Chapter 5

Transient and Steady-State Response Analysis

Outline

- Introduction
- Time response of first-order systems
  - Properties of first-order system
- Time response of a prototype 2nd-order system
  - Properties of 2nd-order systems
  - The effect of pole locations on 2nd-order systems
  - Effect of adding poles and zeros
- Stability of linear systems (in complex plane)
  - Routh-Hurwithz Criterion
- Steady-state error

Introduction

Time-Response Behavior

- Since time is used as an independent variable in most control systems, it is usually of interest to evaluate the output response with respect to time, or simply, the time response.
- When you design a system, the time response behavior may well be the most important aspect of its behavior
- Many design criteria are based on the system response to test signals
- Typical test signals are step, ramp, acceleration, impulse, sinusoidal functions, and white noise

Time-Response Behavior (2)

Points you might worry about include

- How quickly a system responds is important.
    - If you have a control system that's controlling a temperature, how long it takes the temperature to reach a new steady state is important
- Overshoot and how close a system comes to instability.
    - Say you're trying to control a temperature, and you want the temperature to be 200° C. If the temperature goes to 250º C before it settles out, you'll want to know that
- Oscillations in a system are not usually desirable
    - If you're trying to control speed of an automobile at 55mph and the speed keeps varying between 50mph and 60mph, your design isn't very good

Time-Response Behavior (3)

Points you might worry about include

- These are but a few of many different aspects of time behavior of a system that are important in control system design.

The points above really are talking about aspects like:

- Speed of response
- Relative stability of the system
- Stability of the system

Stability and Steady-State Error

- The time response of a control system consists of two parts: the transient response and the steady-state response
- The most important characteristic of the dynamic behavior of a control system is absolute stability
- A control system is in equilibrium if, in the absence of any disturbance or input, the output stays in the same state
- An LTI control system is stable if the output eventually comes back to its equilibrium state when the system is subjected to an initial condition
- An LTI control system is critically stable if oscillations of the output continue forever
- It is unstable if the output diverges without bound from its equilibrium state when the system is subjected to an initial condition

Stability and Steady-State Error (2)

- Other important system behaviors include relative stability and steady-state error
- The transient response of a practical control system often exhibits damped oscillations before reaching a steady state
- If the output of a system at steady state does not exactly agree with the input, the system is said to have steady-state error
- In analyzing a control system, we must examine transient-response behavior and steady-state behavior

Time Response of First-Order Systems

Time Response of First-Order Systems

- What is a first-order system?
  - First order systems are described by first order differential equations.

Example

First-order differential equation:

$$\tau \frac{dy(t)}{dt} + y(t) = Ku(t)$$

***y*(*t*)---output response of the system;** ***u*(*t*)---input to the system**

Using Laplace transform and assuming zero initial conditions, we get:

$$\tau s Y(s) + Y(s) = KU(s)$$

Transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1}$$

τ - time constant

K - DC gain

Time Response of First-Order Systems (2)

- Why learn about first order systems?
  - First-order systems are the simplest systems, and they make a good place to begin a study of system dynamics.
  - First-order system concepts form the foundation for understanding more complex systems.
  - Everything starts here…

Time Response of First-Order Systems (3)

- A number of goals
  - First, if you have a first-order system, you need to be able to predict and understand how it responds to an input, so you need to be able to do this.

Given a first-order system

↓

Determine the impulse and step response of the system.

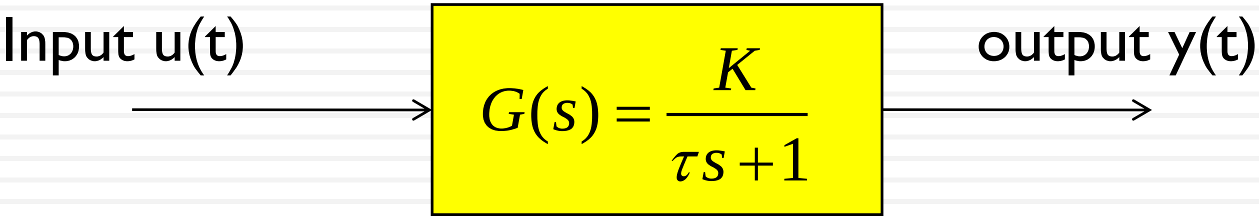
Time Response of First-Order Systems (4)

- A number of goals
  - Secondly, you may go into a lab and measure a system, and if it is first order, you need to be able to do this.

Given the time response of a first order system

Determine the parameters of the system (time constant and DC gain)

Parameters of First-order Systems



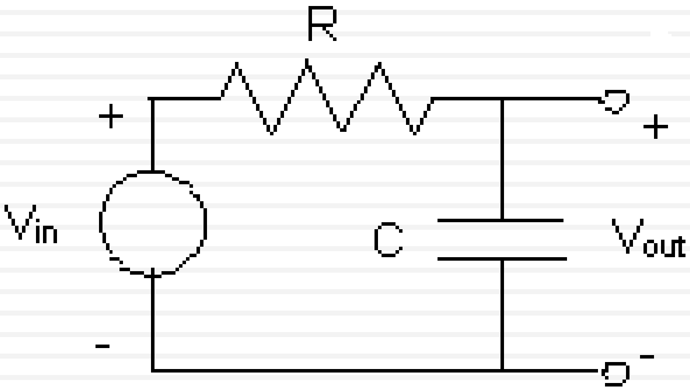
$$G(s) = \frac{K}{\tau s + 1}$$

τ, the time constant, will determine how quickly the system moves toward steady state.

K, the DC gain of the system, will determine the size of steady state response when the input settles out to a constant value.

Example Systems

A Resistor-Capacitor Circuit



Differential equation:

$$RC\frac{dV_{out}}{dt} + V_{out} = V_{in}$$

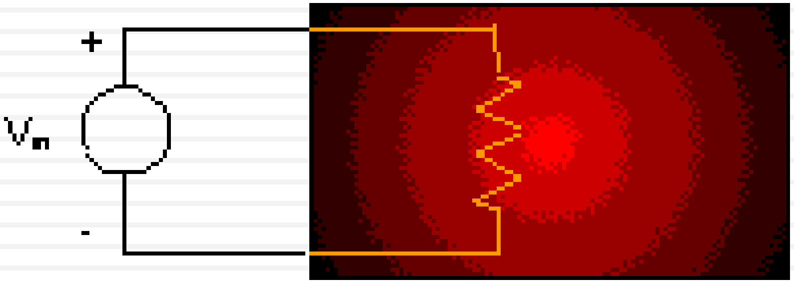
Transfer function:

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs+1}$$

Example Systems

A Simple Thermal System

In this system, heat flows into a heated space and the temperature within the heated space follows a first order linear differential equation.

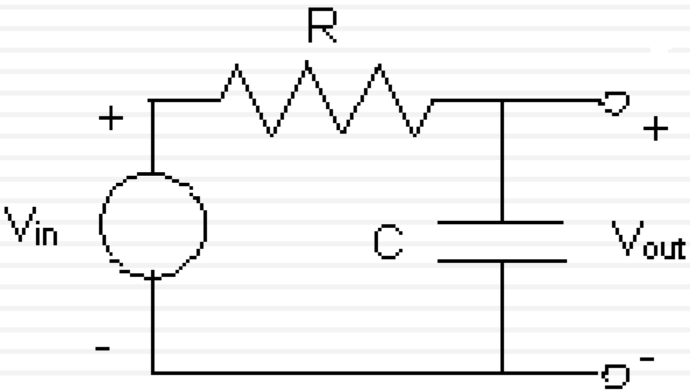


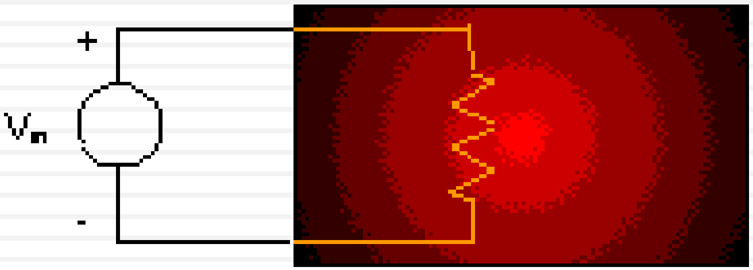
Differential equation:

$$\tau \frac{dTemp}{dt} = -Temp + K \cdot Heat_{in}$$

Transfer function:

$$G(s) = \frac{Temp(s)}{Heat_{in}(s)} = \frac{K}{\tau s + 1}$$





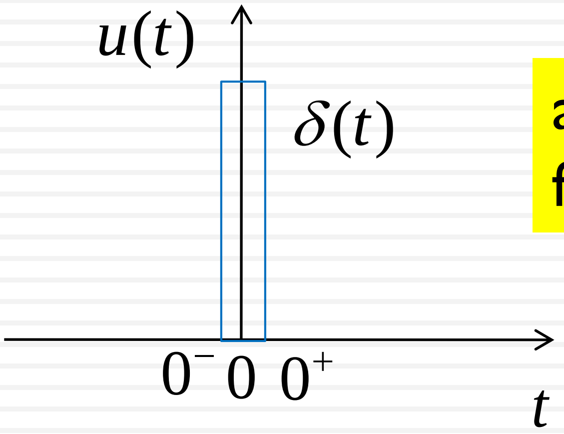
- The systems above come from very diverse places, including circuit theory, thermal dynamics, etc.
- However, there is a common mathematical description for all of those systems.
- That's what you need to learn the properties of a general first-order system.

- When you learn about first-order system dynamics you are learning a topic that:
  - Has applicability to a wide variety of areas
  - Is a good introduction to more complex system dynamics, like second-order systems and more complex systems of higher order.

- We'll start by learning how a first order system responds to two inputs:
  - unit impulse input
  - unit step input
  - Ramp input

Impulse Response of a First-Order System

- The impulse response is the response to a unit impulse input $\delta(t)$.



a very intense force
for a very short time

- The unit impulse has a Laplace transform of unity (1).

$$L[\delta(t)]=1$$

- That gives the unit impulse a unique stature.

For a first-order systems satisfy this generic differential equation

$$\tau \frac{dy(t)}{dt} + y(t) = Ku(t)$$

τ - time constant

K - DC gain

For a unit impulse input $\delta(t)$ and assuming zero initial conditions, could you calculate its output?

$$y(t) = \frac{K}{\tau} e^{-t/\tau}$$

The impulse response is the inverse transform of the transfer function of the above system:

$$G(s) = \frac{K}{\tau s + 1}$$

Now, we need to examine what the impulse response looks like…

Example 1

Consider a first-order system with the following parameters.

Time constant $\tau = 0.1s\ ;$ DC gain $K = 20$

The problem is to determine the unit impulse response of a system that has these parameters.

Solution. **Using the general form of the impulse response of first –order systems**

$$y(t) = \frac{K}{\tau} e^{-t/\tau}$$

With the parameters above, the impulse response is:

$$y(t) = \frac{K}{\tau} e^{-t/\tau} = \frac{20}{0.1} e^{-t/0.1} = 200e^{-10t}$$

What value does the impulse response start from?

Example 1

Consider a first-order system with the following parameters.

Time constant $\tau = 0.1s$; DC gain $K = 20$

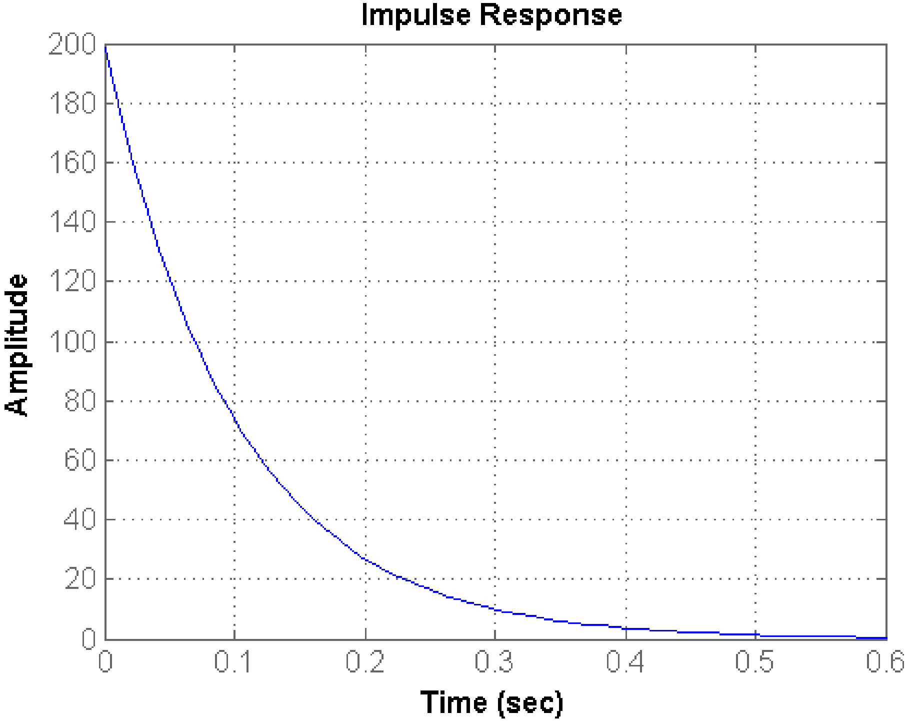
The problem is to determine the unit impulse response of a system that has these parameters.

Solution. $y(t) = \frac{K}{\tau} e^{-t/\tau} = \frac{20}{0.1} e^{-t/0.1} = 200e^{-10t}$

Using MATLAB to get its impulse response

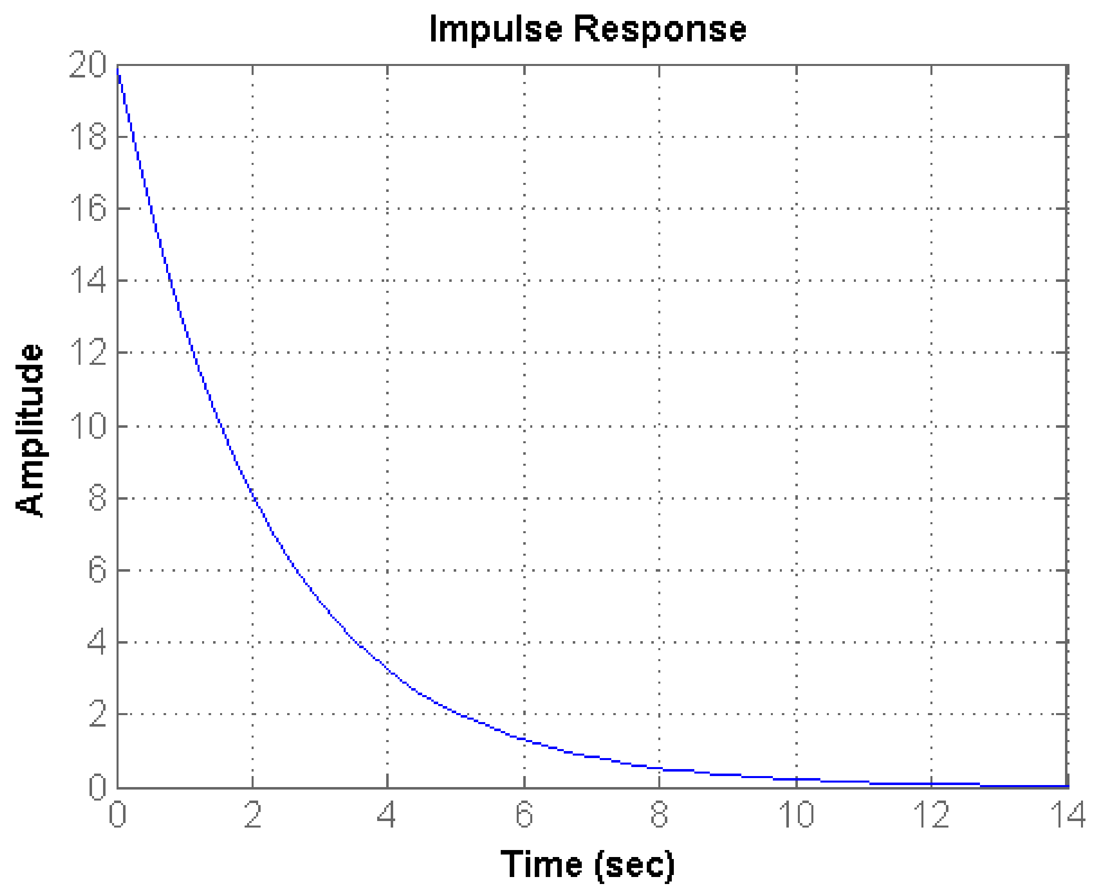
$$G(s) = \frac{K}{\tau s + 1} = \frac{20}{0.1s + 1}$$

```
>> num=20;
>> den=[0.1  1];
>> impulse(num,den)
```



Example 2

Below is the impulse response of a system - i.e. the response to a unit impulse.



The system starts with an initial condition of zero just before the impulse comes along at t = 0, so y(0-) = 0.

Could you compute the parameters of the system?

Solution.

