boldsymbol{\mathcal{C}}} = \left[\underline{\boldsymbol{B}} \;\vdots\; \underline{\boldsymbol{A}}\underline{\boldsymbol{B}} \;\vdots\; \underline{\boldsymbol{A}}^2 \underline{\boldsymbol{B}} \right]$$

- **Matlab**: C = ctrb(A,B); rank(C);

Observability

- Consider the *n*-dimensional state space equations with *r* inputs and *m* outputs:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t)$$
$$\underline{y}(t) = \underline{C}\underline{x}(t) + \underline{D}\underline{u}(t)$$

- The state space equations above are said to be "observable" if for any unknown initial state $\underline{x}(t_0) = \underline{x}_0$, there exists a finite $t_1 > 0$ such that the knowledge of the input $\underline{u}(t)$ and the output $\underline{y}(t)$ over the time interval $[t_0, t_1]$ suffices to determine uniquely the initial state $\underline{x}(t_0)$.
- Otherwise, the state space equations are said to be "unobservable".

Observability Matrix

- The observability of state space equations can be checked using the [*nm*×*n*] observability matrix:

$$\underline{\mathcal{O}} = \begin{bmatrix} \underline{C} \\ \cdots\cdots \\ \underline{C}\,\underline{A} \\ \cdots\cdots \\ \underline{C}\,\underline{A}^2 \\ \cdots\cdots \\ \vdots \\ \cdots\cdots \\ \underline{C}\,\underline{A}^{n-1} \end{bmatrix}$$

- A state space described by the pair ($\underline{\boldsymbol{A}}$,$\underline{\boldsymbol{C}}$) is observable if the row rank of $\underline{\mathcal{O}} = n$, or equivalently, if matrix $\underline{\mathcal{O}}$ has *n* linearly independent rows.

Example 2

A state space is given as

$$\underline{\dot{\boldsymbol{x}}}(t)=\begin{bmatrix}1 & -1\\ 0 & -2\end{bmatrix}\underline{\boldsymbol{x}}(t)+\begin{bmatrix}1\\ 3\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}1 & 0\end{bmatrix}\underline{\boldsymbol{x}}(t)$$

Check its controllability and observability.

$$n=2$$

$$\underline{\mathcal{C}}=\left[\underline{\boldsymbol{B}}\ \vdots\ \underline{\boldsymbol{AB}}\right]=\left[\begin{bmatrix}1\\ 3\end{bmatrix}\ \vdots\ \begin{bmatrix}1 & -1\\ 0 & -2\end{bmatrix}\begin{bmatrix}1\\ 3\end{bmatrix}\right]=\begin{bmatrix}1 & \vdots & -2\\ 3 & \vdots & -6\end{bmatrix}$$

- **Column rank of $\underline{\mathcal{C}}$ = 1 ≠ *n***
- **The state space is "<u>un</u>controllable"**

$$\underline{\mathcal{O}}=\begin{bmatrix}\underline{\boldsymbol{C}}\\ \cdots\\ \underline{\boldsymbol{C}}\,\underline{\boldsymbol{A}}\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\\ \cdots\cdots\\ \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & -2\end{bmatrix}\end{bmatrix}=\begin{bmatrix}1 & 0\\ \cdots & \cdots\\ 1 & -1\end{bmatrix}$$

- **Row rank of $\underline{\mathcal{O}}$ = 2 = *n***
- **The state space is "observable"**

State Feedback

- Feedback control is characterized by a comparison of backward connection of output signal(s) to set point(s).
- The feedback signal can be classified into:
  1. **Output feedback**, where only some output variables are measured and can be used for feedback
  2. **State feedback**, where all state variables are measured and can be used for feedback.

