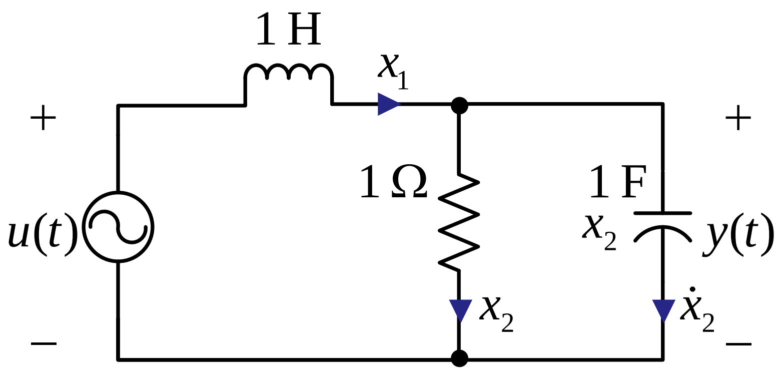
State Space Solutions and Realizations

"Linear System Theory and Design", Chapter 4

http://zitompul.wordpress.com

2014

Equivalent State Equations



$$v_L = L\frac{di_L}{dt} = \dot{x}_1$$

$$i_R = \frac{v_R}{R} = x_2$$

$$i_C = C\frac{dv_C}{dt} = \dot{x}_2$$

$$x_2 = u - \dot{x}_1$$

$$\dot{x}_2 = x_1 - x_2$$

$$y = x_2$$

State variables:

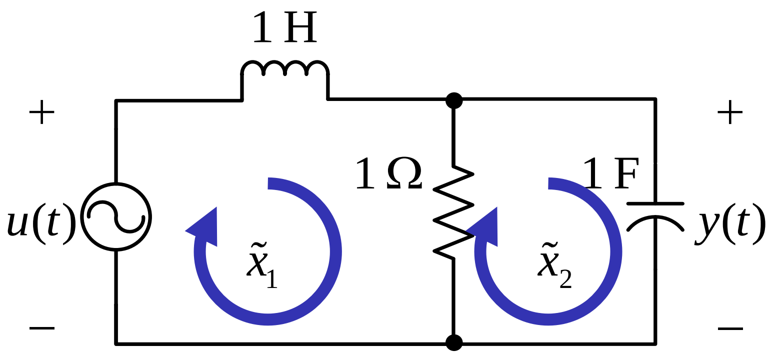
- x_1 : inductor current i_L
- x_2 : capacitor voltage v_C

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Homework 2: Equivalent State Equations

1. Prove that for the same system, with different definition of state variables, we can obtain a state space in the form of:

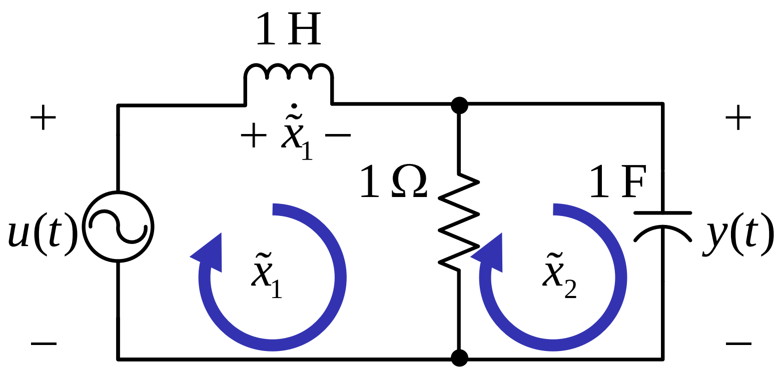


$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

State variables:
- $\tilde{x}_1$: current of left loop
- $\tilde{x}_2$: current of right loop

Homework 2: Equivalent State Equations



$$-u + \dot{\tilde{x}}_1 + (\tilde{x}_1 - \tilde{x}_2) = 0$$

$$\boxed{\dot{\tilde{x}}_1 = u - \tilde{x}_1 + \tilde{x}_2}$$

$$\boxed{v_C = \tilde{x}_1 - \tilde{x}_2 = y}$$

$$i_C = C\frac{dv_C}{dt}$$

State variables:

- $\tilde{x}_1$: loop current left
- $\tilde{x}_2$: loop current right

$$\tilde{x}_2 = \dot{\tilde{x}}_1 - \dot{\tilde{x}}_2$$
$$= (u - \tilde{x}_1 + \tilde{x}_2) - \dot{\tilde{x}}_2$$