The general time response of a first-order system is

$$y(t) = \frac{K}{\tau} e^{-t/\tau} \qquad (1)$$

From the right figure, we have

$$\text{at } t = 0,\ y(0) = 20$$

So we can get

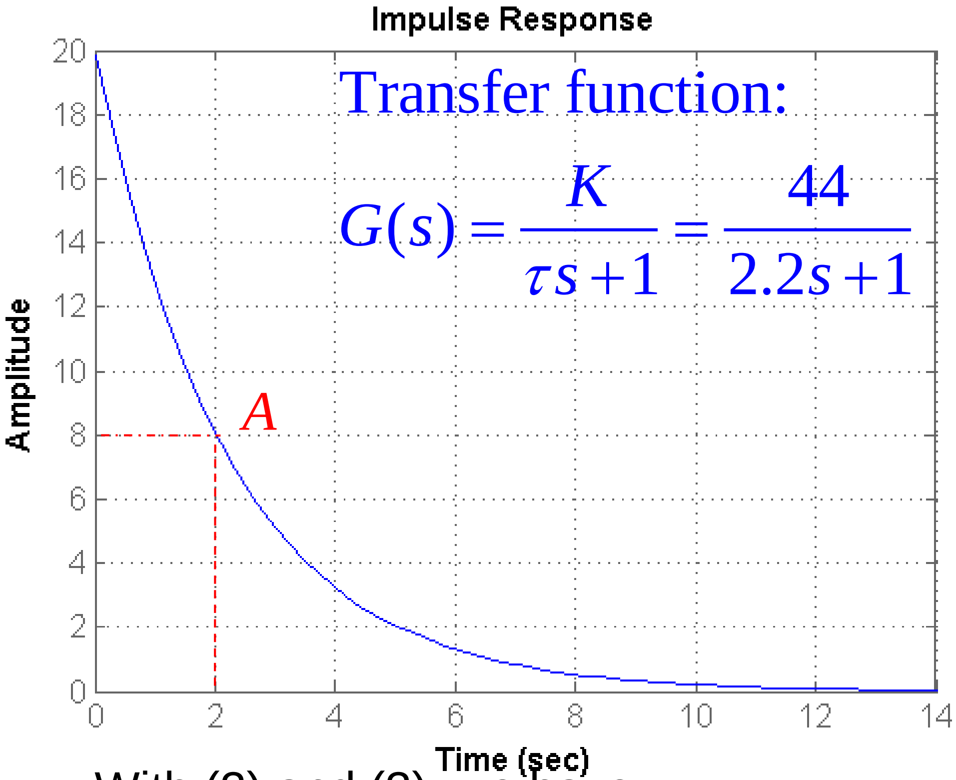
$$\frac{K}{\tau} = 20 \qquad (2)$$

How to get *K* ?

Pick up a point on the response curve:

***A*:** $\text{at } t = 2\,\text{sec}; y(2) = 8$

$$y(2) = \frac{K}{\tau} e^{-2/\tau} = 8 \qquad (3)$$



Transfer function:

$$G(s) = \frac{K}{\tau s + 1} = \frac{44}{2.2s + 1}$$

With (2) and (3), we have

$$y(2) = 20e^{-2/\tau} = 8$$

Solving the equation yields: $e^{-2/\tau} = 0.4$

$$-2/\tau = \ln(0.4) = -0.9163$$

$$\tau \approx 2.2\,\text{sec} \qquad K = 20\tau = 44$$

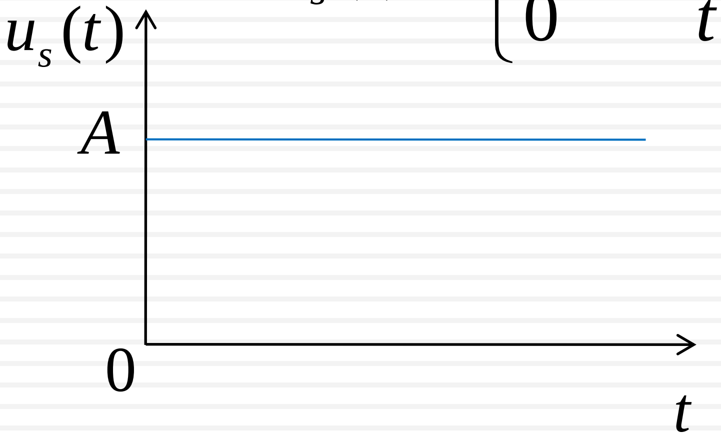
Few Conclusions From the Examples

- Calculating the impulse response is straight-forward.
    - Given the system parameters it is not difficult to calculate - predict - the response of the system.
- The inverse problem is somewhat more difficult.
    - Given a response, you will have to be more inventive to determine what the system was that produced the given response - the system identification problem.
- The underlying theory is the same.
    - You use the same general principles to solve both problems, but the way you have to use the information makes the identification problem more difficult.

Step Response of First-Order Systems

- The step response is the response to a step input $u_s(t)$.

$$u_s(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$



When the magnitude of the step input is 1, it is called a unit-step input, denoted by $1(t)$.

- The Laplace transform of the unit step $1(t)$ is

$$L[1(t)] = \frac{1}{s}$$

For a first-order systems satisfy this generic differential equation

$$\tau \frac{dy(t)}{dt} + y(t) = Ku(t)$$

τ - time constant

K - DC gain

For a unit step input $u(t) = 1(t)$ and assuming zero initial conditions, could you calculate its output?

$$\left.\begin{aligned} &\tau sY(s) + Y(s) = KU(s) \\ &U(s) = L[1(t)] = \frac{1}{s} \end{aligned}\right\} \Rightarrow (\tau s + 1)Y(s) = \frac{K}{s}$$

$$\Rightarrow Y(s) = \frac{K}{s(\tau s + 1)} = K\left(\frac{1}{s} - \frac{\tau}{\tau s + 1}\right) \Rightarrow y(t) = K(1 - e^{-t/\tau})$$

Can you imagine what the response looks like?

$t = 0 \quad y(t) = 0$

$t \to \infty \quad y(t) \to K$

Example 3

Consider a first-order system with the following parameters.

Time constant $\tau = 0.1s$; DC gain $K = 20$

The problem is to determine the unit step response of a system that has these parameters.

Solution.

Using the general form of the unit-step response of first –order systems

$$y(t) = K(1 - e^{-t/\tau})$$

With the parameters above, the impulse response is:

$$y(t) = 20(1 - e^{-t/0.1}) = 20(1 - e^{-10t})$$

Example 3 Time constant $\tau = 0.1s$; DC gain $K = 20$

Solution. **Unit-step time response:** $y(t) = 20(1 - e^{-10t})$

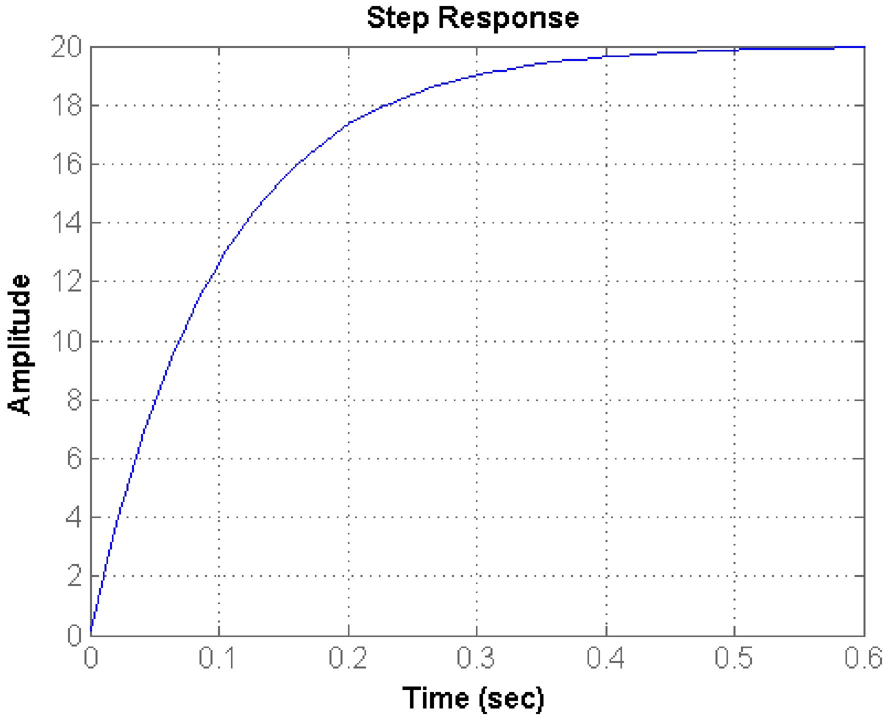
Transfer function of the first-order system:

$$G(s) = \frac{K}{\tau s + 1} = \frac{20}{0.1s + 1}$$

Using MATLAB to get its step response

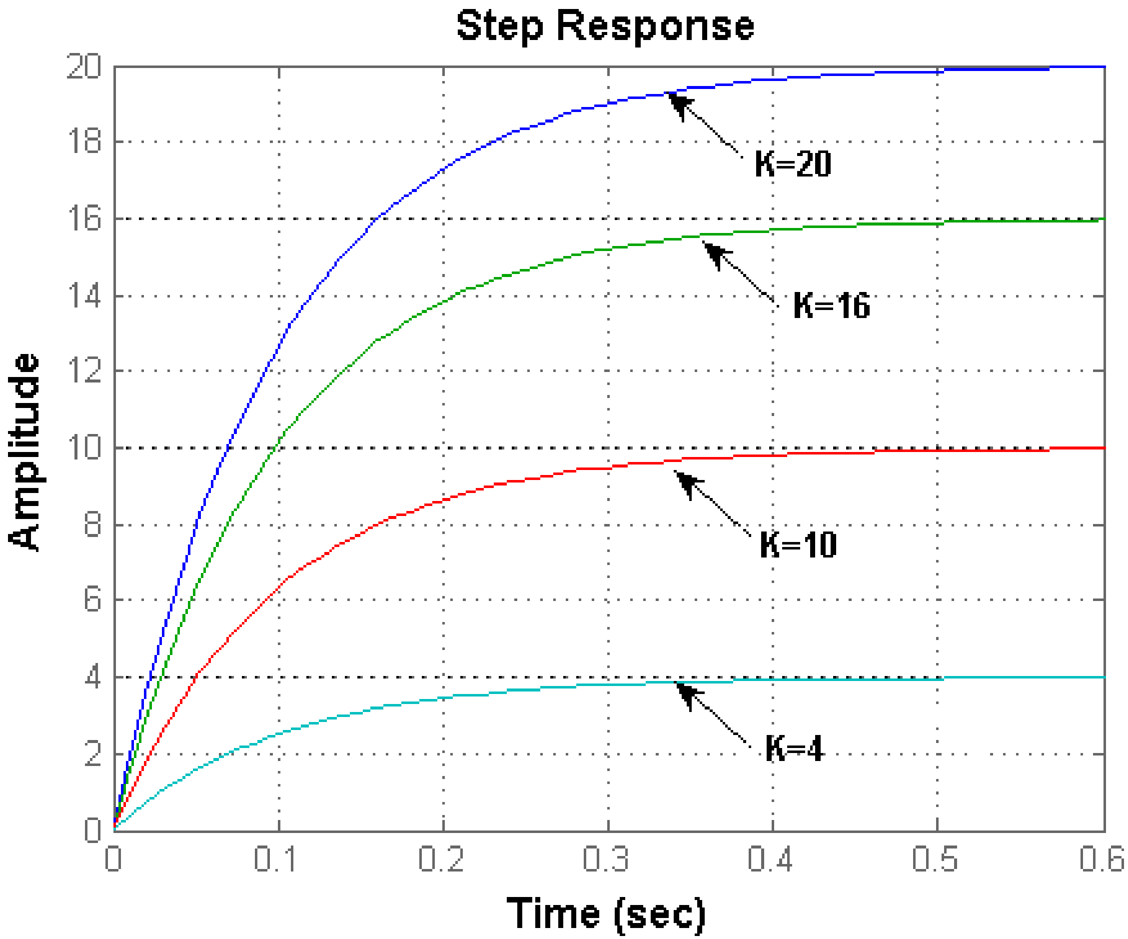
```
>> num=20;
>> den=[0.1 1];
>> step(num,den)
```

If we keep $\tau = 0.1s$ but change K, what will happen?



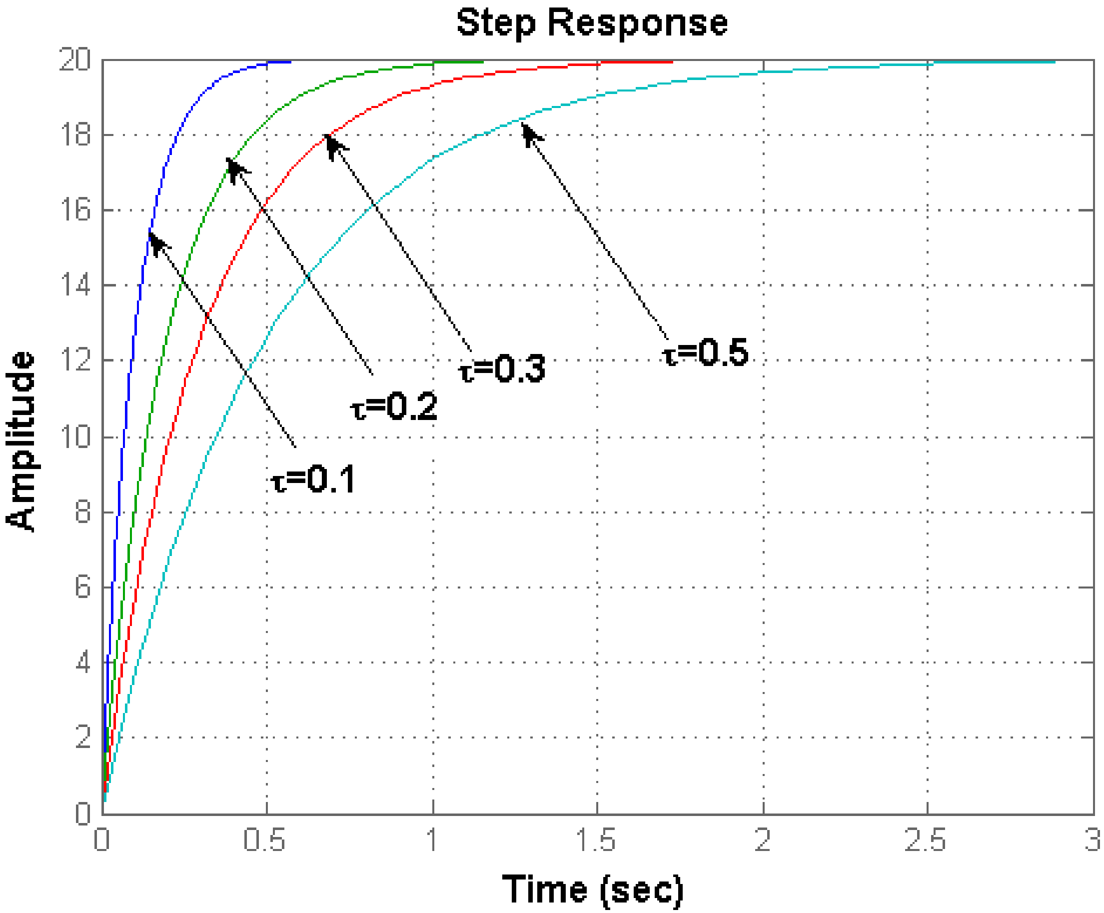
Time constant $\tau = 0.1s$;

DC gain $K = 20 \rightarrow 16 \rightarrow 10 \rightarrow 4$



DC gain $K = 20$

Time constant $\tau = 0.1s \rightarrow 0.2s \rightarrow 0.3s \rightarrow 0.5s$



Note: the time constant reflects the system inertia. The smaller is the system inertia, the shorter is τ and the quicker is the response, *vice versa.*

Two important properties of the unit-step response of a first-order system:

a) The time constant τ can be used to calculate the system output

$$t=\tau,\ y(\tau)=0.632y(\infty)=63.3\%\text{ final value};$$
$$t=2\tau,\ y(2\tau)=0.865y(\infty)=86.5\%\text{ final value};$$
$$t=3\tau,\ y(3\tau)=0.950y(\infty)=95.0\%\text{ final value};$$
$$t=4\tau,\ y(4\tau)=0.982y(\infty)=98.2\%\text{ final value};$$

experiment methods

-- estimate the time constant

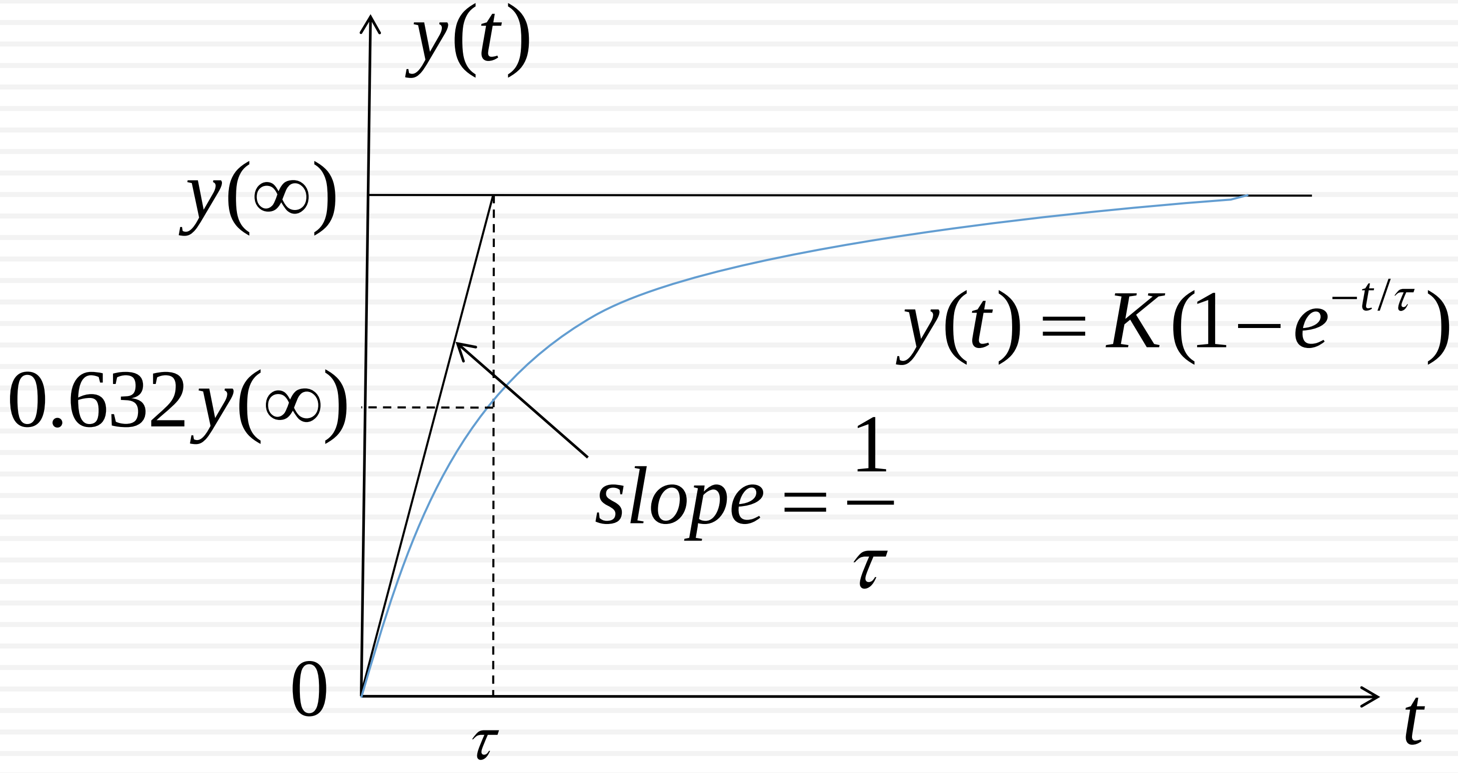
--judge whether a system is first-order or not

b) The initial slope of the response curve is $1/\tau$ and the slope decreases with time

$$\left.\frac{dy(t)}{dt}\right|_{t=0}=\frac{1}{\tau},\quad \left.\frac{dy(t)}{dt}\right|_{t=\tau}=0.368\frac{1}{\tau},\quad \left.\frac{dy(t)}{dt}\right|_{t=\infty}=0.$$

It is also a common method to obtain the time constant though the initial slope in control engineering.