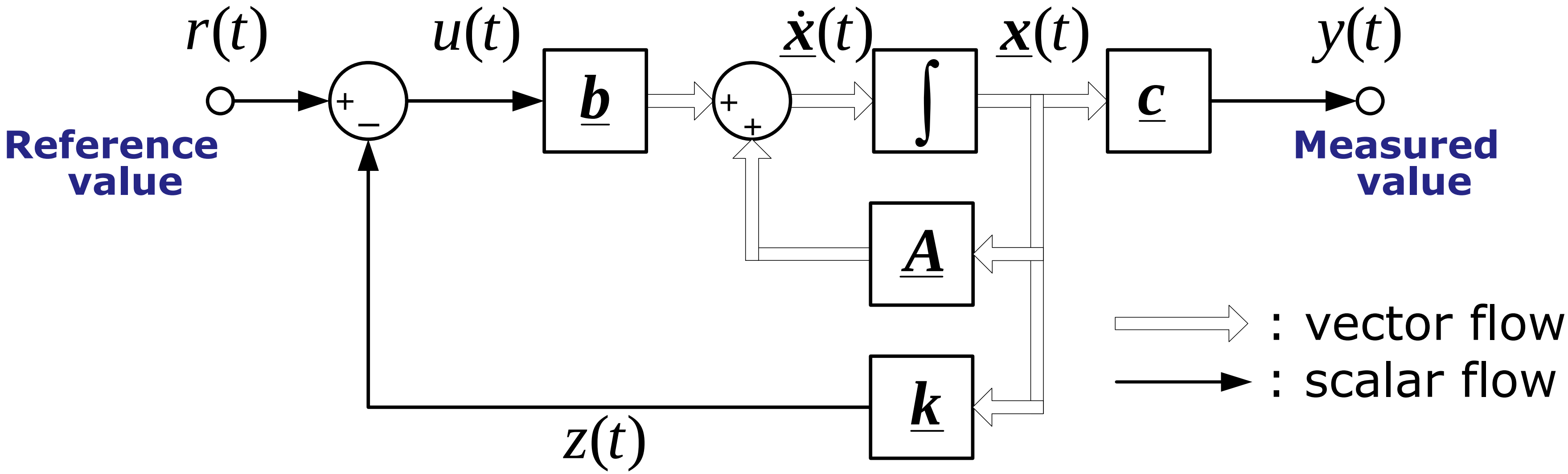
- Consider the *n*-dimensional single-variable state space equations:

$$\dot{\underline{\mathbf{x}}}(t) = \underline{\mathbf{A}}\underline{\mathbf{x}}(t) + \underline{\mathbf{b}}u(t)$$

$$y(t) = \underline{\mathbf{c}}\underline{\mathbf{x}}(t)$$

- **Main idea**: Using measurements of state variables $\underline{\mathbf{x}}(t)$, determine an input $u(t)=f(\underline{\mathbf{x}}(t))$ such that the dynamic properties of the system can be changed to fulfill a certain criteria.

State Feedback



- The states $\underline{\boldsymbol{x}}(t)$ are fed back through a feedback gain $\underline{\boldsymbol{k}}$.
- The input $u(t)$ is given by:

$$u(t) = r(t) - \underline{\boldsymbol{k}}\,\underline{\boldsymbol{x}}(t)$$
$$= r(t) - \sum_{i=1}^{n} k_i x_i(t)$$

$$\underline{\boldsymbol{k}} = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$$
$$\underline{\boldsymbol{x}}(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix}^{\mathrm{T}}$$

State Feedback

- Substituting $u(t)$ to the original state space equations,

$$\dot{\underline{\boldsymbol{x}}}(t) = \left(\underline{\boldsymbol{A}} - \underline{\boldsymbol{b}}\underline{\boldsymbol{k}}\right)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{b}}r(t)$$

$$y(t) = \underline{\boldsymbol{c}}\underline{\boldsymbol{x}}(t)$$

Example 3

Consider a state space

$$\underline{\dot{x}}(t) = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \underline{x}(t)$$

The controllability and observability matrices are:

$$\underline{\mathcal{C}} = [\underline{B} \;\vdots\; \underline{A}\underline{B}] = \begin{bmatrix} 0 & \vdots & 2 \\ 1 & \vdots & 1 \end{bmatrix}$$

- **Column rank of $\underline{\mathcal{C}}$ = 2 → "controllable"**

$$\underline{\mathcal{O}} = \begin{bmatrix} \underline{C} \\ \cdots \\ \underline{C}\underline{A} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ \cdots & \cdots \\ 7 & 4 \end{bmatrix}$$

- **Row rank of $\underline{\mathcal{O}}$ = 2 → "observable"**

Example 4

Let us now introduce a state feedback:

$$u(t) = r(t) - \begin{bmatrix} 3 & 1 \end{bmatrix} \underline{\boldsymbol{x}}(t)$$

