# State Space Solutions and Realizations

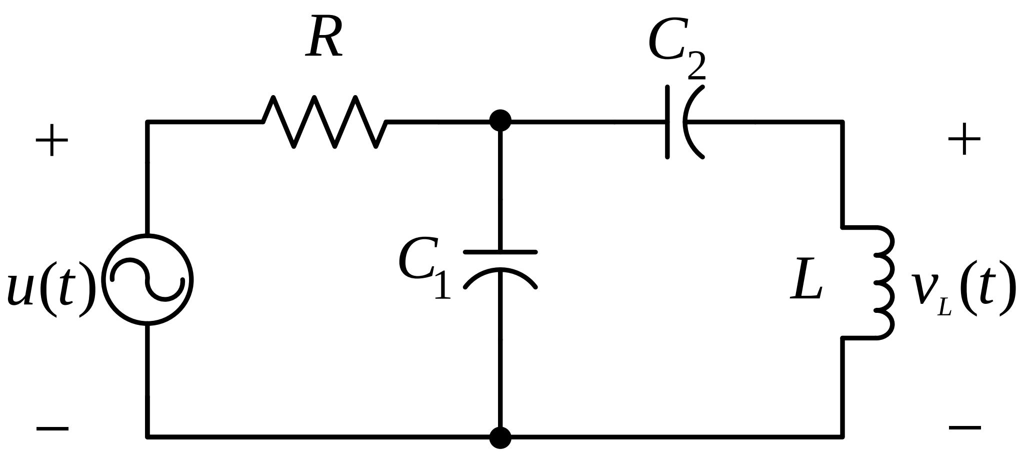
"Linear System Theory and Design", Chapter 4

**http://zitompul.wordpress.com**

**2 0 1 4**

# Homework 1: Electrical System

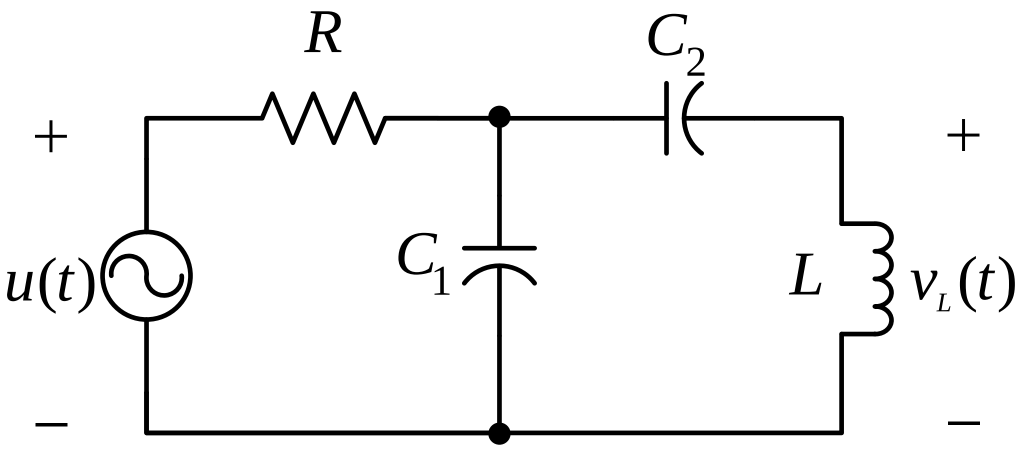
- Derive the state space representation of the following electric circuit:




Input variable $u$:
- Input voltage $u(t)$

Output variable $y$:
- Inductor voltage $v_L(t)$

# Solution of Homework 1: Electrical System




$$v_R = Ri_R$$

$$v_L = L\frac{di_L}{dt} = L\dot{i}_L$$

$$i_C = C\frac{dv_C}{dt} = C\dot{v}_C$$

State variables:

- $x_1$ is the voltage across $C_1$
- $x_2$ is the voltage across $C_2$
- $x_3$ is the current through $L$

$$(x_1 - u)/R + C_1\dot{x}_1 + C_2\dot{x}_2 = 0$$

$$C_2\dot{x}_2 = x_3$$

$$x_1 - x_2 = L\dot{x}_3$$

$$\dot{x}_1 = -1/RC_1 \cdot x_1 - 1/C_1 \cdot x_3 + 1/RC_1 \cdot u$$

$$\dot{x}_2 = 1/C_2 \cdot x_3$$

$$\dot{x}_3 = 1/L \cdot x_1 - 1/L \cdot x_2$$

# Solution of Homework 1: Electrical System

- The state space equation can now be written as:

$$\dot{x}_1 = -1/RC_1 \cdot x_1 - 1/C_1 \cdot x_3 + 1/RC_1 \cdot u$$

$$\dot{x}_2 = 1/C_2 \cdot x_3$$

$$\dot{x}_3 = 1/L \cdot x_1 - 1/L \cdot x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1/RC_1 & 0 & -1/C_1 \\ 0 & 0 & 1/C_2 \\ 1/L & -1/L & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/RC_1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0 \cdot u$$

# Example: Transfer Function

- Given the following transfer function

$$Y(s) = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} U(s)$$

and assuming zero initial conditions, construct a state space equations that can represent the given transfer function.

$$s^3 Y(s) + a_2 s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = U(s)$$

$$\dddot{y}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = u(t)$$

$$\left.\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \ddot{y} \end{aligned}\right\rangle \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \dddot{y} = -a_0 x_1 - a_1 x_2 - a_2 x_3 + u(t) \end{aligned}$$

# Example: Transfer Function

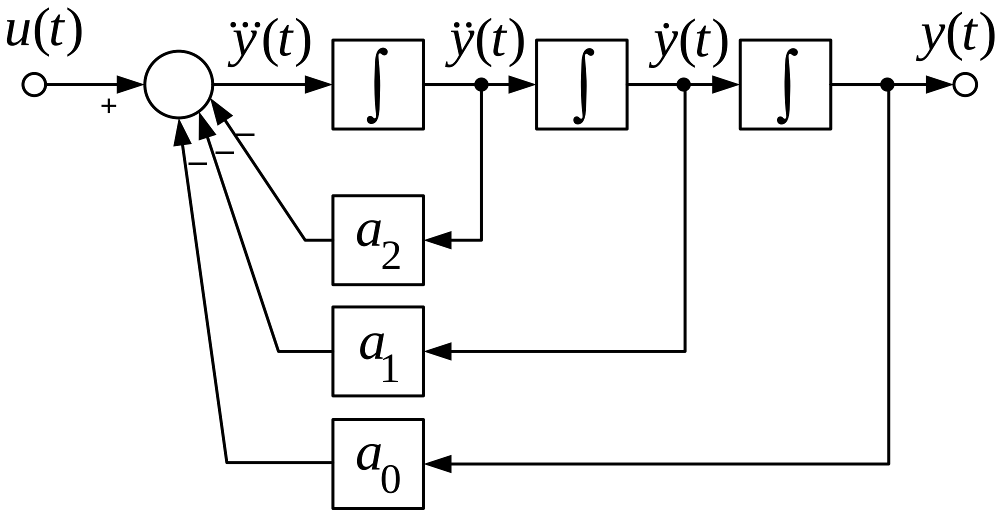
The state space equation can now be given as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

$$Y(s) = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} U(s)$$

The state space equation can also be given using block diagram:



# Vector Case and Scalar Case

- The general form of state space in vector case, where there are multiple inputs and multiple outputs, is given as:

$$\dot{\underline{\boldsymbol{x}}}(t) = \underline{\boldsymbol{A}}\,\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{B}}\,\underline{\boldsymbol{u}}(t)$$

$$\underline{\boldsymbol{y}}(t) = \underline{\boldsymbol{C}}\,\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{D}}\,\underline{\boldsymbol{u}}(t)$$

- In scalar case, where the input and the output are scalar or single, the state space is usually written as:

$$\dot{\underline{\boldsymbol{x}}}(t) = \underline{\boldsymbol{A}}\,\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{b}}u(t)$$

$$y(t) = \underline{\boldsymbol{c}}^{\mathrm{T}}\,\underline{\boldsymbol{x}}(t) + du(t)$$

# Solution of State Equations

- Consider the state equations in vector case.

$$\dot{\underline{\boldsymbol{x}}}(t) = \underline{\boldsymbol{A}}\,\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{B}}\,\underline{\boldsymbol{u}}(t)$$