Unit-step response of a first-order system

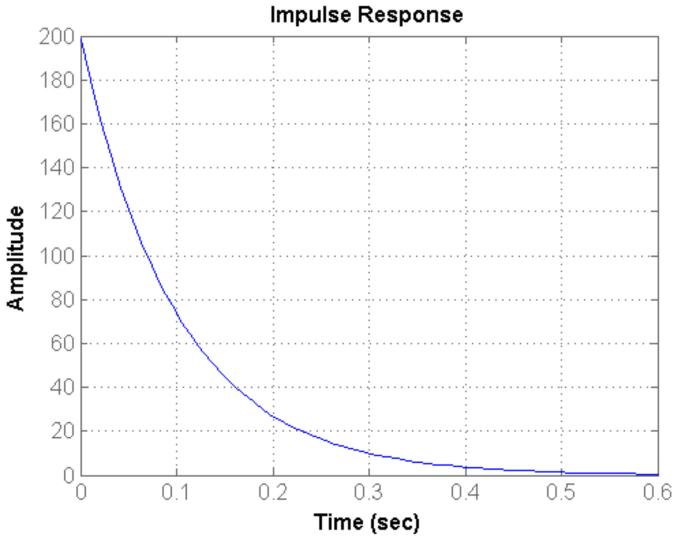


Encountering I-order Systems

$$G(s) = \frac{K}{\tau s + 1}$$

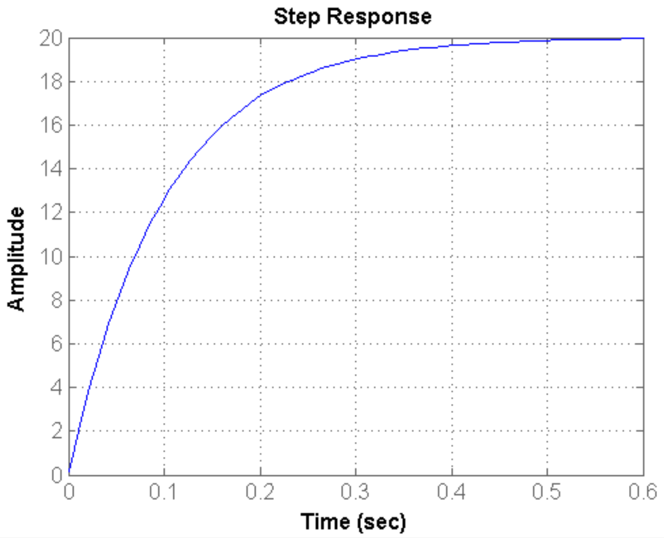
Unit-impulse response:

$$y(t) = \frac{K}{\tau} e^{-t/\tau}$$



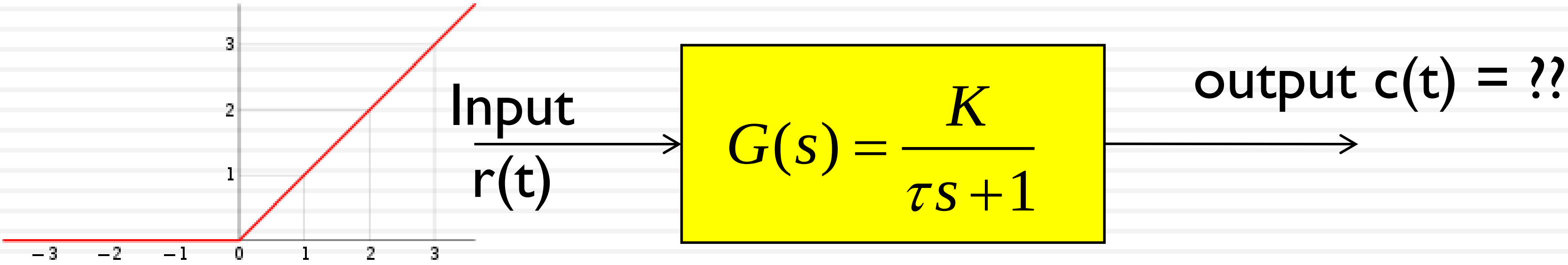
Unit-step response:

$$y(t) = K(1 - e^{-t/\tau})$$



Unit-Ramp Response of First-Order Systems

- Laplace transform of the unit-ramp function is $\frac{1}{s^2}$



Input

r(t)

$$G(s) = \frac{K}{\tau s + 1}$$

output c(t) = ??

- The system output is

$$C(s) = \frac{1}{1+\tau s}\frac{1}{s^2} = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{1+\tau s}$$

using partial fraction method

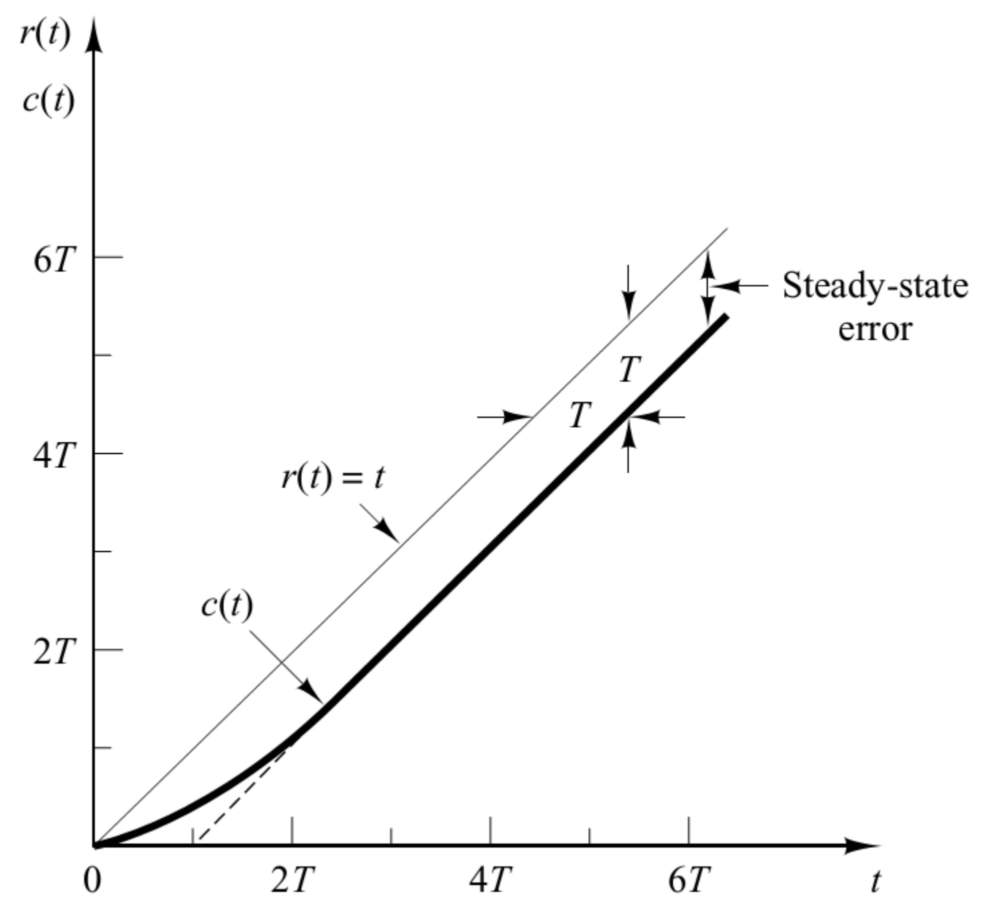
$$c(t) = t - \ \tau + \tau e^{-\frac{t}{\tau}}$$

- The error $e(t)$ signal equals $input\ (t) - output(t)$

$$e(t) = r(t) - c(t) = \tau(1 - e^{-t/\tau})$$

Unit-Ramp Response of First-Order Systems (2)

- As t approaches infinity, $e^{-t/T}$ approaches zero, and thus the error signal $e(t)$ approaches T



Time Response of Second-Order Systems

Servo System (A 2nd-order System)

- The servo system shown consists of a proportional controller and load elements

$$J\ddot{c} + B\dot{c} = T$$

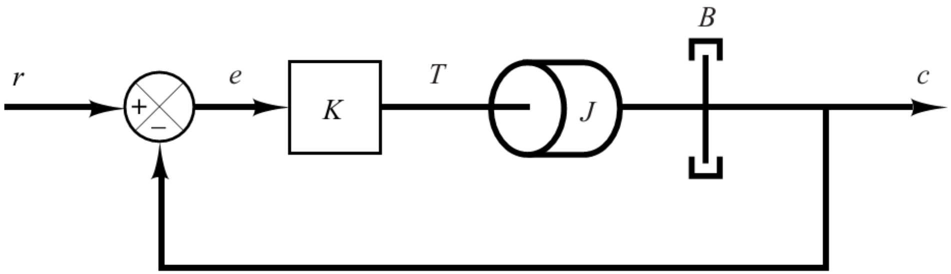
$$Js^2C(s) + BsC(s) = T(s)$$

Open-loop response

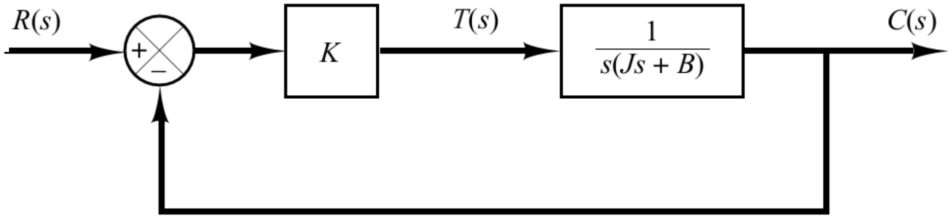
$$\frac{C(s)}{T(s)} = \frac{1}{s(Js + B)}$$

Closed-loop response

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K}$$



(a)



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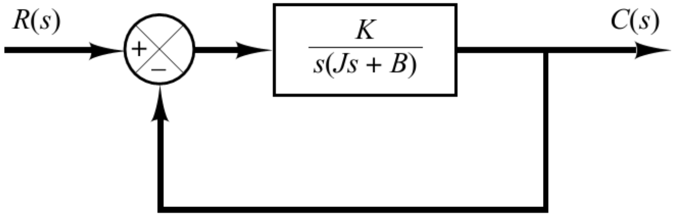


(c)

Servo System (A 2nd-order System)

- ☐ The closed-loop transfer function of the 2ndorder system is

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K}$$

- ☐ This Eqn can be rewritten as

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

- ☐ In the transient-response analysis, it is convenient to write

$$\frac{K}{J} = \omega_n^2, \qquad \frac{B}{J} = 2\zeta\omega_n = 2\sigma$$

where σ is called the attenuation ; ω_n, the undamped natural frequency; and ζ, the damping ratio of the system

Time Response of Second-Order Systems

- What is a second-order system?
  - Second-order systems are described by second-order differential equations.

Example

A prototype second-order differential equation:

$$\frac{d^2}{dt^2}y(t)+2\zeta\omega_n\frac{d}{dt}y(t)+\omega_n^2y(t)=\omega_n^2u(t)$$

y(t)---output response of the system;

u(t)---input to the system

Time response of 2nd-order systems

Using Laplace transform and assuming zero initial conditions, we get:

$$s^2Y(s)+2\zeta\omega_n sY(s)+\omega_n^2Y(s)=\omega_n^2U(s)$$

Transfer function of a second-order system:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

ζ - damping ratio , will determine how much the system oscillates as the response decays toward steady state.

ω_n - undamped natural frequency , will determine how fast the system oscillates during any transient response

Relationship between the characteristic-equation roots and the step response

A second-order system: $$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Its characteristic equation: $$D(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The value of ζ determines the location of the roots of $D(s) = 0.$

$\zeta > 1:$ $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ *overdamped*

$\zeta = 1:$ $s_{1,2} = -\omega_n$ *critically damped*

$0 < \zeta < 1:$ $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$ *underdamped*

$\zeta = 0:$ $s_{1,2} = \pm j\omega_n$ *undamped*

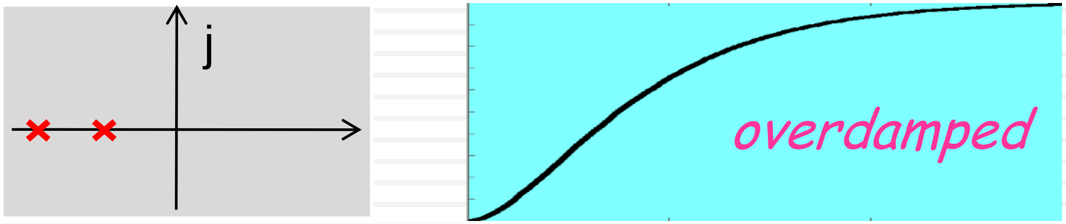
$\zeta < 0:$ $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$ *negatively damped*

A second-order system:

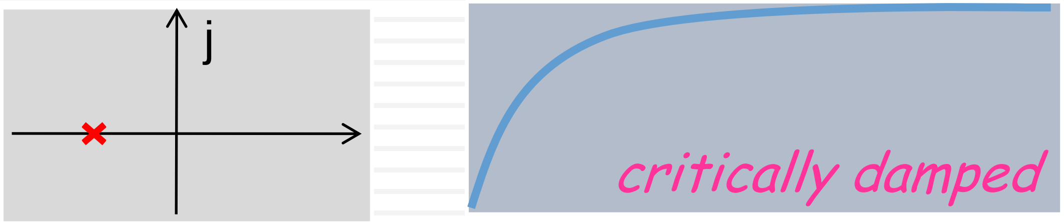
$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$\zeta > 1:$

$$s_{1,2}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

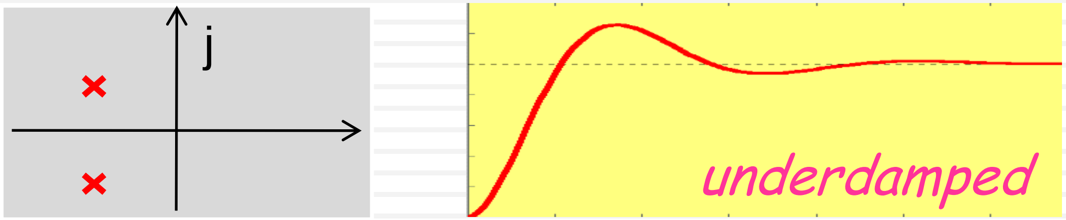


$\zeta = 1:$ $s_{1,2}=-\omega_n$

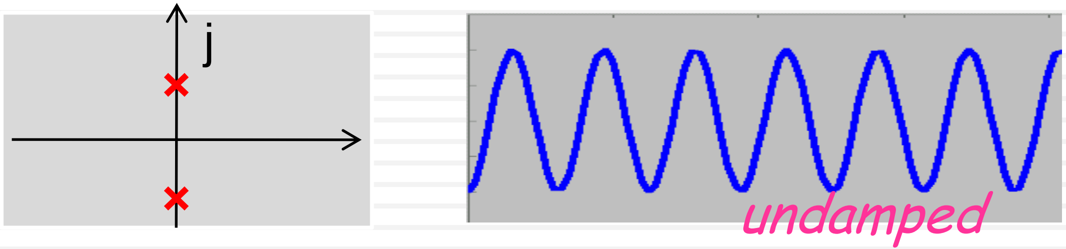


$0 < \zeta < 1:$

$$s_{1,2}=-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



$\zeta = 0:$ $s_{1,2}=\pm j\omega_n$



Step Response of Second-Order Systems

A 2nd-order system: $G(s)=\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

Input : $u(t)=1(t);$ $U(s)=\dfrac{1}{s}$

Case 1: $0\le\zeta<1$ (underdamped), including $\zeta=0$ (undamped)

$$\boxed{y(t)=1-\frac{1}{\beta}e^{-\zeta\omega_n t}\sin(\beta\omega_n t+\theta),}$$

where $\beta=\sqrt{1-\zeta^2}$

$\theta=\tan^{-1}(\beta/\zeta)$

Case 2: $\zeta>1$ (overdamped)

$$\boxed{y(t)=1+k_1e^{-t/\tau_1}+k_2e^{-t/\tau_2}}$$

where $\tau_{1,2}=\dfrac{1}{\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}}$

Case 3: $\zeta=1$ (critically damped)

$$\boxed{y(t)=1+k_1e^{-t/\tau}+k_2e^{-t/\tau}}$$

where $\tau=1/\omega_n$

Step Response of Second-Order Systems (2)

A 2nd-order system: $$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Case 1: $0\le\zeta<1$ (underdamped),