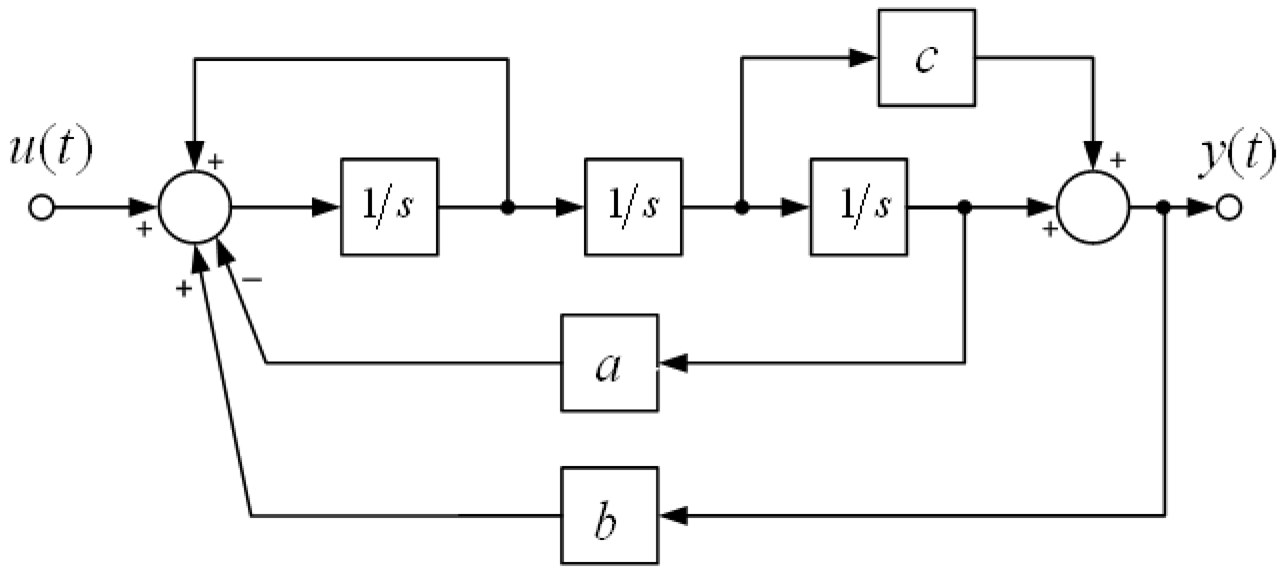
$$\boxed{\dot{\tilde{x}}_2 = u - \tilde{x}_1}$$

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

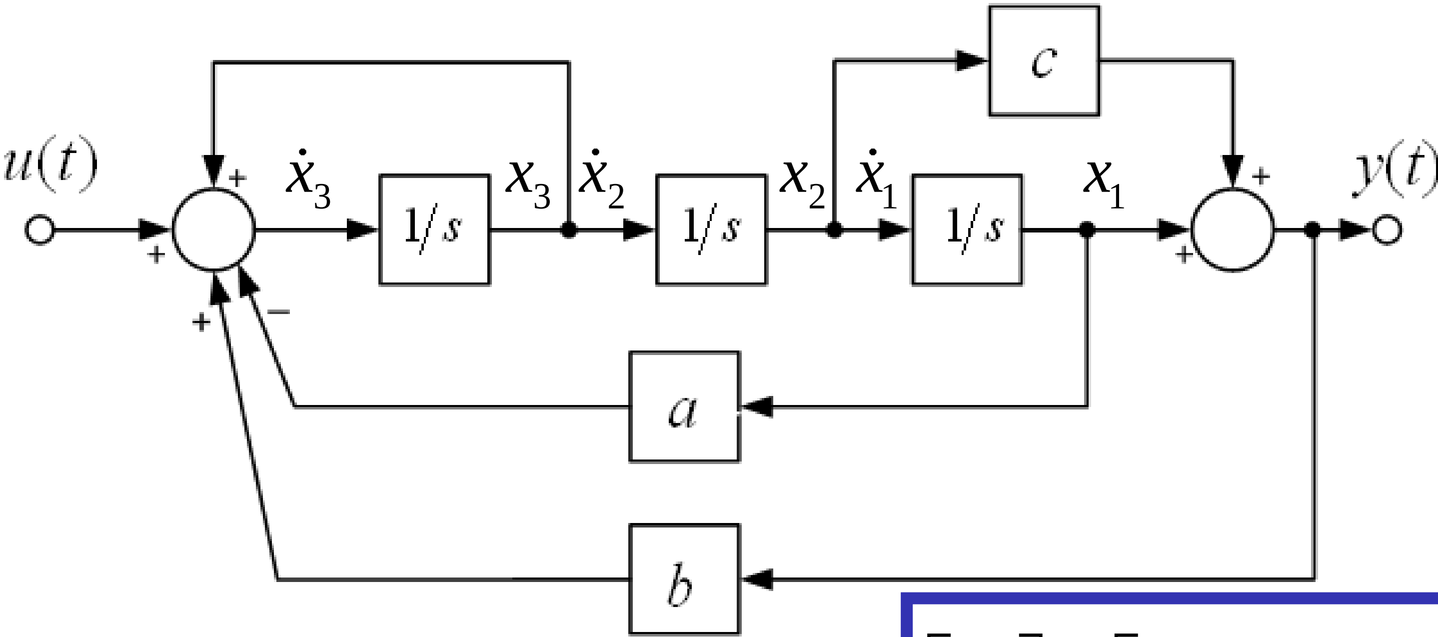
$$y = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