- Multiplying each term with $e^{-\underline{\boldsymbol{A}}t}$,

$$e^{-\underline{\boldsymbol{A}}t}\,\dot{\underline{\boldsymbol{x}}}(t) = e^{-\underline{\boldsymbol{A}}t}\,\underline{\boldsymbol{A}}\,\underline{\boldsymbol{x}}(t) + e^{-\underline{\boldsymbol{A}}t}\,\underline{\boldsymbol{B}}\,\underline{\boldsymbol{u}}(t)$$

$$e^{-\underline{\boldsymbol{A}}t}\,\dot{\underline{\boldsymbol{x}}}(t) - e^{-\underline{\boldsymbol{A}}t}\,\underline{\boldsymbol{A}}\,\underline{\boldsymbol{x}}(t) = e^{-\underline{\boldsymbol{A}}t}\,\underline{\boldsymbol{B}}\,\underline{\boldsymbol{u}}(t)$$

$$\frac{d}{dt}\left(e^{-\underline{\boldsymbol{A}}t}\,\underline{\boldsymbol{x}}(t)\right) = e^{-\underline{\boldsymbol{A}}t}\,\underline{\boldsymbol{B}}\,\underline{\boldsymbol{u}}(t)$$

$$\frac{d}{dt}\left(e^{-\underline{\boldsymbol{A}}t}\right) = -\underline{\boldsymbol{A}}e^{-\underline{\boldsymbol{A}}t}$$

- The last equation will be integrated from 0 to *t*:

$$e^{-\underline{\boldsymbol{A}}t}\,\underline{\boldsymbol{x}}(\tau)\Big]_0^t = \int_0^t e^{-\underline{\boldsymbol{A}}\tau}\,\underline{\boldsymbol{B}}\,\underline{\boldsymbol{u}}(\tau)\,d\tau$$

# Solution of State Equations

$$e^{-\underline{A}t}\,\underline{\boldsymbol{x}}(\tau)\Big]_0^t = \int_0^t e^{-\underline{A}t}\,\underline{\boldsymbol{B}}\underline{\boldsymbol{u}}(\tau)d\tau$$

$$e^{-\underline{A}t}\,\underline{\boldsymbol{x}}(t) - e^{-\underline{A}0}\,\underline{\boldsymbol{x}}(0) = \int_0^t e^{-\underline{A}\tau}\,\underline{\boldsymbol{B}}\underline{\boldsymbol{u}}(\tau)d\tau$$

$$\underline{\boldsymbol{x}}(t) = e^{\underline{A}t}\,\underline{\boldsymbol{x}}(0) + \int_0^t e^{\underline{A}(t-\tau)}\,\underline{\boldsymbol{B}}\underline{\boldsymbol{u}}(\tau)d\tau$$

**Solution of State Equations**

- At $t=0$, $\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$, which are the initial conditions of the states.

# Solution of Output Equations

- We know substitute the solution of state equations into the output equations:

$$\underline{\boldsymbol{y}}(t) = \underline{\boldsymbol{C}}\,\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{D}}\,\underline{\boldsymbol{u}}(t)$$

$$\underline{\boldsymbol{y}}(t) = \underline{\boldsymbol{C}}\left\{ e^{\underline{\boldsymbol{A}}t}\,\underline{\boldsymbol{x}}(0) + \int_0^t e^{\underline{\boldsymbol{A}}(t-\tau)}\,\underline{\boldsymbol{B}}\,\underline{\boldsymbol{u}}(\tau)\,d\tau \right\} + \underline{\boldsymbol{D}}\,\underline{\boldsymbol{u}}(t)$$

**Solution of Output Equations**

# Solutions of State Space in Frequency Domain

- The solution of state equations and output equations can also be written in frequency domain:

$$\underline{\dot{\boldsymbol{x}}}(t) = \underline{\boldsymbol{A}}\,\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{B}}\,\underline{\boldsymbol{u}}(t)$$

$$s\underline{\boldsymbol{X}}(s) - \underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{A}}\,\underline{\boldsymbol{X}}(s) + \underline{\boldsymbol{B}}\,\underline{\boldsymbol{U}}(s)$$

$$(s\underline{\boldsymbol{I}} - \underline{\boldsymbol{A}})\underline{\boldsymbol{X}}(s) = \underline{\boldsymbol{x}}(0) + \underline{\boldsymbol{B}}\,\underline{\boldsymbol{U}}(s)$$

$$\boxed{\underline{\boldsymbol{X}}(s) = (s\underline{\boldsymbol{I}} - \underline{\boldsymbol{A}})^{-1}\underline{\boldsymbol{x}}(0) + (s\underline{\boldsymbol{I}} - \underline{\boldsymbol{A}})^{-1}\underline{\boldsymbol{B}}\,\underline{\boldsymbol{U}}(s)}$$