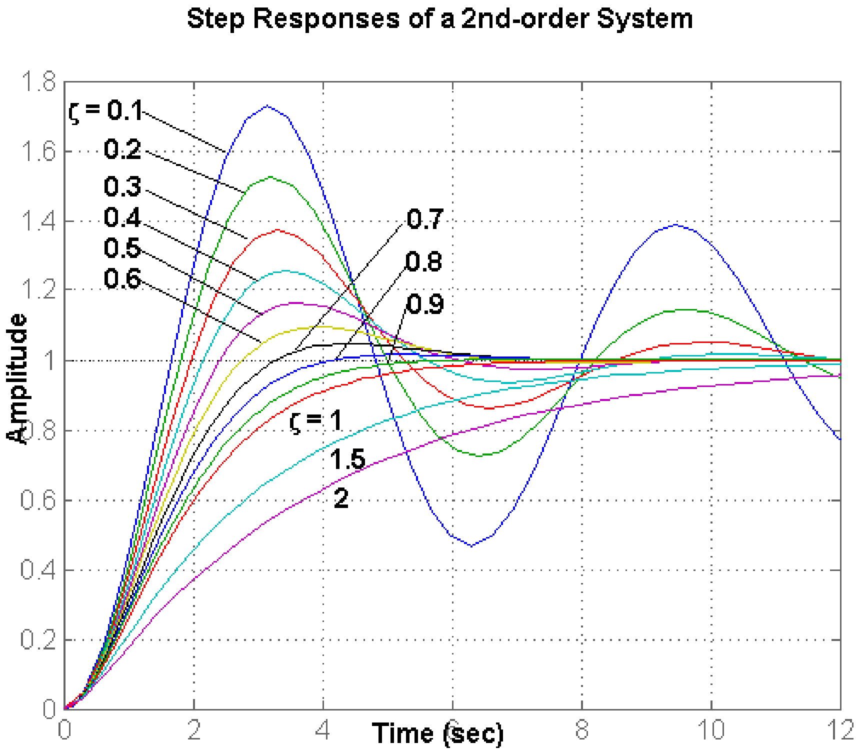
$$\boxed{y(t)=1-\frac{1}{\beta}e^{-\zeta\omega_n t}\sin(\beta\omega_n t+\theta)}$$

Case 2: $\zeta>1$ (overdamped)

$$\boxed{y(t)=1+k_1e^{-t/\tau_1}+k_2e^{-t/\tau_2}}$$

Case 3: $\zeta=1$(critically damped)

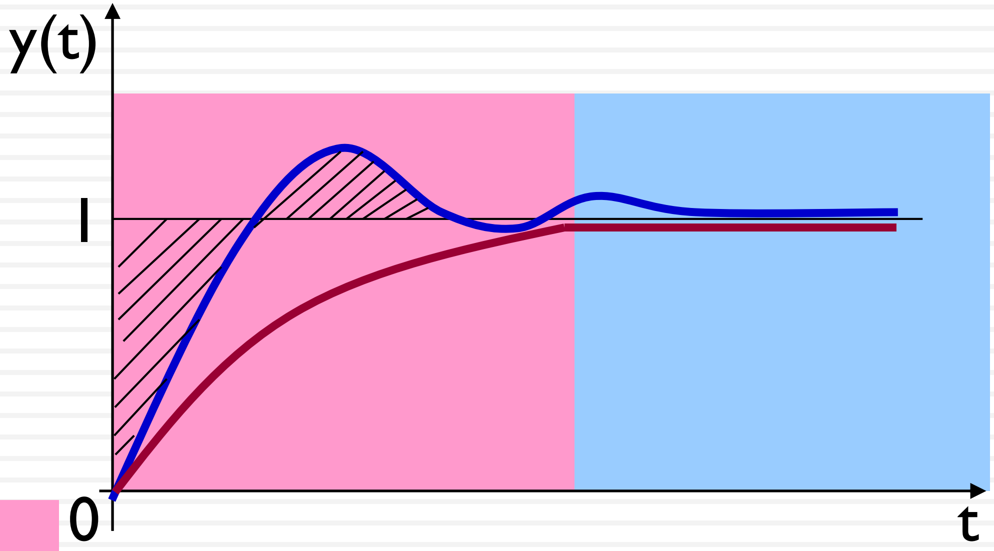
$$\boxed{y(t)=1+k_1e^{-t/\tau}+k_2e^{-t/\tau}}$$



Transient and steady-state response

The time response of a control system is usually divided into two parts:

$$y(t) = y_t(t) + y_{ss}(t)$$



Transient response $y_t(t)$

-- defined as the part of the time response that goes to zero as time becomes very large.

$$\lim_{t\to\infty} y_t(t) = 0$$

Steady-state response $y_{ss}(t)$

-- the part of the time response that remains after the transient has died out.

Time-domain Specifications

1. Steady-state value: y_{ss}

2. Maximum overshoot:

$$M_p = y_{\max} - y_{ss};$$

Percent overshoot $= \dfrac{M_p}{y_{ss}} \times 100\%$

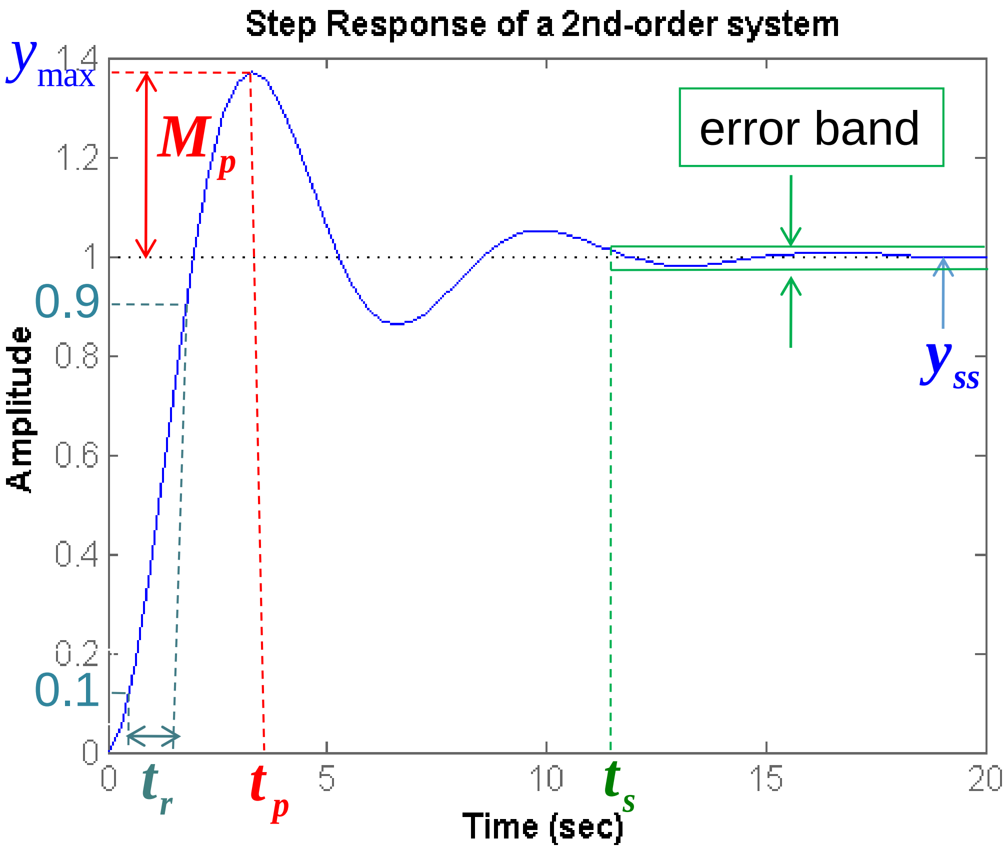
3. Peak time: t_p

How to calculate t_p ?

4. Rise time: t_r

5. Settling time: t_s

Is there overshoot in the time response of a first-order system?



Time-domain Specifications (2)

In practical applications, the following criteria are often used :

- Rise time: evaluate the response speed of the system （quickness)
- Overshoot: evaluate the damping of the system (smoothness)
- Settling time: reflect both response speed and damping

A second-order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Effects of damping ratio ζ

(for a given ω_n)

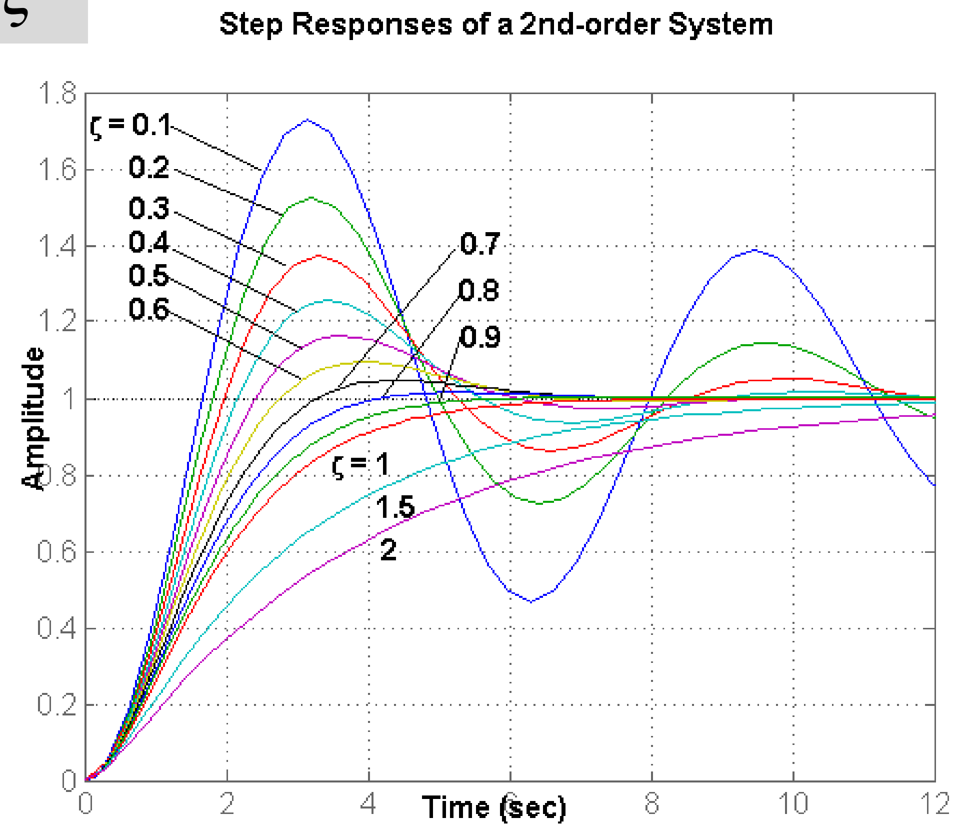
ζ ↑ ⇨

Overshoot M_p ↓

The oscillation is smaller

rise time t_r ↑

The speed of the response is slower



We are confronted with a necessary compromise between the speed of response and the allowable overshoot.

Few Comments about 2nd-Order Systems

Note: In control engineering, except those systems that do not allow any oscillation, usually a control system is desirable with

- moderate damping (allowing some overshoot)
- quick response speed
- short settling time

Therefore, a second-order control system is usually designed as an underdamped system.

Underdamped second-order system

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

ζ, damping ratio $0<\zeta<1$

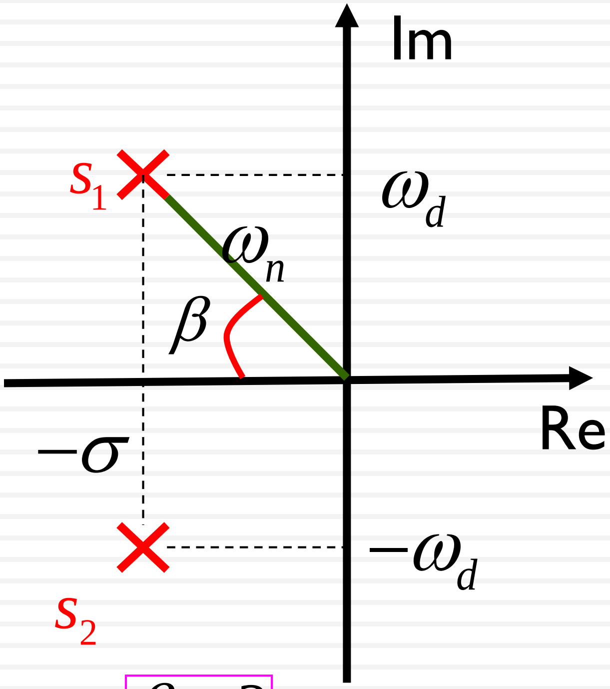
ω_n, natural undamped frequency

$$s_{1,2}=-\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}$$

$$=-\sigma\pm j\omega_d$$

$\sigma=\xi\omega_n$, damping factor

$\omega_d=\omega_n\sqrt{1-\zeta^2}$, damped frequency



$\beta=?$

$\beta=\arccos\zeta$

$$\beta=\arctan\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Unit-step response:

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \beta), \quad 0<\zeta < 1$$

1. Rise Time

$$y(t_r) = 1, \quad \text{that is,} \quad 1 - \frac{e^{-\xi w_n t_r}}{\sqrt{1-\xi^2}} \cdot \sin(w_d t_r + \beta) = 1$$

$$\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \cdot \sin(\omega_d t_r + \beta) = 0$$

$$\frac{e^{-\zeta\cdot\omega_n\cdot t_r}}{\sqrt{1-\zeta^2}} \neq 0, \quad \Rightarrow \omega_d t_r + \beta = n\pi (n = 0, \pm 1, \pm 2, \cdots)$$

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - \beta}{\omega_n\sqrt{1-\zeta^2}}$$

t_r is the time needed for the response to reach the steady-state value for the first time, so n=1.

For a given w_n, $\zeta\downarrow$, $t_r\downarrow$;
For a given ζ, $w_n\uparrow$, $t_r\downarrow$.

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \beta), \quad 0<\zeta<1$$

$$\frac{dy(t)}{dt} = 0$$

2 . Peak time

$$\frac{dy(t)}{dt} = \left(\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \omega_n\sqrt{1-\zeta^2}\right)e^{-\zeta\cdot\omega_n\cdot t}\sin\omega_d t$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\cdot\omega_n\cdot t}\sin\omega_d t \quad =0$$

$$\therefore \sin\omega_d t_p = 0 \Rightarrow \omega_d t_p = n\pi (n = 0, \pm 1, \pm 2, \cdots)$$

t_p is the time needed for the response to reach the maximum value for the first time, so n=1.

$$\omega_d t_p = \pi \quad \Rightarrow \quad t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

For a given **wn**, ζ↓, **tp**↓ ;

For a given ζ, **wn**↑, **tp**↓

$$y(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\beta),\quad 0<\zeta<1$$

$$t_p=\frac{\pi}{w_d}=\frac{\pi}{w_n\sqrt{1-\zeta^2}}$$

3. Overshoot

$$\sigma\%=\frac{y(t_p)-y(\infty)}{y(\infty)}\times 100\%$$

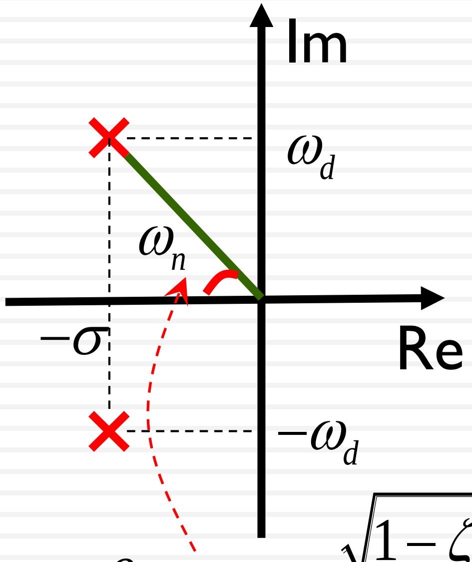
$$y(t_p)=1-\frac{e^{-\zeta\cdot w_n\cdot t_p}}{\sqrt{1-\zeta^2}}\sin(\pi+\beta)$$

$$\sin(\pi+\beta)=-\sin\beta=-\sqrt{1-\zeta^2}$$

$$\therefore\ y(t_p)=1+e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

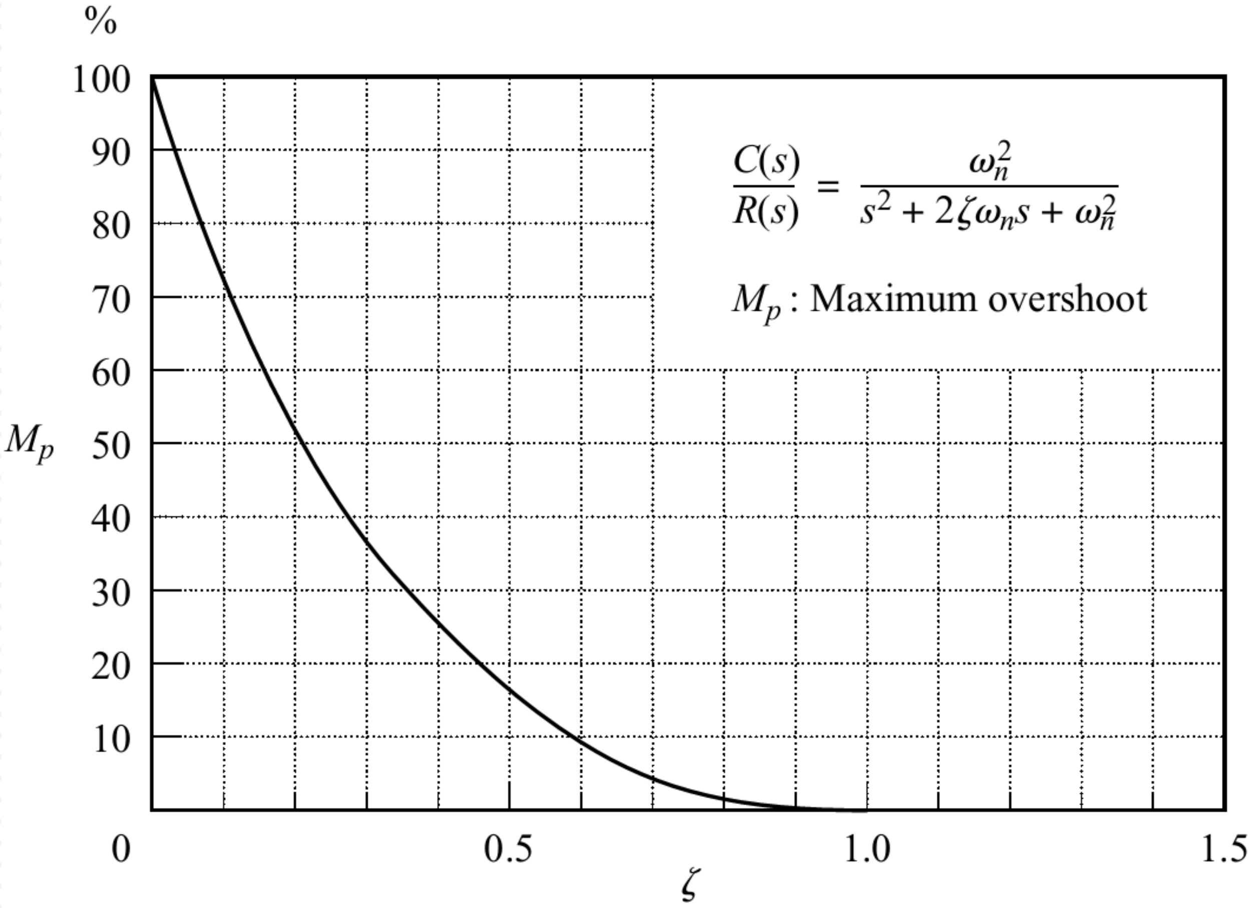
Suppose that $y(\infty)=1$

Thus $\sigma\%=e^{-\pi\zeta/\sqrt{1-\zeta^2}}\cdot 100\%$



$$\beta=\arctan\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Overshoot is a function of damping ratio ζ, independent of wn.