Homework 2: Equivalent State Equations

2. Derive a state-space description for the following diagram



Homework 2: Equivalent State Equations



$$y = x_1 + cx_2$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -ax_1 + x_3 + by + u$$

$$= -ax_1 + x_3 + b(x_1 + cx_2) + u$$

$$= (-a + b)x_1 + bcx_2 + x_3 + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a+b & bc & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & c & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

Equivalent State Equations

- Consider an *n*-dimensional state space equations:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t)$$
$$\underline{y}(t) = \underline{C}\underline{x}(t) + \underline{D}\underline{u}(t)$$

- Let $\underline{P}$ be an $n \times n$ real nonsingular matrix, and let $\underline{\tilde{x}} = \underline{P}\,\underline{x}$. Then, the state space equations

$$\dot{\underline{\tilde{x}}}(t) = \underline{\tilde{A}}\underline{\tilde{x}}(t) + \underline{\tilde{B}}\underline{u}(t)$$
$$\underline{y}(t) = \underline{\tilde{C}}\underline{\tilde{x}}(t) + \underline{\tilde{D}}\underline{u}(t)$$

  where

$$\underline{\tilde{A}} = \underline{P}\,\underline{A}\,\underline{P}^{-1}, \quad \underline{\tilde{B}} = \underline{P}\,\underline{B}, \quad \underline{\tilde{C}} = \underline{C}\,\underline{P}^{-1}, \quad \underline{\tilde{D}} = \underline{D}.$$

  is said to be algebraically equivalent with the original state space equations.

- $\underline{\tilde{x}} = \underline{P}\,\underline{x}$ is called an equivalence transformation.

Equivalent State Equations

- *Proof*:

Substituting $\underline{\mathbf{x}}(t) = \underline{\boldsymbol{P}}^{-1}\tilde{\underline{\mathbf{x}}}(t)$

$$\underline{\boldsymbol{P}}^{-1}\dot{\tilde{\underline{\mathbf{x}}}}(t) = \underline{\boldsymbol{A}}\,\underline{\boldsymbol{P}}^{-1}\tilde{\underline{\mathbf{x}}}(t) + \underline{\boldsymbol{B}}\,\underline{\boldsymbol{u}}(t)$$

$$\dot{\tilde{\underline{\mathbf{x}}}}(t) = \underbrace{\underline{\boldsymbol{P}}\,\underline{\boldsymbol{A}}\,\underline{\boldsymbol{P}}^{-1}}_{\tilde{\underline{\boldsymbol{A}}}}\tilde{\underline{\mathbf{x}}}(t) + \underbrace{\underline{\boldsymbol{P}}\,\underline{\boldsymbol{B}}}_{\tilde{\underline{\boldsymbol{B}}}}\underline{\boldsymbol{u}}(t)$$

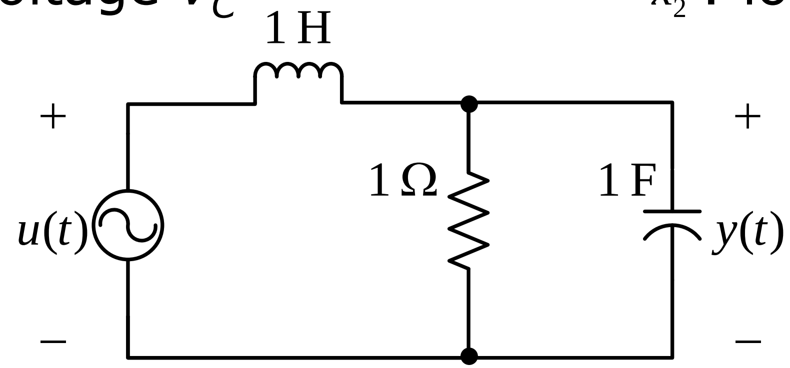
$$\underline{\boldsymbol{y}}(t) = \underbrace{\underline{\boldsymbol{C}}\,\underline{\boldsymbol{P}}^{-1}}_{\tilde{\underline{\boldsymbol{C}}}}\tilde{\underline{\mathbf{x}}}(t) + \underbrace{\underline{\boldsymbol{D}}}_{\tilde{\underline{\boldsymbol{D}}}}\underline{\boldsymbol{u}}(t)$$