**Solution of State Equations**

$$\underline{\boldsymbol{y}}(t) = \underline{\boldsymbol{C}}\,\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{D}}\,\underline{\boldsymbol{u}}(t)$$

$$\underline{\boldsymbol{Y}}(s) = \underline{\boldsymbol{C}}\,\underline{\boldsymbol{X}}(s) + \underline{\boldsymbol{D}}\,\underline{\boldsymbol{U}}(s)$$

$$\boxed{\underline{\boldsymbol{Y}}(s) = \underline{\boldsymbol{C}}\left\{(s\underline{\boldsymbol{I}} - \underline{\boldsymbol{A}})^{-1}\underline{\boldsymbol{x}}(0) + (s\underline{\boldsymbol{I}} - \underline{\boldsymbol{A}})^{-1}\underline{\boldsymbol{B}}\,\underline{\boldsymbol{U}}(s)\right\} + \underline{\boldsymbol{D}}\,\underline{\boldsymbol{U}}(s)}$$

**Solution of Output Equations**

# Relation between $e^{\underline{A}t}$ and $(s\underline{I}-\underline{A})$

- Taylor series expansion of exponential function is given by:

$$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \ldots + \frac{\lambda^n t^n}{n!}$$

**Scalar Function**
- **Exact solution, around $t$ = 0, infinite number of terms**

$$e^{\underline{A}t} = \underline{I} + t\underline{A} + \frac{t^2}{2!}\underline{A}^2 + \ldots + \frac{t^n}{n!}\underline{A}^n$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!}\underline{A}^k$$

**Vector Function**

- It can be shown that $\mathcal{L}\left[\frac{t^k}{k!}\right] = s^{-(k+1)}$ so that:

$$\mathcal{L}\left[e^{\underline{A}t}\right] = \mathcal{L}\left[\sum_{k=0}^{\infty} \frac{t^k}{k!}\underline{A}^k\right] = \sum_{k=0}^{\infty} s^{-(k+1)}\underline{A}^k$$

# Relation between $e^{\underline{A}t}$ and $(s\underline{I}-\underline{A})$

- Deriving further,

$$\mathcal{L}\left[e^{\underline{A}t}\right]=\sum_{k=0}^{\infty} s^{-(k+1)}\underline{A}^{k}$$

$$=s^{-1}\underline{I}+s^{-2}\underline{A}+s^{-3}\underline{A}^{2}+\ldots$$

$$=\frac{s^{-1}\underline{I}}{\underline{I}-s^{-1}\underline{A}}$$

$$=s^{-1}(\underline{I}-s^{-1}\underline{A})^{-1}$$

$$=\left(s(\underline{I}-s^{-1}\underline{A})\right)^{-1}$$

$$\boxed{\mathcal{L}\left[e^{\underline{A}t}\right]=(s\underline{I}-\underline{A})^{-1}}$$

$$\boxed{e^{\underline{A}t}=\mathcal{L}^{-1}\left[(s\underline{I}-\underline{A})^{-1}\right]}$$

# State Transition Matrix

- Writing again the general form of the state space equations:

$$\underline{\dot{\boldsymbol{x}}}(t) = \underline{\boldsymbol{A}}\,\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{B}}\,\underline{\boldsymbol{u}}(t)$$

$$\underline{\boldsymbol{y}}(t) = \underline{\boldsymbol{C}}\,\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{D}}\,\underline{\boldsymbol{u}}(t)$$

- The behavior of x(t) and y(t) can be classified into:
  - Homogenous solution (zero input, initial state applied)
  - Non-homogenous solution (input applied, initial state applied)