Damping ratio ξ

$\zeta \uparrow, \quad M_p\% \downarrow$

Usually ζ is set between 0.4 and 0.8 to get better smoothness and quickness, the corresponding overshoot is between 25% and 2.5%.

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \beta), \quad 0 < \zeta < 1$$

4 Settling Time

With the definition of error band

$$\left| \frac{e^{-\zeta \cdot \omega_n \cdot t}}{\sqrt{1-\zeta^2}} \cdot \sin(\sqrt{1-\zeta^2} \cdot \omega_n t_s + \beta) \right| \le 0.05 \; or \; 0.02$$

t_s can not be obtained directly, but we can get the relationship between $w_n t_s$ and ζ.

Relationship between (ζ,ω_n) and $(t_r,t_p,\sigma\%,t_s)$

$$t_r=\frac{\pi-\beta}{\omega_d}=\frac{\pi-\beta}{\omega_n\sqrt{1-\zeta^2}}$$

$$t_p=\frac{\pi}{\omega_d}=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

For a given ω_n

$\zeta\downarrow \;\Rightarrow\; t_r\downarrow\; t_p\downarrow$

$\zeta\downarrow \;\Rightarrow\; \sigma\%\uparrow$

$$\sigma\%=e^{-\pi\zeta/\sqrt{1-\zeta^2}}\cdot 100\%$$

ζ can be calculated by the requirement on the overshoot $\sigma\%$. $\zeta\in[0.4,0.8]$.

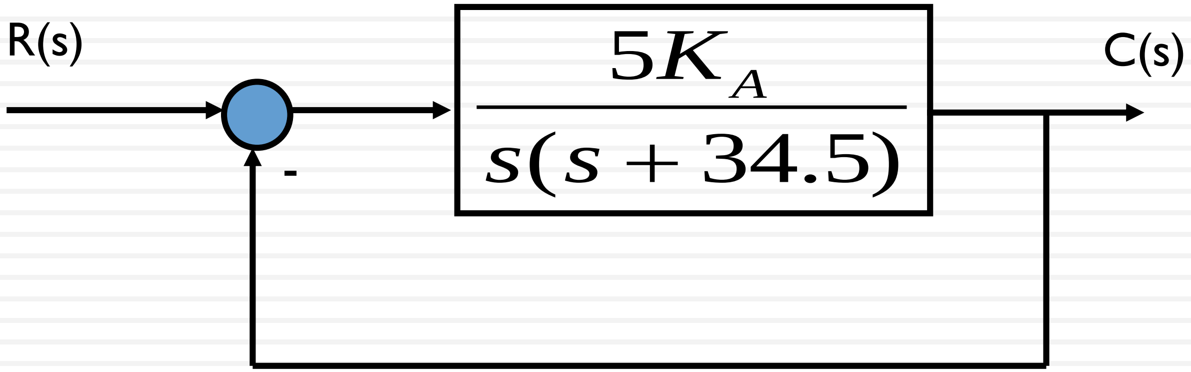
$$t_s\approx\frac{3}{\zeta\omega_n}(\Delta=5\%)$$

$$t_s\approx\frac{4}{\zeta\omega_n}(\Delta=2\%)$$

$\zeta\omega_n\uparrow \;\Rightarrow\; t_s\downarrow$

Once ζ is determined, ω_n can be determined based on the requirement on error band $\Delta\%$.

Example 1: Consider the following unit-feedback system



System input is the unit-step function, When the amplifier gains are K_A=200, K_A=1500, K_A=13.5 respectively, can you calculate the time-domain specifications of the unit-step response ?

Investigate the effect of the amplifier gain K_A on the system response

Solution: The closed-loop transfer function is

$$\phi(s) = \frac{G(s)}{1+G(s)} = \frac{5K_A}{s^2 + 34.5s + 5K_A}$$

$$K_A = 200, \therefore \phi(s) = \frac{1000}{s^2 + 34.5s + 1000}$$

$$\therefore \omega_n^2 = 1000, \quad 2\zeta\omega_n = 34.5$$

$$\therefore \omega_n = 31.6(rad/s), \zeta = \frac{34.5}{2\omega_n} = 0.545$$

According to the formula to calculate the performance indices, it follows that

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = 0.12(\text{sec})$$

$$t_s \approx \frac{3}{\zeta\omega_n} = 0.174(\text{sec})$$

$$\sigma\% = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\% = 13\%$$

$K_A = 1500$

$$\text{If } K_A = 200,\ \text{then } \omega_n = 34.5(rad/s); \quad \zeta = 0.545$$
$$\therefore t_p = 0.12(s), \quad t_s = 0.174(s), \quad \sigma\% = 13\%$$

$$\text{If } K_A = 1500,\ \text{then } \omega_n = 86.2(rad/s); \quad \zeta = 0.2$$
$$\therefore t_p = 0.037(s), \quad t_s = 0.174(s), \quad \sigma\% = 52.7\%$$

Thus, the greater the K_A, the less the ξ, the greater the w_n, the less the t_p, the greater the σ%, while the settling time t_s has no change.

$K_A = 13.5$

$$\zeta > 1$$

Overdamped

$$\text{When } \ K_A = 13.5, \omega_n = 8.22(rad/s), \zeta = 2.1$$

When $K_A = 13.5, \omega_n = 8.22(rad/s), \zeta = 2.1$

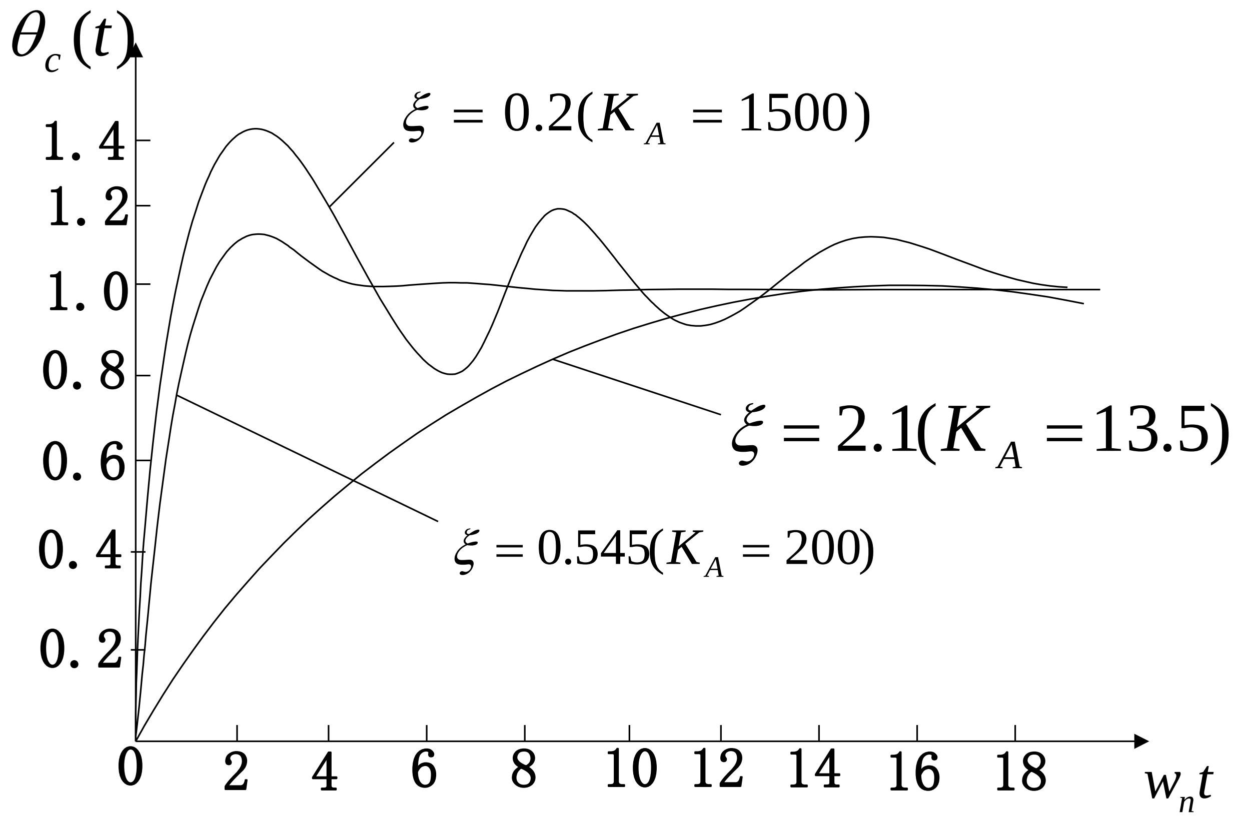
When system is over-damped, there is no peak time, overshoot and oscillation.

The settling time can be calculated approximately:

$$t_s \approx 3T = 1.46(\text{sec})$$

$$\frac{1}{T} = \omega_n(\zeta - \sqrt{\zeta^2 - 1})$$

The settling time is greater than previous cases, although the response has no overshoot, the transition process is very slow, the curves are as follows:

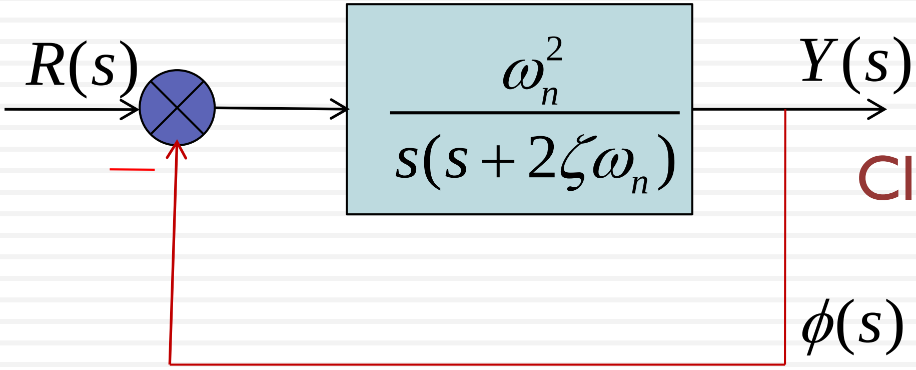


Note: When K_A increases， t_p decreases， t_r decreases， the speed of response increases, meanwhile, the overshoot increases. Therefore, to improve the dynamic performance indexes of a system, we adopt PD-control or velocity feedback control， namely, PD compensation

Effects of Adding Poles and Zeros to Transfer Function

Effects of Adding Poles

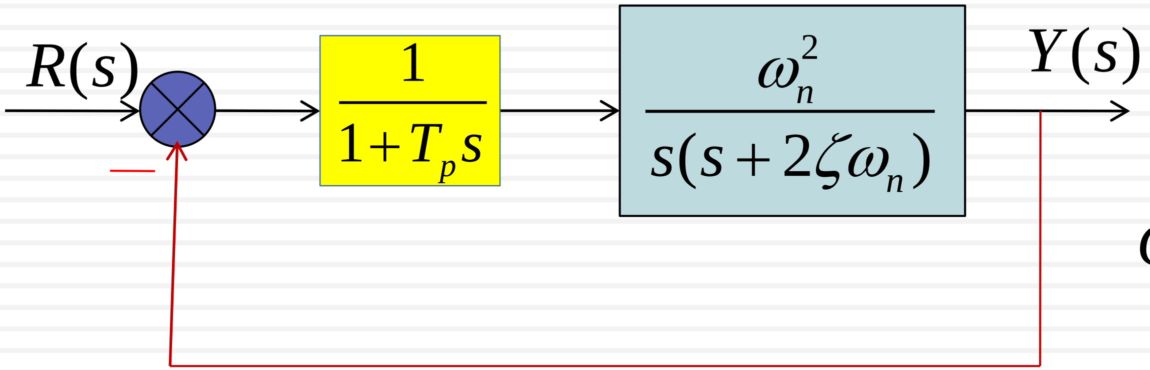
Open-loop TF: $G(s)=\dfrac{\omega_n^2}{s(s+2\zeta\omega_n)}$



Closed-loop TF:

$$\phi(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

1.Adding a pole at $s=-1/T_p$ to the open-loop TF



Open-loop TF:

$$G(s)=\frac{\omega_n^2}{s(s+2\zeta\omega_n)(1+T_p s)}$$

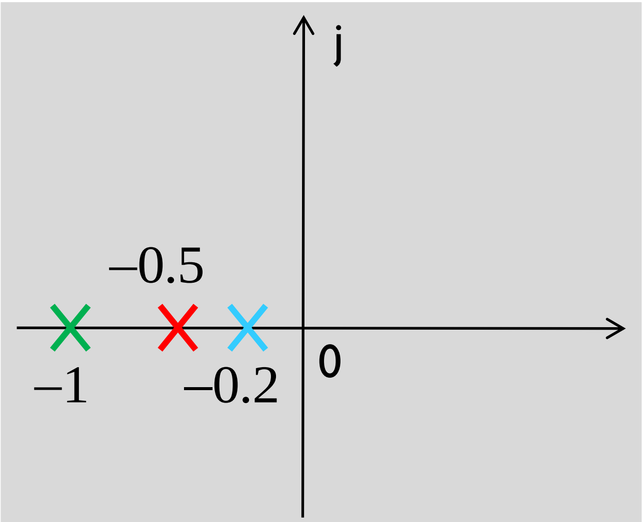
Closed-loop TF: $\phi(s)=\dfrac{\omega_n^2}{T_p s^3+(1+2\zeta\omega_n T_p)s^2+2\zeta\omega_n s+\omega_n^2}$

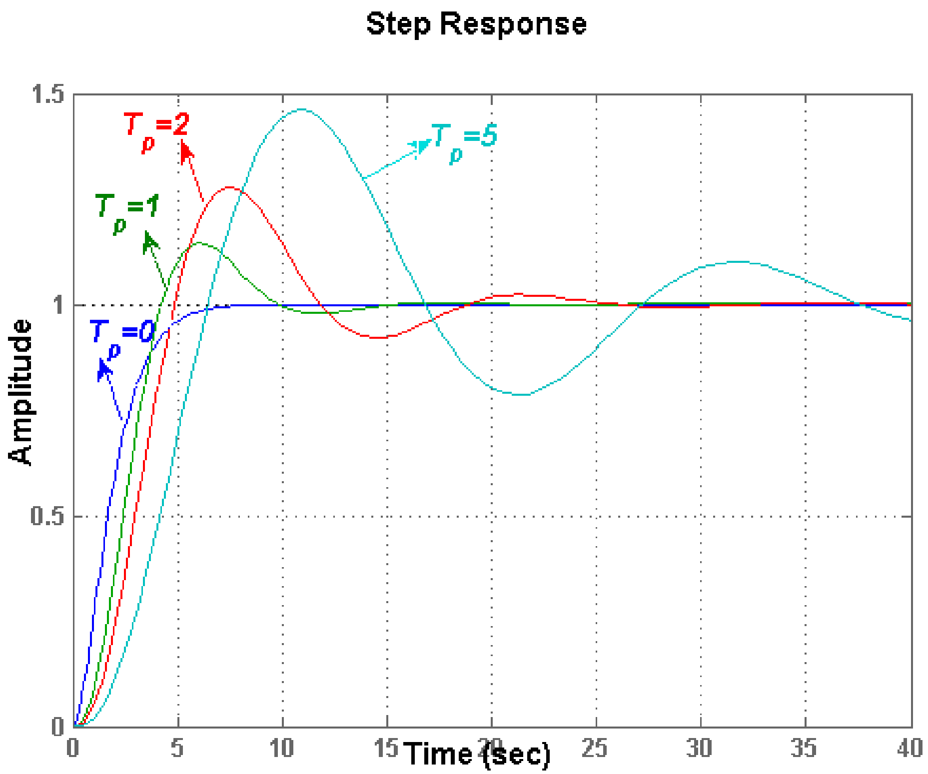
I. Adding a pole at $s = -1/T_p$ to the open-loop TF

How does it affect closed-loop system step-response performance?

$$T_p = 0,\ 1,\ 2,\ 5$$

$$poles:\ s = \infty,\ -1,\ -0.5,\ -0.2$$

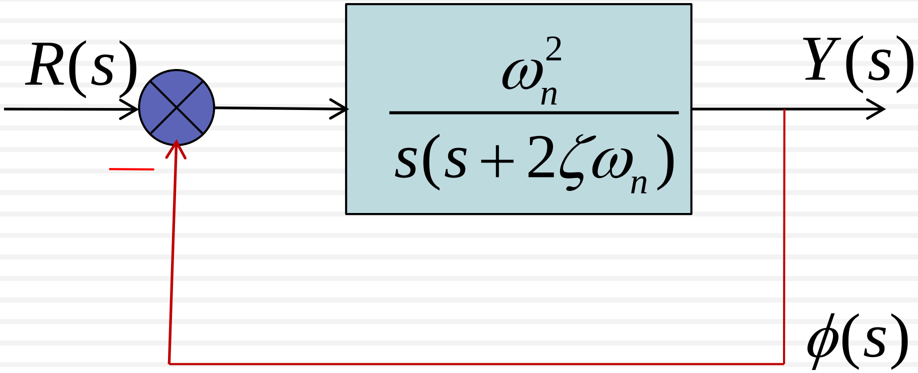




-- Increasing the maximum overshoot of the closed-loop system;

-- Increasing the rise time of the closed-loop system.