Equivalent State Equations

- From the last electrical circuit,

State variables:
- x_1 : inductor current i_L
- x_2 : capacitor voltage v_C

State variables:
- $\tilde{x}_1$: loop current left
- $\tilde{x}_2$: loop current right



- The two sets of states can be related in the way:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

or

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Math Preliminaries

- **Linear Independence**
- The n-dimensional vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n\}$ are linearly dependent if there are n scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that:

$$\alpha_1 \underline{\boldsymbol{v}}_1 + \alpha_2 \underline{\boldsymbol{v}}_2 + \ldots + \alpha_n \underline{\boldsymbol{v}}_n = 0$$

- They are linearly independent if:

$$\sum_{i=1}^{n} \alpha_i \underline{\boldsymbol{v}}_i = 0 \Leftrightarrow \alpha_i = 0 \text{ for } i = 1, 2, \ldots, n$$

Math Preliminaries

- The vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n\}$ are linearly independent *if and only if*

$$\operatorname{rank}\left[\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n\right] = n$$

- Let matrix $\underline{\boldsymbol{A}}$ be $n \times n$. If $\operatorname{rank}(\underline{\boldsymbol{A}}) = r < n$, then there are $(n-r)$ linearly independent vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_{n-r}$ such that

$$\underline{\boldsymbol{A}}\underline{\boldsymbol{v}}_i = \underline{\mathbf{0}}, \text{ for } i = 1, 2, \ldots, n-r$$

- A square matrix is non-singular *if and only if* all its columns are linearly independent.

- Rank of a matrix $\underline{\boldsymbol{A}}$ is the maximum number of linearly independent column (the **column rank**) in $\underline{\boldsymbol{A}}$ or the maximum number of linearly independent rows in $\underline{\boldsymbol{A}}$ (the **row rank**).
- For every matrix, the column rank is equal to the row rank.

Math Preliminaries

- **Basis**
- A basis $\boldsymbol{B} = \{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n\}$ of a vector space $\mathcal{V}$ over a scalar field **F** is a linearly independent subset of $\mathcal{V}$, *i.e.*,

$$\sum \alpha_i \underline{\mathbf{v}}_i = 0 \leftrightarrow \alpha_i = 0 \text{ for } i = 1, 2, \ldots, n$$

that spans $\mathcal{V}$, *i.e*, for every $x \in \mathcal{V}$ there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in$ **F** such that

$$x = \sum_{i=1}^{n} \alpha_i \underline{\mathbf{v}}_i$$

- Every set of *n* linearly-independent vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n\}$ in an *n*-dimensional space $\mathcal{V}$ is a basis of $\mathcal{V}$.
- Way to prove: It can be shown, that the independent set can be reduced (using Gauss-Jordan elimination) to unit vectors of $\mathcal{V}$, and thus spans $\mathcal{V}$.

Math Preliminaries

- **Eigenvalues and Eigenvectors**
- For an $n \times n$ matrix $\underline{\boldsymbol{A}}$, the eigenvalues and eigenvectors are defined by:

$$\underline{\boldsymbol{A}}\underline{\boldsymbol{v}}_i = \lambda_i \underline{\boldsymbol{v}}_i$$

- The matrix has n eigenvalues λ_1, λ_2, …, λ_n with corresponding eigenvectors.
- The eigenvalues are the roots of the characteristic equation

$$\det(s\underline{\boldsymbol{I}} - \underline{\boldsymbol{A}}) = 0$$

- If the eigenvalues are distinct, then the eigenvectors are linearly independent.

Example 1

Find the eigenvalues and the eigenvectors of the matrix below:

$$\underline{\boldsymbol{A}} = \begin{bmatrix} -5 & -4 & -2 \\ -0.5 & -3 & 1 \\ 10 & 14 & 2 \end{bmatrix}$$

Transfer Function and Transfer Matrix

- Consider a state space equations for SISO systems:

$$\dot{\underline{\mathbf{x}}}(t) = \underline{\mathbf{A}}\,\underline{\mathbf{x}}(t) + \underline{\mathbf{B}}u(t)$$

$$y(t) = \underline{\mathbf{C}}\,\underline{\mathbf{x}}(t) + Du(t)$$

- Using Laplace transform, we will obtain:

$$s\underline{\mathbf{X}}(s) - \underline{\mathbf{x}}(0) = \underline{\mathbf{A}}\,\underline{\mathbf{X}}(s) + \underline{\mathbf{B}}U(s)$$

$$Y(s) = \underline{\mathbf{C}}\,\underline{\mathbf{X}}(s) + DU(s)$$

- For zero initial conditions, $\underline{\mathbf{x}}(0) = \underline{\mathbf{0}}$,

$$\underline{\mathbf{X}}(s) = (s\underline{\mathbf{I}} - \underline{\mathbf{A}})^{-1}\underline{\mathbf{B}}U(s)$$

$$Y(s) = \left(\underline{\mathbf{C}}(s\underline{\mathbf{I}} - \underline{\mathbf{A}})^{-1}\underline{\mathbf{B}} + D\right)U(s)$$