# State Transition Matrix

- **Homogenous Solution**:

$$\dot{\underline{\boldsymbol{x}}}(t) = \underline{\boldsymbol{A}}\,\underline{\boldsymbol{x}}(t)$$

$$s\underline{\boldsymbol{X}}(s) - \underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{A}}\,\underline{\boldsymbol{X}}(s)$$

$$\underline{\boldsymbol{X}}(s) = (s\underline{\boldsymbol{I}} - \underline{\boldsymbol{A}})^{-1}\underline{\boldsymbol{x}}(0) \quad \Rightarrow \quad \underline{\boldsymbol{x}}(t) = \mathcal{L}^{-1}\left[(s\underline{\boldsymbol{I}} - \underline{\boldsymbol{A}})^{-1}\right]\underline{\boldsymbol{x}}(0)$$

$$\underline{\boldsymbol{x}}(t) = e^{\underline{\boldsymbol{A}}t}\,\underline{\boldsymbol{x}}(0)$$

- $e^{\underline{\boldsymbol{A}}t}$ is called the state transition matrix, able to give the current state $\underline{\boldsymbol{x}}(t)$ out of the initial state $\underline{\boldsymbol{x}}(0)$,

$$\underline{\boldsymbol{\Phi}} = e^{\underline{\boldsymbol{A}}t} = \mathcal{L}^{-1}\left[(s\underline{\boldsymbol{I}} - \underline{\boldsymbol{A}})^{-1}\right]$$

# State Transition Matrix

- Since

$$\underline{\boldsymbol{x}}(t) = e^{\underline{\boldsymbol{A}}t}\,\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{\Phi}}(t)\underline{\boldsymbol{x}}(0)$$

- We can write

$$\underline{\boldsymbol{x}}(t_0) = e^{\underline{\boldsymbol{A}}t_0}\,\underline{\boldsymbol{x}}(0) \Rightarrow \underline{\boldsymbol{x}}(0) = e^{-\underline{\boldsymbol{A}}t_0}\,\underline{\boldsymbol{x}}(t_0)$$

$$\underline{\boldsymbol{x}}(t) = e^{\underline{\boldsymbol{A}}t}e^{-\underline{\boldsymbol{A}}t_0}\,\underline{\boldsymbol{x}}(t_0) = e^{\underline{\boldsymbol{A}}(t-t_0)}\,\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{\Phi}}(t-t_0)\underline{\boldsymbol{x}}(t_0)$$

- Some properties of state transition matrix:
  1. $\underline{\boldsymbol{\Phi}}(0) = \underline{\boldsymbol{I}}$
  2. $\underline{\boldsymbol{\Phi}}^{-1}(t) = \underline{\boldsymbol{\Phi}}(-t)$
  3. $\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{\Phi}}(-t)\underline{\boldsymbol{x}}(t)$
  4. $\underline{\boldsymbol{\Phi}}(t_2 - t_1)\underline{\boldsymbol{\Phi}}(t_1 - t_0) = \underline{\boldsymbol{\Phi}}(t_2 - t_0)$
  5. $\underline{\boldsymbol{\Phi}}(t)^k = \underline{\boldsymbol{\Phi}}(kt)$

# State Transition Matrix

- **Non-Homogenous Solution**:

$$s\underline{\boldsymbol{X}}(s)-\underline{\boldsymbol{x}}(0)=\underline{\boldsymbol{A}}\underline{\boldsymbol{X}}(s)+\underline{\boldsymbol{B}}\underline{\boldsymbol{U}}(s)$$

$$(s\underline{\boldsymbol{I}}-\underline{\boldsymbol{A}})\underline{\boldsymbol{X}}(s)=\underline{\boldsymbol{x}}(0)+\underline{\boldsymbol{B}}\underline{\boldsymbol{U}}(s)$$

$$\underline{\boldsymbol{X}}(s)=(s\underline{\boldsymbol{I}}-\underline{\boldsymbol{A}})^{-1}\underline{\boldsymbol{x}}(0)+(s\underline{\boldsymbol{I}}-\underline{\boldsymbol{A}})^{-1}\underline{\boldsymbol{B}}\underline{\boldsymbol{U}}(s)$$