Effects of Adding Poles



Open-loop TF: $G(s)=\dfrac{\omega_n^2}{s(s+2\zeta\omega_n)}$

Closed-loop TF:

$$\phi(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

2. Adding a pole at $s=-1/T_p$ to the closed-loop TF

Closed-loop TF:

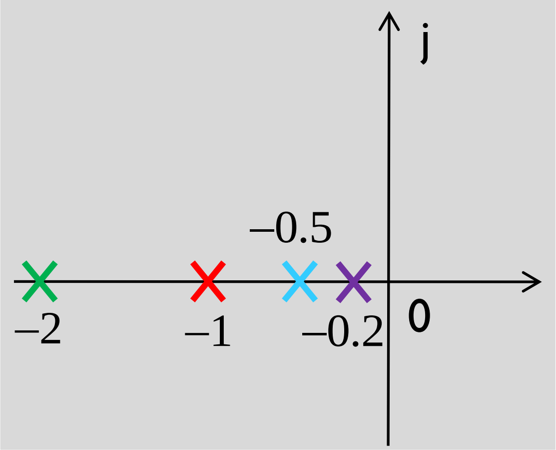
$$\phi(s)=\frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)(1+T_p s)}$$

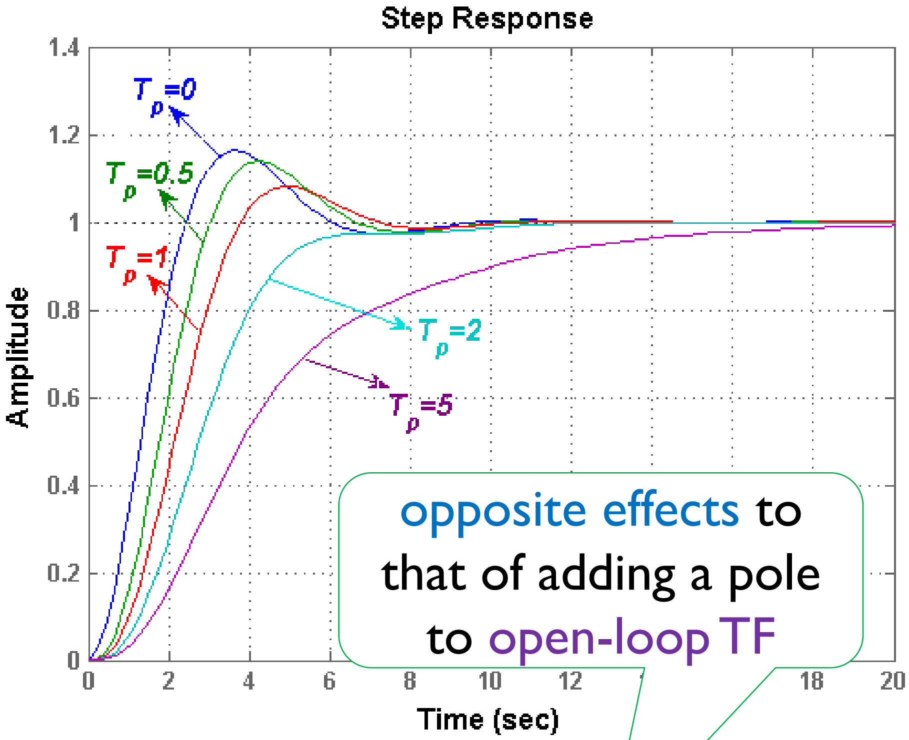
$$=\frac{\omega_n^2}{T_p s^3+(1+2\zeta\omega_n T_p)s^2+(2\zeta\omega_n+\omega_n^2 T_p)s+\omega_n^2}$$

2. Adding a pole at $s = -1/T_p$ to the closed-loop TF

$$T_p = 0,\ 0.5,\ 1,\ 2,\ 5$$

$$s = \infty,\ -2,\ -1,\ -0.5,\ -0.2$$





opposite effects to that of adding a pole to open-loop TF

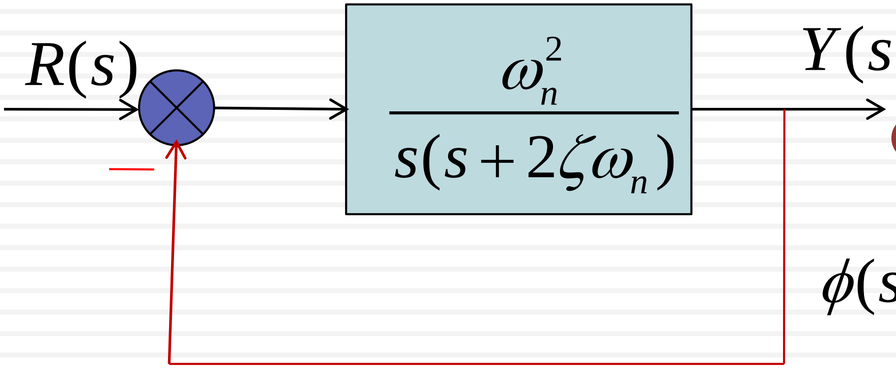
As the pole at $s = -1/T_p$ is moved toward the origin in the s-plane

-- the maximum overshoot of the closed-loop system decreases;

-- the rise time of the closed-loop system increases.

1. Adding a pole to the closed-loop system has the effect as increasing the damping ratio;
2. An originally underdamped system can be made into overdamped by adding a closed-loop pole.

Effects of Adding Zeros



Open-loop TF: $G(s)=\frac{\omega_n^2}{s(s+2\zeta\omega_n)}$

Closed-loop TF:

$$\phi(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

1.Adding a zero at $s=-1/T_z$ to the closed-loop TF

Closed-loop TF:

$$\phi(s)=\frac{Y(s)}{R(s)}=\frac{\omega_n^2(1+T_z s)}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}+\frac{\omega_n^2 T_z s}{s^2+2\zeta\omega_n s+\omega_n^2}$$

1. Adding a zero at $s=-1/T_z$ to the closed-loop TF

Closed-loop TF:

$$\phi(s)=\frac{Y(s)}{R(s)}=\frac{\omega_n^2(1+T_zs)}{s^2+2\zeta\omega_ns+\omega_n^2}=\frac{\omega_n^2}{s^2+2\zeta\omega_ns+\omega_n^2}+\frac{\omega_n^2T_zs}{s^2+2\zeta\omega_ns+\omega_n^2}$$

For a unit-step input $1(t)$, $R(s)=\frac{1}{s}$

The step response of the closed-loop system

$$Y(s)=\phi(s)R(s)=\boxed{\frac{\omega_n^2}{s^2+2\zeta\omega_ns+\omega_n^2}\cdot\frac{1}{s}}+\boxed{T_zs\frac{\omega_n^2}{s^2+2\zeta\omega_ns+\omega_n^2}\cdot\frac{1}{s}}$$

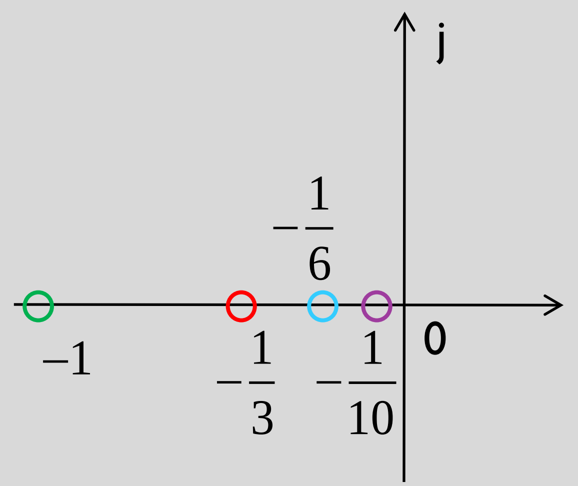
$$y(t)=L^{-1}[Y(s)]=\boxed{y_1(t)}+\boxed{T_z\frac{dy_1(t)}{dt}}$$

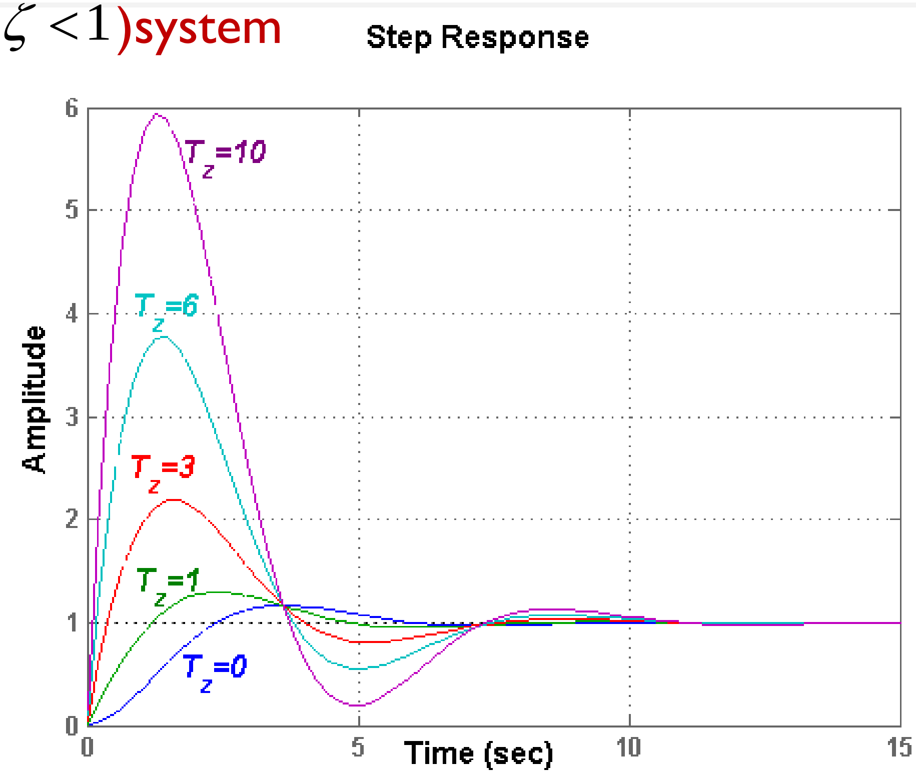
1. Adding a zero at $s=-1/T_z$ to the closed-loop TF

a) Its effects on an underdamped $(0<\zeta<1)$ system

$$T_Z = 0,\ 1,\ 3,\ 6,\ 10$$

$$s=\infty,\ -1,\ -\frac{1}{3},\ -\frac{1}{6},\ -\frac{1}{10}$$





As the zero at $s=-1/T_z$ is moved toward the origin in the s-plane

-- the maximum overshoot of the closed-loop system increases;

-- the rise time of the closed-loop system decreases.

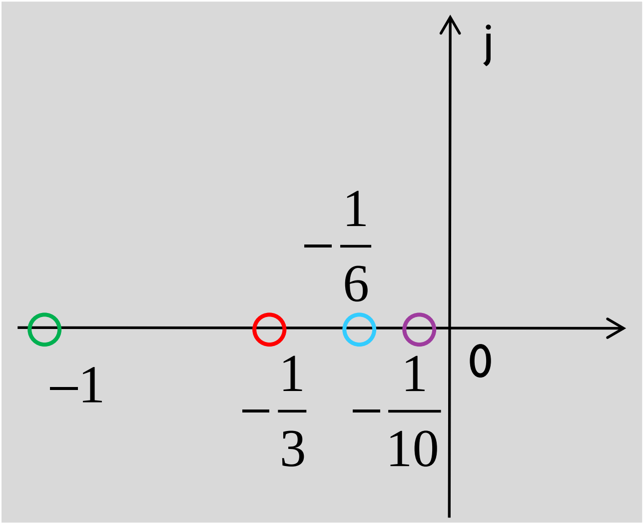
The additional zero has the effect as reducing the damping ratio

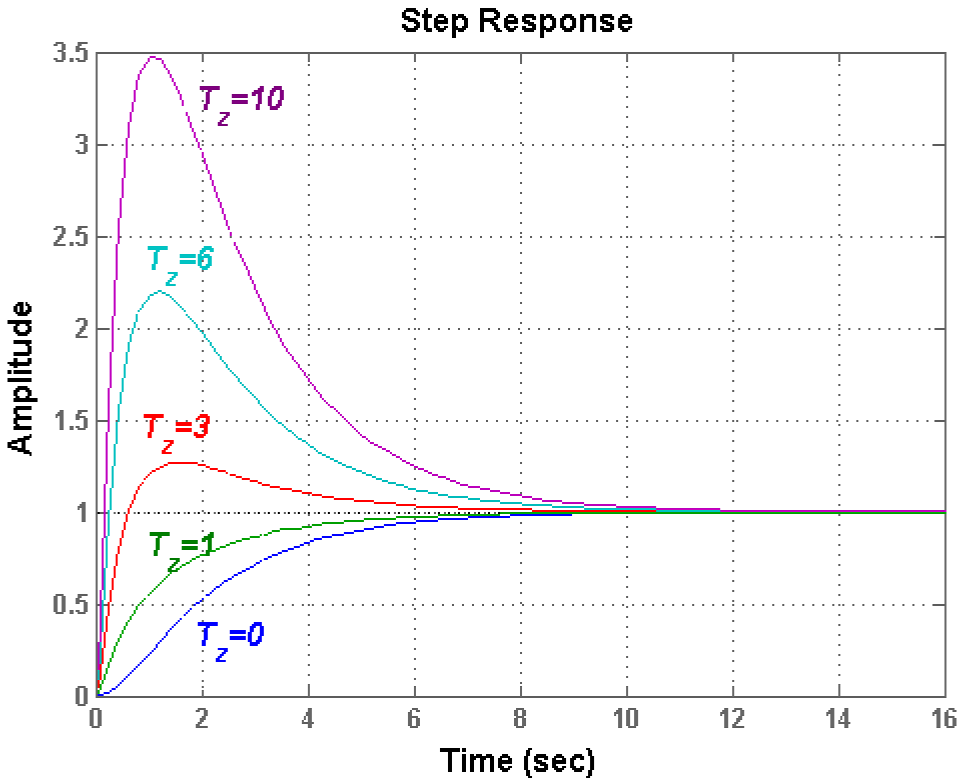
1. Adding a zero at $s = -1/T_z$ to the closed-loop TF

b) Its effects on an overdamped ($\zeta > 1$) system

$$T_Z = 0,\ 1,\ 3,\ 6,\ 10$$

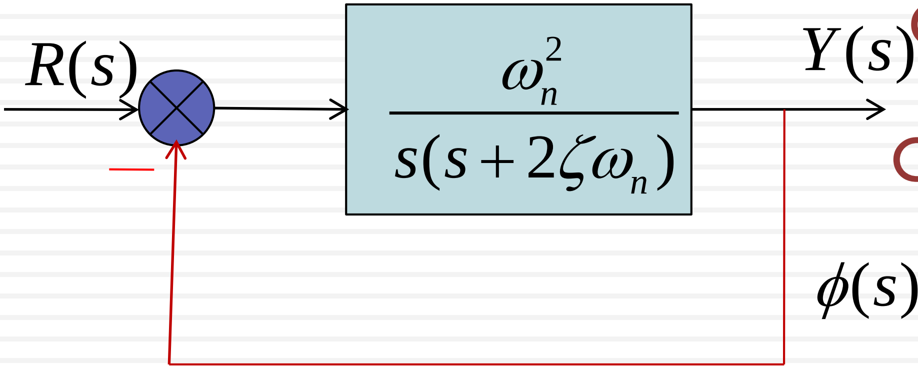
$$s = \infty,\ -1,\ -\frac{1}{3},\ -\frac{1}{6},\ -\frac{1}{10}$$





Adding a zero to an overdamped system can change it into an underdamped system by putting the zero at a proper position.

Effects of Adding Zeros

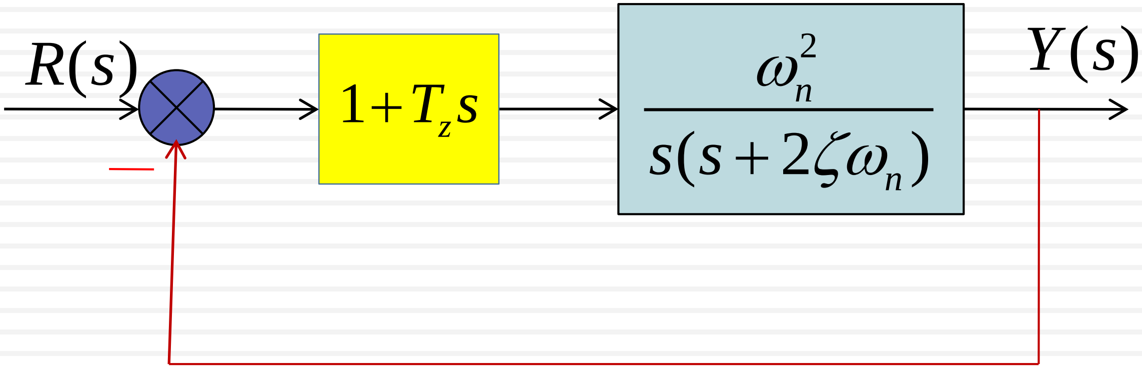


Open-loop TF: $G(s)=\frac{\omega_n^2}{s(s+2\zeta\omega_n)}$

Closed-loop TF:

$$\phi(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\boxed{\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

1. Adding a zero at $s=-1/T_z$ to the open-loop TF



Open-loop TF:

$$G(s)=\frac{(1+T_z s)\omega_n^2}{s(s+2\zeta\omega_n)}$$

Closed-loop TF: $\phi(s)=\boxed{\frac{\omega_n^2(1+T_z s)}{s^2+(2\zeta\omega_n+T_z\omega_n^2)s+\omega_n^2}}$

The additional zero changes both numerator and denominator.

1. Adding a zero at $s=-1/T_z$ to the open-loop TF

The additional zero changes both numerator and denominator.