The state space is now:

$$\begin{aligned}
\dot{\underline{\boldsymbol{x}}}(t) &= \left(\underline{\boldsymbol{A}} - \underline{\boldsymbol{b}}\underline{\boldsymbol{k}} \right) \underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{b}} r(t) \\
&= \left(\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \end{bmatrix} \right) \underline{\boldsymbol{x}}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) \\
&= \left(\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix} \right) \underline{\boldsymbol{x}}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)
\end{aligned}$$

$$\dot{\underline{\boldsymbol{x}}}(t) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \underline{\boldsymbol{x}}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \underline{\boldsymbol{x}}(t)$$

$$\underline{\mathcal{C}} = \left[\begin{array}{c:c} 0 & 2 \\ 1 & 0 \end{array} \right]$$

$$\underline{\mathcal{O}} = \left[\begin{array}{cc} 1 & 2 \\ \hdashline 1 & 2 \end{array} \right]$$

- **Column rank of $\underline{\mathcal{C}}$ = 2 → "controllable"**
- **Row rank of $\underline{\mathcal{O}}$ = 1 → "not observable"**
- **State feedback may make a state space become "not observable"**

Example 5

Consider a SISO system with the following state equations:

$$\dot{\underline{\boldsymbol{x}}}(t) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \underline{\boldsymbol{x}}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

The transfer function of the system is:

$$G(s) = \underline{\boldsymbol{C}}(s\underline{\boldsymbol{I}} - \underline{\boldsymbol{A}})^{-1}\underline{\boldsymbol{B}} + D$$

The characteristic equation, or the denominator of $G(s)$, is given by:

$$\begin{aligned} a(s) &= \det(s\underline{\boldsymbol{I}} - \underline{\boldsymbol{A}}) \\ &= \det\left(\begin{bmatrix} s-1 & -3 \\ -3 & s-1 \end{bmatrix}\right) \\ &= (s-1)^2 - (-3)(-3) \\ &= s^2 - 2s - 8 \\ &= (s-4)(s+2) \end{aligned}$$

- **λ = 4, positive**
- **Unstable eigenvalues or unstable pole**

Example 5

Let us now introduce a state feedback:

$$u(t) = r(t) - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \underline{\boldsymbol{x}}(t)$$

The state space is now:

$$\dot{\underline{\boldsymbol{x}}}(t) = \begin{bmatrix} 1-k_1 & 3-k_2 \\ 3 & 1 \end{bmatrix} \underline{\boldsymbol{x}}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r(t)$$

The characteristic equation becomes:

$$a(s) = \det\left(\begin{bmatrix} s-(1-k_1) & -(3-k_2) \\ -3 & s-1 \end{bmatrix} \right)$$

$$= (s-1+k_1)(s-1) - (-3+k_2)(-3)$$

$$= s^2 + \underbrace{(k_1-2)}s + \underbrace{(3k_2-k_1-8)}$$

- **The roots of the new characteristic equation can be placed in any location by assigning appropriate value of k_1 and k_2**
- **Condition: complex eigenvalues must be given in pairs**

Homework 4

Again, consider a SISO system with the state equations:

$$\underline{\dot{\boldsymbol{x}}}(t) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \underline{\boldsymbol{x}}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \underline{\boldsymbol{x}}(t)$$

a. If the state feedback in the form of:

$$u(t) = r(t) - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \underline{\boldsymbol{x}}(t)$$

is implemented to the system and it is wished that the poles of the system will be –3 and –4, determine the value of k_1 and k_2.

b. Find the transfer function of the system and again, check the location of the poles of the transfer function.

Homework 4A

Consider a SISO system with the state equations:

$$\dot{\underline{\boldsymbol{x}}}(t) = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \underline{\boldsymbol{x}}(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} \underline{\boldsymbol{x}}(t)$$

a. If the state feedback in the form of:

$$u(t) = r(t) - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \underline{\boldsymbol{x}}(t)$$

is implemented to the system and it is wished that the damping factor ζ of the system is equal to 0.8 while keeping the system stable. Determine the required value of k_1 and k_2. **Hint**: Take one reasonable value of ω.

b. Find the transfer function of the system and again, check the location of the poles of the transfer function.