- Then,

$$x(t)=\mathcal{L}^{-1}\left[(s\underline{\boldsymbol{I}}-\underline{\boldsymbol{A}})^{-1}\right]\underline{\boldsymbol{x}}(0)+\mathcal{L}^{-1}\left[(s\underline{\boldsymbol{I}}-\underline{\boldsymbol{A}})^{-1}\underline{\boldsymbol{B}}\underline{\boldsymbol{U}}(s)\right]$$

$$\underline{\boldsymbol{x}}(t)=\underbrace{\underline{\boldsymbol{\Phi}}(t)\underline{\boldsymbol{x}}(0)}_{\text{Homogenous Solution}}+\int_0^t\underline{\boldsymbol{\Phi}}(t-\tau)\underline{\boldsymbol{B}}\underline{\boldsymbol{u}}(\tau)d\tau$$

# Example 1: Solution of State Equations

Compute $(s\underline{\boldsymbol{I}}-\underline{\boldsymbol{A}})^{-1}$ if $\underline{\boldsymbol{A}}=\begin{bmatrix}0 & -1\\ 1 & -2\end{bmatrix}$.

$$(s\underline{\boldsymbol{I}}-\underline{\boldsymbol{A}})=\begin{bmatrix}s & 1\\ -1 & s+2\end{bmatrix}$$

$$(s\underline{\boldsymbol{I}}-\underline{\boldsymbol{A}})^{-1}=\frac{1}{(s)(s+2)-(1)(-1)}\begin{bmatrix}s+2 & -1\\ 1 & s\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{s+2}{s^2+2s+1} & \dfrac{-1}{s^2+2s+1}\\ \dfrac{1}{s^2+2s+1} & \dfrac{s}{s^2+2s+1}\end{bmatrix}$$

# Example 2: Solution of State Equations

Given $\dot{\underline{x}}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$, find the solution for $\underline{\boldsymbol{x}}(t)$.

$$\underline{\boldsymbol{x}}(t) = e^{\boldsymbol{A}t}\underline{\boldsymbol{x}}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\underline{\boldsymbol{B}}u(\tau)d\tau$$

$$e^{\underline{A}t} = \mathcal{L}^{-1}\left[(s\underline{\boldsymbol{I}} - \underline{\boldsymbol{A}})^{-1}\right]$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+2}{(s+1)^2} & \dfrac{-1}{(s+1)^2} \\ \dfrac{1}{(s+1)^2} & \dfrac{s}{(s+1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

| $f(t)$ | $F(s)$ |
|---|---|
| $\frac{t^{n-1}e^{-at}}{(n-1)!}1(t), n \geq 1$ | $\frac{1}{(s+a)^n}$ |

# Example 2: Solution of State Equations

Now, we substitute $e^{\underline{\mathbf{A}}t}$ to obtain the solution for $\underline{\mathbf{x}}(t)$:

$$\underline{\mathbf{x}}(t) = \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix} \underline{\mathbf{x}}(0) + \int_0^t \begin{bmatrix} (1+(t-\tau))e^{-(t-\tau)} & -(t-\tau)e^{-(t-\tau)} \\ (t-\tau)e^{-(t-\tau)} & (1-(t-\tau))e^{-(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau)d\tau$$

$$= \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix} \underline{\mathbf{x}}(0) + \begin{bmatrix} -\int_0^t (t-\tau)e^{-(t-\tau)}u(\tau)d\tau \\ \int_0^t (1-(t-\tau))e^{-(t-\tau)}u(\tau)d\tau \end{bmatrix}$$

# Example 3: Solution of State Equations

If $\underline{\boldsymbol{x}}(0)=\underline{\mathbf{0}}$ and $u(t)$ is a step function, determine $\underline{\boldsymbol{x}}(t)$.

$$\underline{\boldsymbol{x}}(t)=\begin{bmatrix}(1+t)e^{-t} & -te^{-t}\\ te^{-t} & (1-t)e^{-t}\end{bmatrix}\underline{\mathbf{0}}+\begin{bmatrix}-\int_0^t (t-\tau)e^{-(t-\tau)}1(\tau)d\tau\\ \int_0^t (1-(t-\tau))e^{-(t-\tau)}1(\tau)d\tau\end{bmatrix}$$

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\begin{bmatrix}-\int_0^t (t-\tau)e^{-(t-\tau)}d\tau\\ \int_0^t (1-(t-\tau))e^{-(t-\tau)}d\tau\end{bmatrix}$$