Closed-loop TF:

$$\phi(s)=\frac{\omega_n^2(1+T_z s)}{s^2+(2\zeta\omega_n+T_z\omega_n^2)s+\omega_n^2}=\frac{\omega_n^2(1+T_z s)}{s^2+2\zeta_z\omega_n s+\omega_n^2}$$

An additional zero $s=-1/T_z$ will increase overshoot

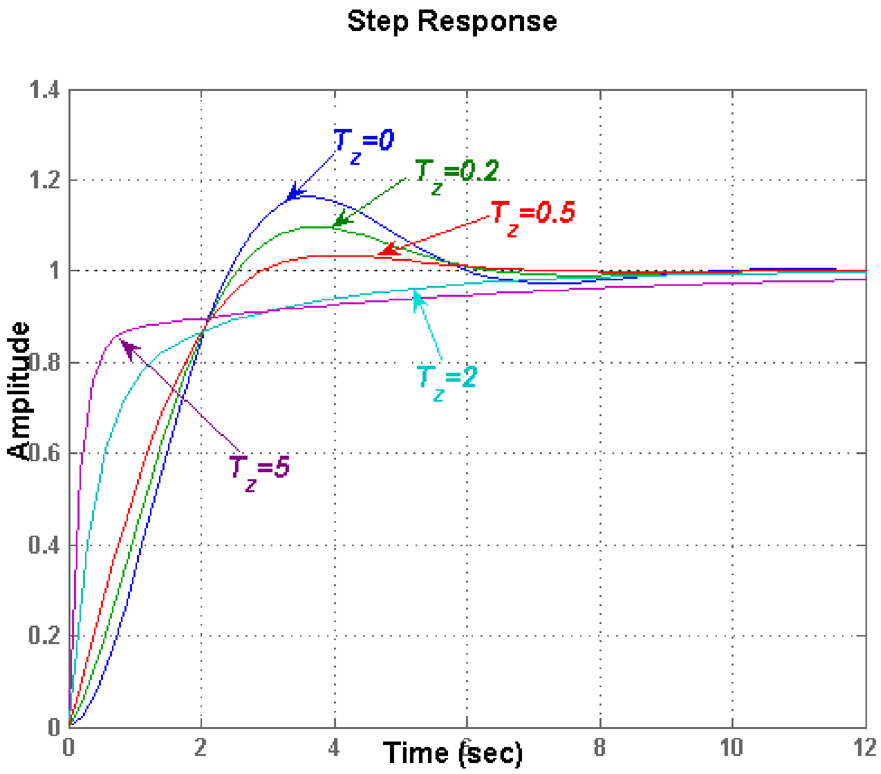
$T_z \uparrow$ overshoot $\sigma\% \uparrow$

The equivalent damping ratio:

$$\zeta_z=\zeta+\frac{T_z\omega_n}{2}>\zeta$$

$T_z \uparrow$ $\zeta_z \uparrow$ overshoot $\sigma\% \downarrow$

when $\zeta_z>1$, the closed-loop system becomes overdamped (no overshoot no matter how large T_z is)



Dominant Poles of a Transfer Function

Dominant poles: those poses that have a dominant effect on the transient response.

By identifying dominant poles, high-order systems can be approximated by lower ones as the transient response is concerned.

e.g.

$$Y(s) = \frac{1}{(s+p_1)(s+p_2)(s+p_3)} = \frac{c_1}{s+p_1} + \frac{c_2}{s+p_3} + \frac{c_3}{s+p_3}$$

$$y(t) = c_1 e^{-p_1 t} + c_2 e^{-p_2 t} + c_3 e^{-p_3 t}$$

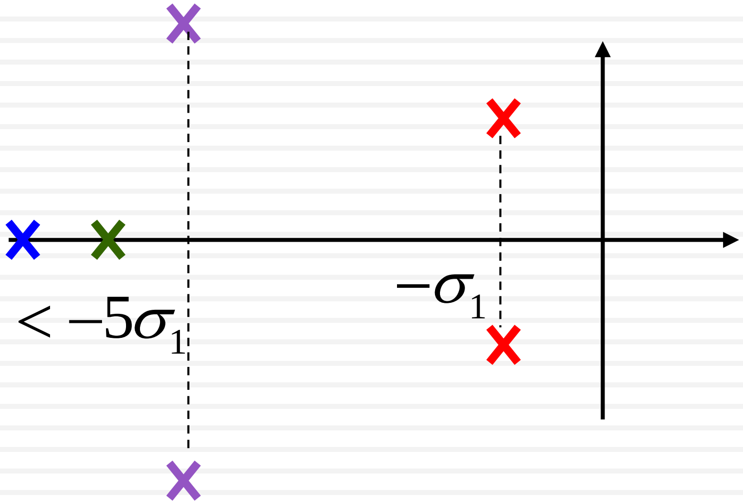
poles: $-p_1, -p_2, -p_3$

If $p_1 > p_2 > p_3$, $c_1 e^{-p_1 t}$ decays fastest, $c_3 e^{-p_3 t}$ decays slowest.

Position of Poles in the left-half s-plane	Their effects on transient response
close to the imaginary axis	decaying relatively slowly
far away from the imaginary axis	decaying fast

Dominant Poles of Transfer Function

If the ratio of real parts exceed 5 and no zeros nearby, the closed-loop poles nearest the imaginary-axis will dominate in the transient response behavior.



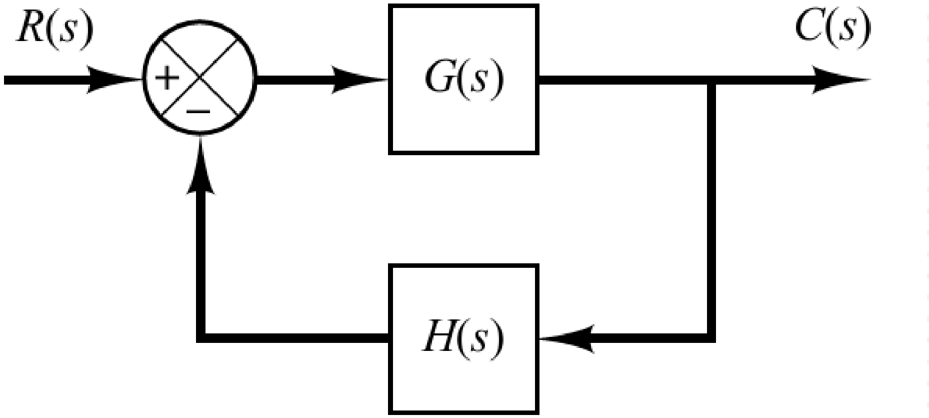
The dominant poles can be a real pole, but a pair of complex conjugate poles are more preferable in control engineering（why?）.

In order to apply second-order system in approximating the dynamic performance of higher-order system

Higher-Order Systems

Transient Response of Higher-Order Systems

- The response of a higher-order system is the sum of the responses of first-order and second-order systems
- The closed-loop transfer function is



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$= \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \qquad (m \le n)$$

or

$$\frac{C(s)}{R(s)} = \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)}$$

Transient Response of Higher-Order Systems (2)

- The response behavior of this system to a unit-step input can be written as (using partial fractions):

$$C(s) = \frac{a}{s} + \sum_{i=1}^{n} \frac{a_i}{s+p_i}$$

where a_i is the residue of the pole at $s=-p_i$

- If all closed-loop poles lie in the left-half s plane, the relative magnitudes of the residues determine the relative importance of the components in the expanded form of *C*(*s*)
- If there is a closed-loop zero close to a closed-loop pole, then the residue at this pole is small and the coefficient of the transient-response term corresponding to this pole becomes small
- A pair of closely located poles and zeros will effectively cancel each other

Transient Response of Higher-Order Systems (3)

- If a pole is located very far from the origin, the residue at this pole may be small
- The transients corresponding to such a remote pole are small and last a short time
- Terms in the expanded form of *C(s)* having very small residues contribute little to the transient response, and these terms may be neglected
- If this is done, the higher-order system may be approximated by a lower-order one

Transient Response of Higher-Order Systems (4)

- Consider the case where the poles of C(s)consist of real poles and pairs of complex-conjugate poles
- A pair of complex-conjugate poles yields a second-order term in s
- The step response can be rewritten as

$$C(s) = \frac{a}{s} + \sum_{j=1}^{q} \frac{a_j}{s + p_j} + \sum_{k=1}^{r} \frac{b_k(s + \zeta_k\omega_k) + c_k\omega_k\sqrt{1 - \zeta_k^2}}{s^2 + 2\zeta_k\omega_k s + \omega_k^2} \qquad (q + 2r = n)$$

- The inverse Laplace transform of C(s), is

$$c(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\zeta_k\omega_k t} \cos \omega_k \sqrt{1 - \zeta_k^2}\, t$$
$$+ \sum_{k=1}^{r} c_k e^{-\zeta_k\omega_k t} \sin \omega_k \sqrt{1 - \zeta_k^2}\, t, \qquad \text{for } t \geq 0$$

Transient Response of Higher-Order Systems (5)

- If all closed-loop poles lie in the left-half *s* plane, then the exponential terms and the damped exponential terms approach zero as time *t* increases
- The exponential terms that correspond to poles located far from the *jw* axis decay very rapidly to zero
- Note that the horizontal distance from a closed-loop pole to the *jw* axis determines the settling time of transients due to that pole
- Remember that the type of transient response is determined by the closed-loop poles, while the shape of the transient response is primarily determined by the closed-loop zeros

Dominant Closed-Loop Poles

- The relative dominance of closed-loop poles is determined by the ratio of the real parts of the closed-loop poles, as well as by the relative magnitudes of the residues evaluated at the closed-loop poles
- The magnitudes of the residues depend on both the closed-loop poles and zeros
- If the ratios of the real parts of the closed-loop poles exceed 5 and there are no zeros nearby, then the closed-loop poles nearest the *jw* axis will dominate in the transient-response
- Those closed-loop poles that have dominant effects on the transient-response behavior are called dominant closed-loop poles

Dominant Poles of a Transfer Function

Dominant poles: those poses that have a dominant effect on the transient response.

By identifying dominant poles, high-order systems can be approximated by lower ones as the transient response is concerned.

e.g.

$$Y(s) = \frac{1}{(s+p_1)(s+p_2)(s+p_3)} = \frac{c_1}{s+p_1} + \frac{c_2}{s+p_3} + \frac{c_3}{s+p_3}$$

$$y(t) = c_1 e^{-p_1 t} + c_2 e^{-p_2 t} + c_3 e^{-p_3 t}$$

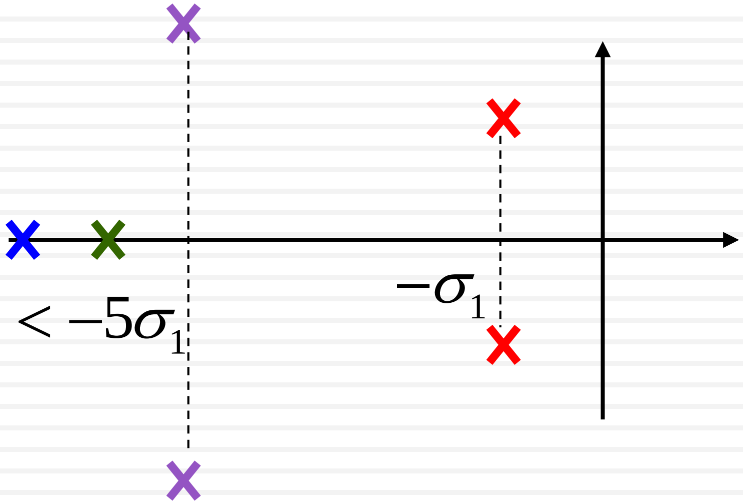
poles: $-p_1, -p_2, -p_3$

If $p_1 > p_2 > p_3$, $c_1 e^{-p_1 t}$ decays fastest, $c_3 e^{-p_3 t}$ decays slowest.

Position of Poles in the left-half s-plane	Their effects on transient response
close to the imaginary axis	decaying relatively slowly
far away from the imaginary axis	decaying fast

Dominant Poles of Transfer Function

If the ratio of real parts exceed 5 and no zeros nearby, the closed-loop poles nearest the imaginary-axis will dominate in the transient response behavior.



The dominant poles can be a real pole, but a pair of complex conjugate poles are more preferable in control engineering（why?）.

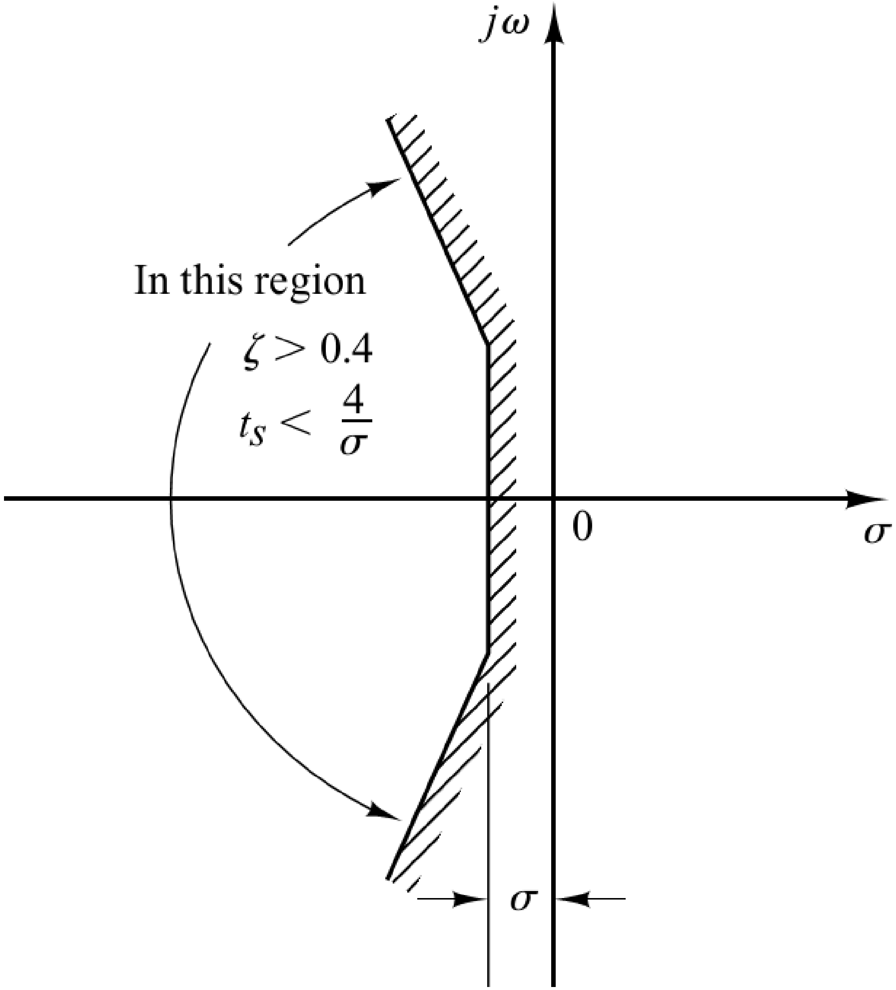
In order to apply second-order system in approximating the dynamic performance of higher-order system

Stability Analysis in the Complex Plane

- The stability of a linear closed-loop system can be determined from the location of the closed-loop poles in the *s* plane
- If any of these poles lie in the right-half *s* plane, then the system is unstable
- If all closed-loop poles lie to the left of the *jw* axis, any transient response eventually reaches quilibrium. This represents a stable system
- The fact that all closed-loop poles lie in the left-half *s* plane does not guarantee satisfactory transient-response characteristics

Stability Analysis in the Complex Plane (2)

- ☐ If dominant complex-conjugate closed-loop poles lie close to the *jw* axis, the transient response may exhibit excessive oscillations or may be very slow
- ☐ To guarantee fast, yet well-damped, transient-response characteristics, it is necessary that the closed-loop poles of the system lie in a particular region in the complex plane as shown



Stability Analysis in the complex plane

The Concept of Stability

Two types of response for LTI systems:

--Zero-state response: the response is due to input only; all the initial conditions are zero;

Bounded-input-bounded-output(BIBO) stability: with zero initial conditions, the system's output y(t) is bounded to a bounded input u(t).

--Zero-input response: the response is due to the initial conditions only; all the inputs are zero;

Asymptotic stability: with zero input, for finite initial conditions $y(t_0), \dot{y}(t_0), \cdots, y^{(n)}(t_0)$, an LTI system is asymptotic stable there exist a positive number M which depends on the initial conditions, such that

$$(1) \quad |y(t)| \le M < \infty \quad \textit{for all } t > t_0; \quad \text{and} \quad (2) \quad \lim_{t \to \infty} |y(t)| = 0.$$

Time-Domain Definition

The initial condition of the system is zero. When system input is unit impulse function δ(t), the system output is g(t).

If $\lim_{t \to \infty} g(t) = 0$, then the system is stable。

$$G(s) = \frac{Y(s)}{U(s)} = \sum_{i=1}^{n} \frac{C_i}{s + p_i} \qquad g(t) = L^{-1}[G(s)] = \sum_{i=1}^{n} C_i e^{-p_i t}$$

$$\lim_{t \to \infty} g(t) = 0 \qquad \Rightarrow e^{-p_i t} \text{ decay with time}$$

All poles should locate in the left side of s-plane

Stability Criterion in Complex Plane

A system is stable if and only if

all roots of the system characteristic equation have negative real parts

or equivalently

all poles of closed-loop transfer functions must locate in the left half of s-plane.

For LTI systems, both BIBO stability and asymptotic stability have the same requirement on pole location. Thus if a system is BIBO stable , it must also be asymptotic stable.

So we simply refer to the stability condition of an LTI system as stable or unstable.

LTI Systems Stability Conditions

Stability Conditions	Location of poles
stable	all poles in LHP
marginally stable	simple poles on the jw-axis and no poles in RHF
unstable	at least one simple pole in RHF or at least one multi-order pole on the jw-axis

Routh-Hurwitz's Stability Criterion

All poles in left s-plane => No poles in right s-plane

- The criterion tests whether any of the roots of the characteristic equation lie in the right half of the s-plane, without actually calculating the roots.
- Information about stability can be obtained directly from the coefficients of the characteristic equation

Routh-Hurwitz's Stability Criterion

A necessary (but not sufficient) condition for stability:

(1) All the coefficients of the characteristic equation have the same sign.

(2) None of the coefficients vanishes.