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \underline{\mathbf{C}}(s\underline{\mathbf{I}} - \underline{\mathbf{A}})^{-1}\underline{\mathbf{B}} + D}$$

Transfer Function

Example 2

Find the transfer function of the following state space:

$$\underline{\dot{\boldsymbol{x}}} = \begin{bmatrix} -4 & 0 \\ 1 & -2 \end{bmatrix} \underline{\boldsymbol{x}} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0.5 & 1 \end{bmatrix} \underline{\boldsymbol{x}}$$

Realization of State Space Equations

- Every linear time-invariant system can be described by the input-output description in the form of:

$$Y(s) = U(s)G(s)$$

- If the system is lumped (i.e., having concentrated parameters), it can also be described by the state space equations

$$\dot{\underline{\mathbf{x}}}(t) = \underline{\mathbf{A}}\underline{\mathbf{x}}(t) + \underline{\mathbf{B}}u(t)$$

$$y(t) = \underline{\mathbf{C}}\underline{\mathbf{x}}(t) + Du(t)$$

- The problem concerning how to describe a system in state space equations, provided that the transfer function of a system, $G(s)$, is available, is called **Realization Problem**.

$$\boldsymbol{G(s)} \Longrightarrow \underline{\boldsymbol{A}}, \underline{\boldsymbol{B}}, \underline{\boldsymbol{C}}, \underline{\boldsymbol{D}}.$$

Realization of State Space Equations

- Three realization methods will be discussed now:
  - Frobenius Form
  - Observer Form
  - Canonical Form

Frobenius Form

$$G(s)=\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

- **Special Case**: No derivation of input

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 u(t)$$

Frobenius Form

- We now define:

$$
\begin{aligned}
x_1(t) &= y(t) \\
x_2(t) &= \dot{y}(t) = \dot{x}_1(t) \\
x_3(t) &= \ddot{y}(t) = \dot{x}_2(t) \\
&\vdots \\
x_n(t) &= y^{(n-1)}(t) = \dot{x}_{n-1}(t)
\end{aligned}
$$

$$
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & & \vdots \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} u(t)
$$

$$
y(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + 0u(t)
$$

Frobenius Form, Special Case

Frobenius Form

- **General Case**: With derivation of input

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)} \qquad m < n$$

- If $m = n-1$ (largest possible value), then

$$Y(s) = \left(b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \cdots + b_1 s + b_0\right) \frac{U(s)}{D(s)}$$

$$Y(s) = b_0 \underbrace{\frac{U(s)}{D(s)}}_{X_1(s)} + b_1 s \underbrace{\frac{U(s)}{D(s)}}_{X_2(s)} + \cdots + b_{n-2} s^{n-2} \underbrace{\frac{U(s)}{D(s)}}_{X_{n-1}(s)} + b_{n-1} s^{n-1} \underbrace{\frac{U(s)}{D(s)}}_{X_n(s)}$$

Frobenius Form

- If $m = n-1$ (largest possible value), then

$$
\begin{aligned}
x_1(t) &= \mathcal{L}^{-1}\left[X_1(s)\right] \\
x_2(t) &= \dot{x}_1(t) \\
&\vdots \\
x_{n-1}(t) &= \dot{x}_{n-2}(t) \\
x_n(t) &= \dot{x}_{n-1}(t)
\end{aligned}
$$

- But $X_1(s) = \dfrac{U(s)}{D(s)} = \dfrac{U(s)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$

$$s^n X_1(s) + a_{n-1}s^{n-1}X_1(s) + \cdots + a_2 s^2 X_1(s) + a_1 s X_1(s) + a_0 X_1(s) = U(s)$$

$$s^n X_1(s) = U(s) - a_{n-1}s^{n-1}X_1(s) - \cdots - a_2 s^2 X_1(s) - a_1 s X_1(s) - a_0 X_1(s)$$

$$x_1^{(n)}(t) = u(t) - a_{n-1}x_1^{(n-1)}(t) - \cdots - a_2\ddot{x}_1(t) - a_1\dot{x}_1(t) - a_0 x_1(t)$$

$$\dot{x}_n(t) = u(t) - a_{n-1}x_n(t) - \cdots - a_2 x_3(t) - a_1 x_2(t) - a_0 x_1(t)$$