# Example 3: Solution of State Equations

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \int_0^t (t-\tau)e^{-(t-\tau)}d(t-\tau) \\ \int_0^t ((t-\tau)-1)e^{-(t-\tau)}d(t-\tau) \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-(t-\tau)}(1+(t-\tau))\Big]_0^t \\ -e^{-(t-\tau)}(t-\tau)\Big]_0^t \end{bmatrix}$$

$$= \begin{bmatrix} -1+e^{-t}(1+t) \\ e^{-t}t \end{bmatrix}$$

$$\frac{d(t-\tau)}{d\tau} = -1$$
$$d(t-\tau) = -d\tau$$

$$\int te^{-t}dt = -e^{-t}(1+t)$$
$$\int e^{-t}dt = -e^{-t}$$

$\Rightarrow$

$$x_1(t) = -1+e^{-t}(1+t)$$
$$x_2(t) = e^{-t}t$$

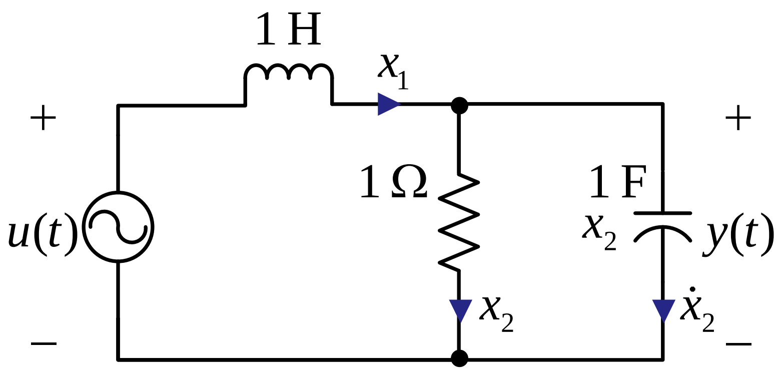
# Example 4: Solution of State Equations

Compute $e^{\underline{\boldsymbol{A}}t}$ if $\underline{\boldsymbol{A}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

# Example 5: Solution of State Equations

Find $e^{\underline{A}t}$ for $\underline{A} = \begin{bmatrix} -1 & 1 \\ -\frac{1}{2} & -2 \end{bmatrix}$.

# Equivalent State Equations




State variables:

- $x_1$ : inductor current $i_L$
- $x_2$ : capacitor voltage $v_C$

$$v_L = L\frac{di_L}{dt} = \dot{x}_1$$

$$i_R = \frac{v_R}{R} = x_2$$

$$i_C = C\frac{dv_C}{dt} = \dot{x}_2$$

$$x_2 = u - \dot{x}_1$$

$$\dot{x}_2 = x_1 - x_2$$

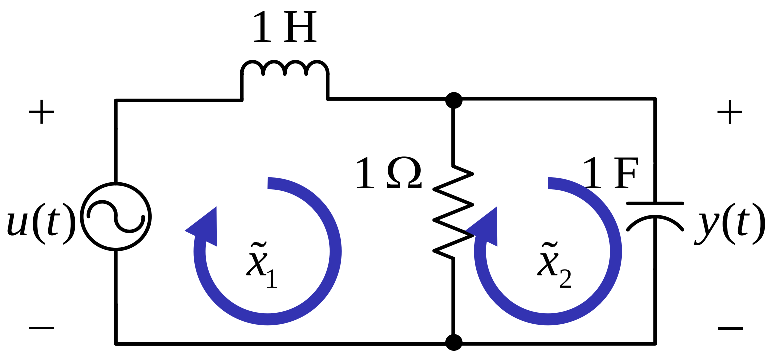
$$y = x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

# Homework 2: Equivalent State Equations

**1.** Prove that for the same system, with different definition of state variables, we can obtain a state space in the form of:




$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

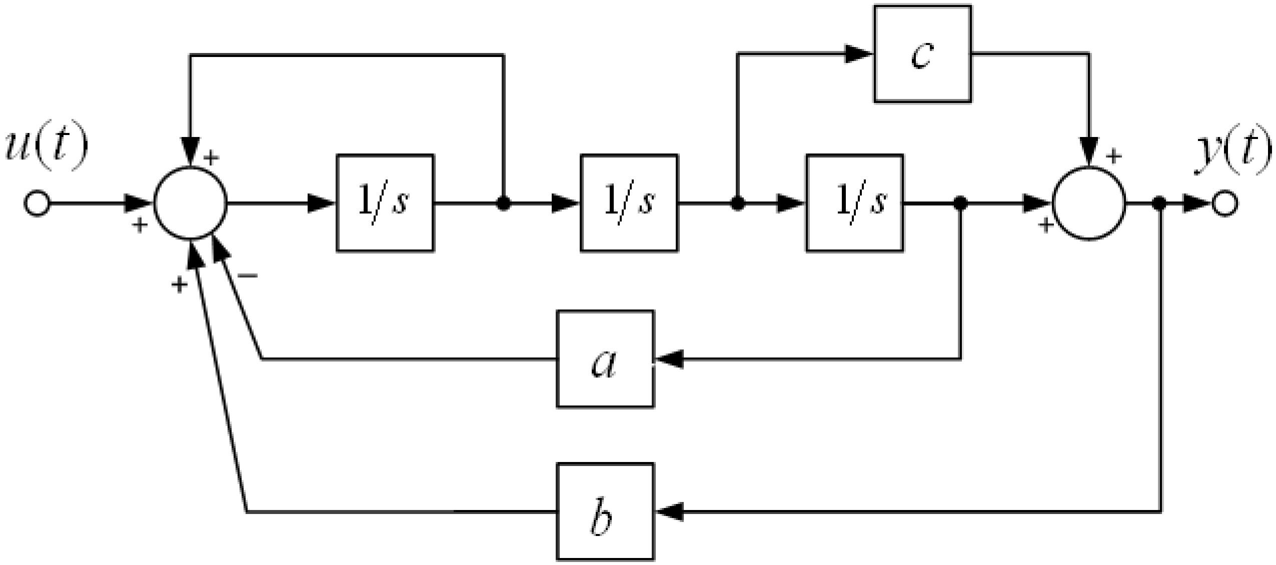
$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

State variables:

- $\tilde{x}_1$ : current of left loop
- $\tilde{x}_2$ : current of right loop

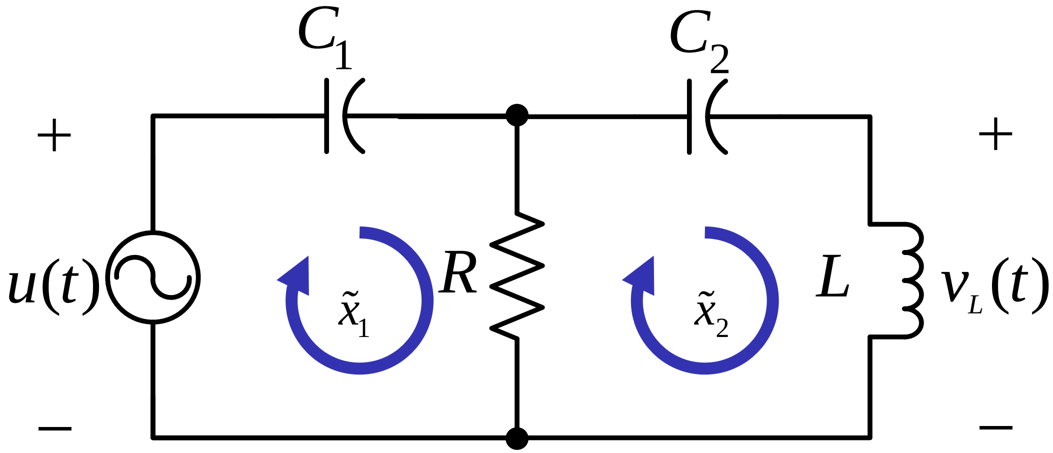
# Homework 2: Equivalent State Equations

**2.** Derive a state-space description for the following diagram



# Homework 2A: Equivalent State Equations

**1.** From Homework 1A, find out whether it is possible to describe the same circuit with different definition of state variables.




State variables:

- $\tilde{x}_1$ : current of left loop
- $\tilde{x}_2$ : current of right loop

# Homework 2A: Equivalent State Equations

**2.** Given the following state space, with zero initial conditions,

$$\underline{\dot{\mathbf{x}}}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \underline{\mathbf{x}}(t) + \begin{bmatrix} -1 \\ 1.5 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & -2 \end{bmatrix} \underline{\mathbf{x}}(t),$$

find the solution for $y(t)$ for a unit step input and draw a sketch of it.