Frobenius Form

- The state space equations can now be written as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & & \vdots \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 & b_1 \cdots b_{n-2} & b_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + 0u(t)$$

Frobenius Form, General Case

Example 3

Find the state-space realization of the following ordinary differential equation, where the initial conditions are zero.

$$\frac{d^3y}{dt^3}+5\frac{d^2y}{dt^2}+\frac{dy}{dt}+2y=u$$

Let
$$x_1 = y$$
$$x_2 = \dot{x}_1 = y'$$
$$x_3 = \dot{x}_2 = y''$$

$$\Rightarrow \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & -1 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Example 3

Find the state-space realization of the following ordinary differential equation, where the initial conditions are zero.

$$\frac{d^3y}{dt^3}+5\frac{d^2y}{dt^2}+\frac{dy}{dt}+2y=u$$

Alternatively,

$$s^3Y(s)+5s^2Y(s)+sY(s)+2Y(s)=U(s)$$

$$G(s)=\frac{Y(s)}{U(s)}=\frac{1}{s^3+5s^2+s+2}$$

$a_2 \quad a_1 \quad a_0$

$$\Rightarrow \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Observer Form

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \qquad n = m+1$$

$$s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_1 s Y(s) + a_0 Y(s) = b_{n-1}s^{n-1}U(s) + b_{n-2}s^{n-2}U(s) + \cdots + b_1 s U(s) + b_0 U(s)$$

$$Y(s) + a_{n-1}\frac{Y(s)}{s} + \cdots + a_1 \frac{Y(s)}{s^{n-1}} + a_0 \frac{Y(s)}{s^n} = b_{n-1}\frac{U(s)}{s} + b_{n-2}\frac{U(s)}{s^2} + \cdots + b_1 \frac{U(s)}{s^{n-1}} + b_0 \frac{U(s)}{s^n}$$

$$Y(s) = \frac{1}{s}\left\{\left(b_{n-1}U(s) - a_{n-1}Y(s)\right) + \frac{1}{s}\left\{\left(b_{n-2}U(s) - a_{n-2}Y(s)\right) + \frac{1}{s}(\cdots) + \underbrace{\frac{1}{s}\left\{b_0 U(s) - a_0 Y(s)\right\}}_{X_1(s)}\right\}\cdots\right\}$$

Observer Form

$$X_1(s) = \frac{1}{s}\left\{b_0 U(s) - a_0 Y(s)\right\} \longrightarrow \dot{x}_1(t) = b_0 u(t) - a_0 y(t)$$

$$X_2(s) = \frac{1}{s}\left\{\left(b_1 U(s) - a_1 Y(s)\right) + X_1(s)\right\} \longrightarrow \dot{x}_2(t) = b_1 u(t) - a_1 y(t) + x_1(t)$$

$$\vdots \qquad\qquad \vdots$$

$$X_n(s) = \frac{1}{s}\left\{\left(b_{n-1} U(s) - a_{n-1} Y(s)\right) + X_{n-1}(s)\right\} \longrightarrow \dot{x}_n(t) = b_{n-1} u(t) - a_{n-1} y(t) + x_{n-1}(t)$$

$$Y(s) = X_n(s) \longrightarrow y(t) = x_n(t)$$

Observer Form

- The state space equations in observer form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & -a_0 \\ 1 & 0 & \cdots & -a_1 \\ \vdots & \ddots & & \vdots \\ 0 & 0 \cdots 1 & & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + 0u(t)$$

Observer Form

Example 4

Find the state-space realization of the following transfer function in Frobenius Form.

$$G(s)=\frac{4s^3+25s^2+45s+34}{2s^3+12s^2+20s+16}$$

$$G(s)=\frac{4s^3+25s^2+45s+34}{2s^3+12s^2+20s+16}=2+\frac{s^2+5s+2}{2s^3+12s^2+20s+16}=2+\frac{\frac{1}{2}s^2+2\frac{1}{2}s+1}{s^3+6s^2+10s+8}$$

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0&0&-a_0\\1&0&-a_1\\0&1&-a_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}b_0\\b_1\\b_2\end{bmatrix}u$$

$$y=\begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+0u(t)$$

$$\Rightarrow$$

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0&0&-8\\1&0&-10\\0&1&-6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}1\\2\frac{1}{2}\\\frac{1}{2}\end{bmatrix}u$$

$$y=\begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+2u(t)$$

Canonical Form

- To construct state space equations in canonical form, we need to perform partial fraction decomposition to the respective transfer function.

$$Y(s)=\frac{N(s)}{D(s)}U(s) \quad =\left\{\sum_{i=1}^{n}\frac{r_i}{s-\lambda_i}+r_0\right\}U(s)$$

- In case **all poles are distinct**, we define:

$$X_1(s)=\frac{1}{s-\lambda_1}U(s) \longrightarrow \dot{x}_1(t)=\lambda_1 x_1(t)+u(t)$$

$$X_2(s)=\frac{1}{s-\lambda_2}U(s) \longrightarrow \dot{x}_2(t)=\lambda_2 x_2(t)+u(t)$$

$$\vdots$$

$$X_n(s)=\frac{1}{s-\lambda_n}U(s) \longrightarrow \dot{x}_n(t)=\lambda_n x_n(t)+u(t)$$

$$Y(s)=r_1X_1(s)+r_2X_2(s)+\cdots+r_nX_n(s)+r_0U(s) \longrightarrow y(t)=r_1x_1(t)+r_2x_2(t)+\cdots+r_nx_n(t)+r_0u(t)$$

Canonical Form

- The state space equations in case all poles are distinct:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

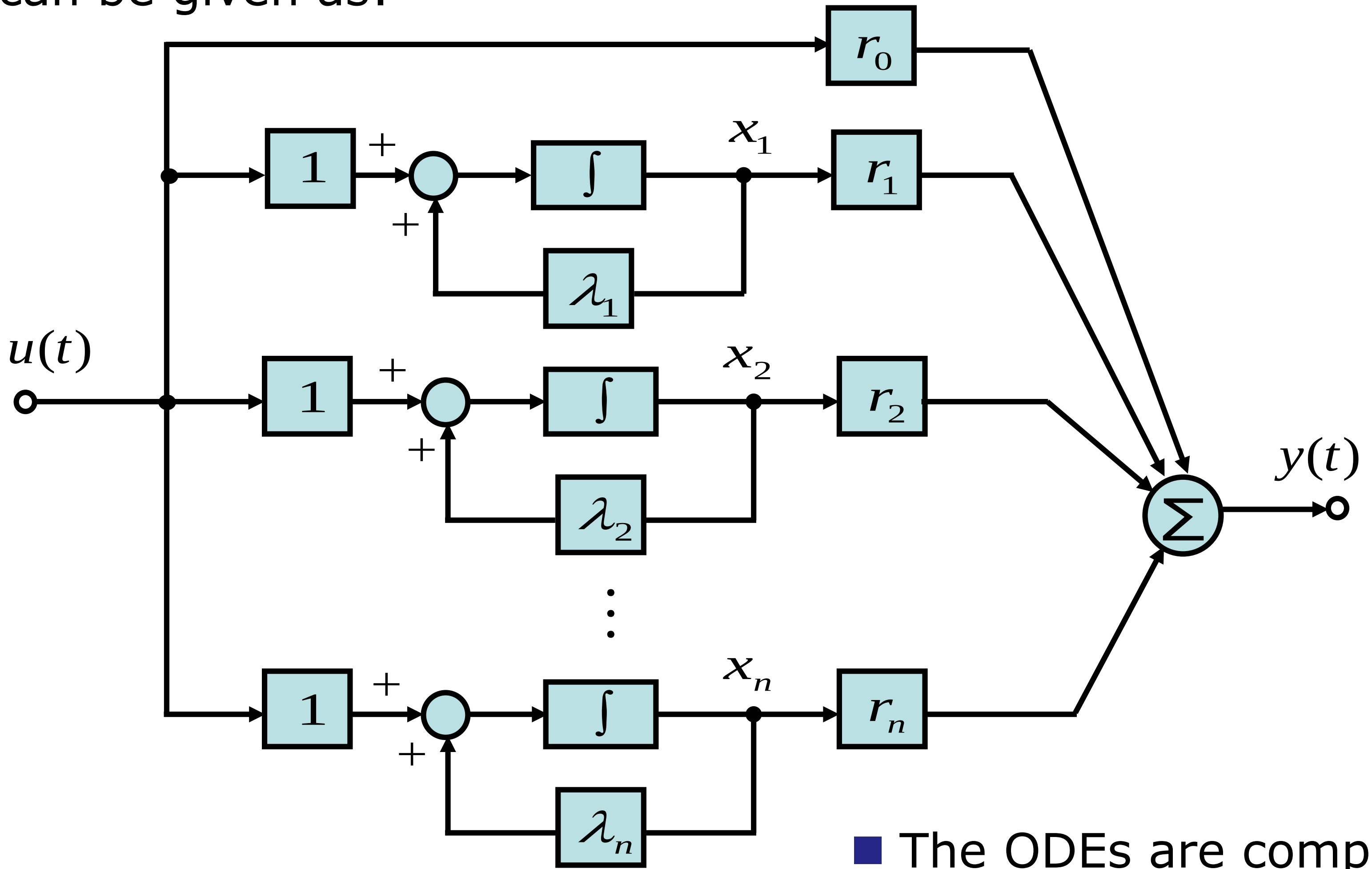
$$y(t) = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + r_0 u(t)$$

Canonical Form, Distinct Poles

- **The resulting matrix $\underline{A}$ is a diagonal matrix.**
- **The ODEs are decoupled, each of them can be solved independently.**

Canonical Form

- The block diagram of the state space equations in Canonical Form can be given as:



- The ODEs are completely decoupled from each other.
- Controllability, Observability

Canonical Form

- In case of **repeating poles**, for example λ_1 is repeated for *p* times, the decomposed equation will be:

$$Y(s)=\left\{r_0+\frac{r_{11}}{s-\lambda_1}+\frac{r_{12}}{(s-\lambda_1)^2}+\cdots+\frac{r_{1p}}{(s-\lambda_1)^p}+\frac{r_2}{s-\lambda_2}+\cdots+\frac{r_{n-p+1}}{s-\lambda_{n-p+1}}\right\}U(s)$$

- We define:

$$X_1(s)=\frac{1}{s-\lambda_1}U(s) \longrightarrow \dot{x}_1(t)=\lambda_1 x_1(t)+u(t)$$

$$X_2(s)=\frac{1}{(s-\lambda_1)^2}U(s)$$

$$=\frac{1}{s-\lambda_1}X_1(s) \longrightarrow \dot{x}_2(t)=\lambda_1 x_2(t)+x_1(t)$$

- $x_1(t)$ coupled with $x_2(t)$

$$\vdots$$

$$X_p(s)=\frac{1}{(s-\lambda_1)^p}U(s)$$

- $x_{p-1}(t)$ coupled with $x_p(t)$

$$=\frac{1}{s-\lambda_1}X_{p-1}(s) \longrightarrow \dot{x}_p(t)=\lambda_1 x_p(t)+x_{p-1}(t)$$

Canonical Form

$$X_{p+1}(s)=\frac{1}{s-\lambda_2}U(s) \longrightarrow \dot{x}_{p+1}(t)=\lambda_2 x_{p+1}(t)+u(t)$$

$$\vdots \qquad\qquad \vdots$$

$$X_n(s)=\frac{1}{s-\lambda_{n-p+1}}U(s) \longrightarrow \dot{x}_n(t)=\lambda_{n-p+1}x_n(t)+u(t)$$

$$\begin{aligned}Y(s)=&\,r_{11}X_1(s)+r_{12}X_2(s)+\\ &\cdots+r_{1p}X_p(s)+r_2X_{p+1}(s)+\\ &\cdots+r_{n-p+1}X_n(s)+r_0U(s)\end{aligned} \longrightarrow \begin{aligned}y(t)=&\,r_{11}x_1(t)+r_{12}x_2(t)+\\ &\cdots+r_{1p}x_p(t)+r_2x_{p+1}(t)+\\ &\cdots+r_{n-p+1}x_n(t)+r_0u(t)\end{aligned}$$

Canonical Form

- The state space equations in case of repeating poles:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_p(t) \\ \dot{x}_{p+1}(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & & \cdots & & & 0 \\ 1 & \lambda_1 & 0 & & \cdots & & 0 \\ 0 & \ddots & \lambda_1 & \ddots & & & \vdots \\ 0 & \cdots & 1 & \lambda_1 & & & \\ \vdots & & & 0 & \lambda_2 & & 0 \\ & & & & \ddots & \ddots & \vdots \\ 0 & & \cdots & & & 0 & \lambda_{n-p+1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_p(t) \\ x_{p+1}(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

Canonical Form, Repeating Poles

Canonical Form

- The state space equations in case of repeating poles:

$$y(t) = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1p} & r_2 & \cdots & r_{n-p+1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_p(t) \\ x_{p+1}(t) \\ \vdots \\ x_n(t) \end{bmatrix} + r_0 u(t)$$

Canonical Form, Repeating Poles

Homework 3: Transfer Function - State Space

- Find the state-space realizations of the following transfer function in *Frobenius Form*, *Observer Form*, and *Canonical Form*.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^3+8s^2+19s+12}$$

- **Hint**: Learn the following functions in Matlab and use the to solve this problem: **roots**, **residue**, **convolution**.

Homework 3A: Transfer Function - State Space

- Perform a step by step transformation (by calculation of transfer matrix) from the following state-space equations to result the corresponding transfer function.

$$\dot{\underline{\boldsymbol{x}}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -2 \end{bmatrix} \cdot \underline{\boldsymbol{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot u(t)$$

$$y(t) = \begin{bmatrix} 5 & 1 & 0 \end{bmatrix} \cdot \underline{\boldsymbol{x}}(t)$$

- Verify your calculation result using Matlab.
- **Hint**: Learn the following functions in Matlab and use the to solve this problem: **ss2tf**, **tf2ss**.