Consider the characteristic equation of a LTI system

$$D(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0, \quad a_0 > 0$$

Note 1: The above conditions are based on the laws of algebra .

$$(s-s_1)(s-s_2)(s-s_3)=0 \quad \rightarrow a_0s^3+a_1s^2+a_2s+a_3=0$$

$$s(s-s_2)(s-s_3)-s_1(s-s_2)(s-s_3)=0$$

$$s(s^2-(s_2+s_3)s+s_2s_3)-s_1(s^2-(s_2+s_3)s+s_2s_3)=0$$

$$s^3-\underline{(s_1+s_2+s_3)}s^2+\underline{(s_1s_2+s_1s_3+s_2s_3)}s-\underline{s_1s_2s_3}=0$$

$$\frac{a_1}{a_0}=-\sum_{i=1}^{3}s_i>0 \qquad \frac{a_2}{a_0}=\sum_{\substack{i,j=1\\ i\neq j}}^{3}s_is_j>0 \qquad \frac{a_3}{a_0}=-\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{3}s_is_js_k>0$$

If all roots of the system characteristic equation have negative real parts, all the coefficients have the same sign

Note 2: These conditions are not sufficient.

Routh's Tabulation

Consider the characteristic equation of a LTI system

$$D(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0, \quad a_0 > 0$$

$$\begin{array}{l|lllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
\vdots & \vdots & \vdots & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_1 & & & &
\end{array}$$

$$b_1 = \frac{-1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} \qquad c_1 = \frac{-1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}$$

$$b_2 = \frac{-1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} \qquad c_2 = \frac{-1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}$$

$$b_3 = \frac{-1}{a_1}\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix} \qquad c_3 = \frac{-1}{b_1}\begin{vmatrix} a_1 & a_7 \\ b_1 & b_4 \end{vmatrix}$$

Routh's Criterion

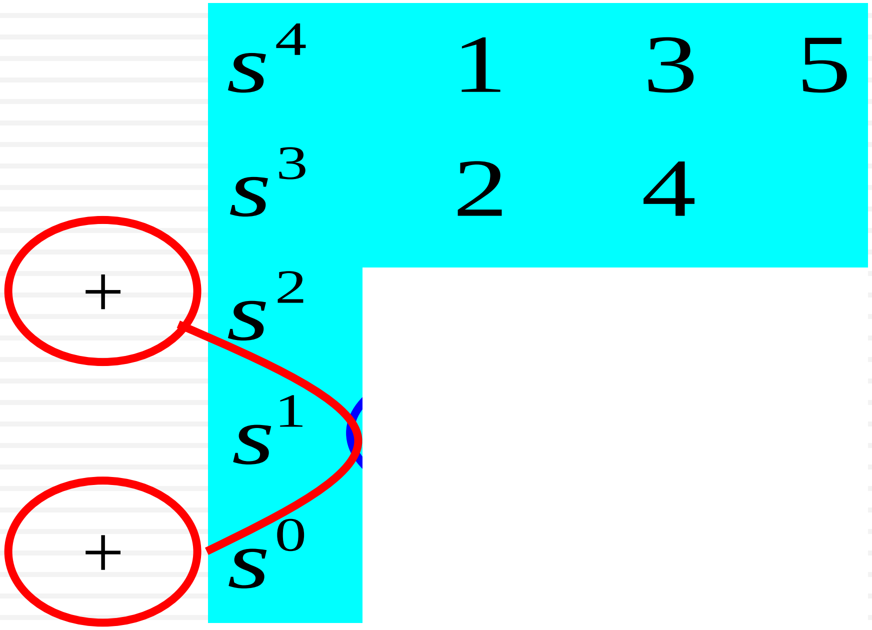
A necessary and sufficient condition for stability:

all the elements of the first column of the Routh' Tabulation are of the same sign.

The number of changes of signs in the elements of the first column equals the number of roots in the right-half s-plane.

Example

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$



s^4	1	3	5
s^3	2	4	
s^2			
s^1			
s^0			

Therefore, the system is unstable and has two roots in the right-half s-plane.

First-order: $a_0 s + a_1 = 0$

If a_0 and a_1 have the same sign, the system is stable.

Second-order: $a_0 s^2 + a_1 s + a_2 = 0$

If a_0 , a_1 and a2 have the same sign, the system is stable.

Third-order:

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

$$\begin{array}{lcc}
s^3 & a_0 & a_2 \\
s^2 & a_1 & a_3 \\
s^1 & -\dfrac{a_0 a_3 - a_1 a_2}{a_1} & \\
s^0 & a_3 &
\end{array}$$

If a0,a1,a2,a3 are all positive and a1a2>a3a0,the system is stable.

Special cases when applying Routh's Tabulation

Case 1: only the first element in one of the rows of Routh's tabulation is zero

Solution: replace the zero with a small positive constant ε and proceed as before by taking the limit as $\varepsilon \to 0$

$$s^4 + 3s^3 + 4s^2 + 12s + 16 = 0$$

$$\begin{array}{llll}
s^4 & 1 & 4 & 16 \\
s^3 & 3 & 12 & \\
s^2 & 0(\varepsilon) & 16 & \\
s^1 & \dfrac{12\varepsilon - 48}{\varepsilon} & 0 & \\
s^0 & 16 & &
\end{array}$$

when $\varepsilon \to 0$

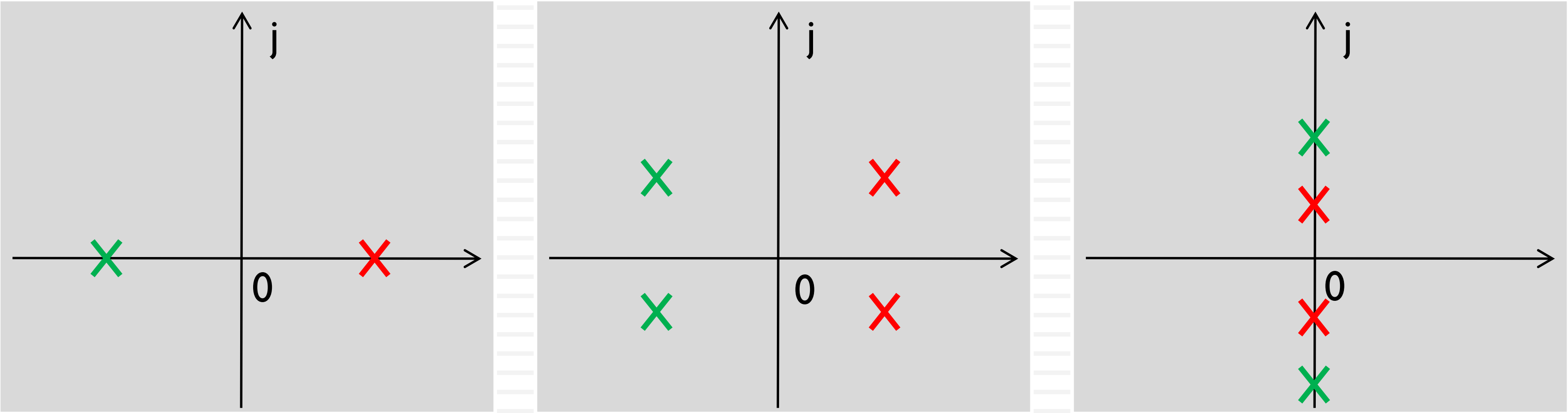
$$\frac{12\varepsilon - 48}{\varepsilon} = 12 - \frac{48}{\varepsilon} < 0$$

The system is unstable and has two roots not in the left-half s-plane.

Case 2: an entire row of Routh's tabulation is zero.

This indicates…

There are complex conjugate pairs of roots that are mirror images of each other with respect to the imaginary axis.



e.g. $s_{1,2} = \pm 1$

e.g. $s_{1,2} = -1 \pm j1$

$s_{3,4} = 1 \pm j1$

e.g. $s_{1,2} = \pm 1j$

$s_{3,4} = \pm 2j$

Example

The characteristic equation of a system is:

$$s^5+3s^4+3s^3+9s^2-4s-12=0$$

Determine whether there are any roots on the imaginary axis or in the RHP.

s^5	1	3	-4
s^4	3	9	-12
s^3	12	18	0
s^2	$9/2$	-12	
s^1	50	0	
s^0	-12		

$= 0$

The sign in the first column changes once, so the system is unstable and there is one root outside LHP.

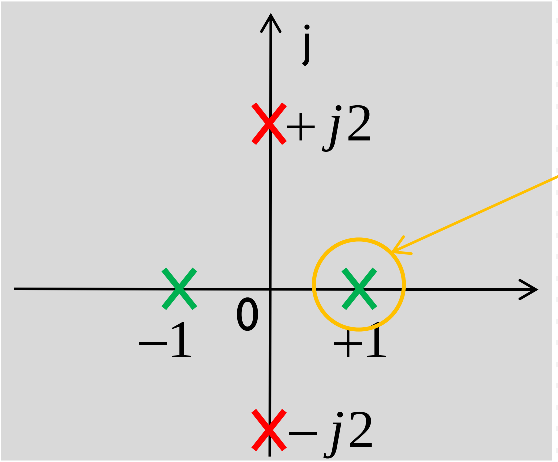
Solving the auxiliary equation

$$A(s) = 3s^4 + 9s^2 - 12 = 0$$

$$s^4 + 3s^2 - 4 = (s^2 - 1)(s^2 + 4) = 0$$

$$s_{1,2} = \pm 1$$

$$s_{3,4} = \pm j2$$



A positive real root locates in the RHP

Exercise

- Determine the stability of the following systems:

$$(1)\ 2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

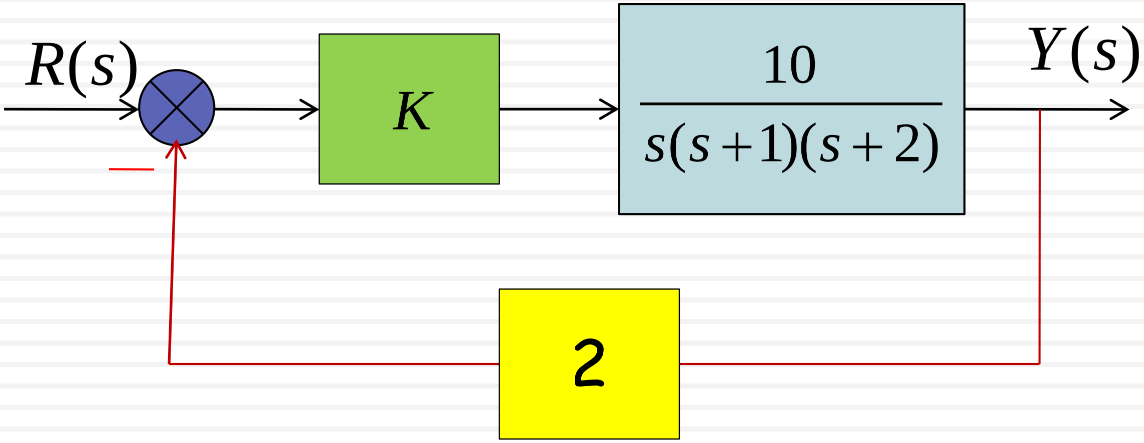
$$(2)\ s^5 + 3s^4 + 12s^3 + 24s^2 + 32s + 48 = 0$$

Application of Routh Tabulation

- Routh's tabulation can not only be used to determine the stability of a system, but also the spread of its characteristic roots.
- For a control system with a regulator, Routh's tabulation can be used to select parameter values so that the system is stable.

Example

- **Determine the range of K so that the system is stable.**



- **Solution:**

$$\phi(s)=\frac{C(s)}{R(s)}=\frac{G_{forward}(s)}{1+G_{loop}(s)}=\frac{\dfrac{10k}{s(s+1)(s+2)}}{1+\dfrac{20k}{s(s+1)(s+2)}}$$

Example (2)

Characteristic equation: $1+G_{\text{l oop}}(s)=0$

$$\therefore 1+\frac{20k}{s(s+1)(s+2)}=0$$

$$s^3+3s^2+2s+20k=0$$

$$\begin{array}{lcc l}
s^3 & 1 & 2 & \\
s^2 & 3 & 20k & \\
s^1 & \dfrac{3\times 2-20k}{3} & 0 & \Rightarrow\ 6\text{-}20k>0\ \Rightarrow\ k<0.3 \\
s^0 & 20k & & \Rightarrow\ k>0
\end{array}$$

So when $0<k<0.3$, the system is stable.

Exercise

Consider that a 3rd-order system has the characteristic equation $0.025s^3 + 0.325s^2 + s + k = 0$

Determine the range of k so that the system is stable.

Solution. Reformulate the characteristic equation as

$$s^3 + 13s^2 + 40s + 40k = 0$$

$$\begin{array}{llll}
s^3 & 1 & 40 & \\
s^2 & 13 & 40k & \\
s^1 & \dfrac{13 \times 40 - 40k}{13} & \Rightarrow & k < 13 \\
s^0 & 40k & \Rightarrow & k > 0
\end{array}$$

Effects of Integral and Derivative Control Actions on System Performance

Integral Control Action

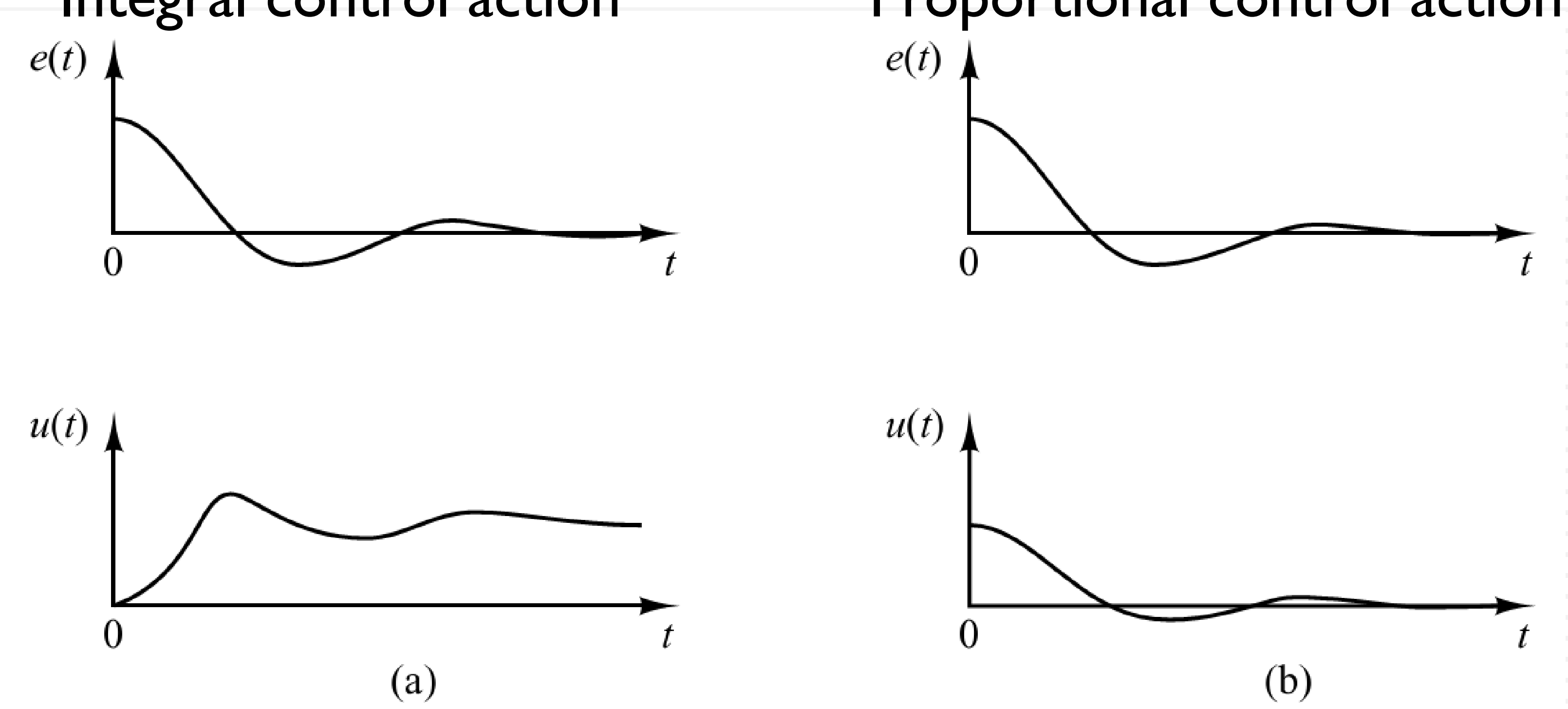
- In the proportional control of a plant whose transfer function does not possess an integrator *1/s*, there is a steady-state error, or offset, in the response to a step input. Such an offset can be eliminated if the integral control action is included in the controller
- In the integral control of a plant, the control signal at any instant is the area under the actuating-error-signal curve up to that instant
- The control signal $u(t)$ can have a nonzero value when the actuating error signal $e(t)$ is zero

Integral Control Action (2)

- Figure shows the curve *e(t)* versus *t* and the corresponding curve *u(t)* versus *t* when the
- controller is of the proportional type

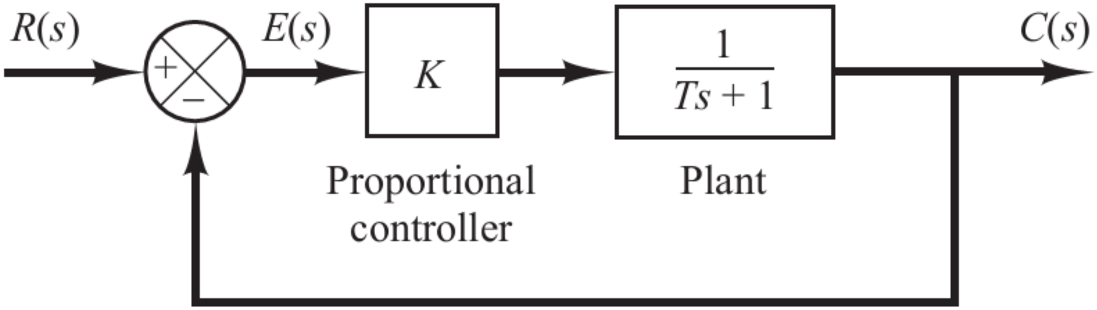
Integral control action

Proportional control action



Proportional Control of Systems

- ☐ We will prove that proportional control of a system without an integrator will result in a steady-state error with a step input.



Let:

$$G(s) = \frac{K}{1+Ts}$$

$$E(s) = R(s) - C(s)$$

$$= R(s) - E(s)G(s)$$

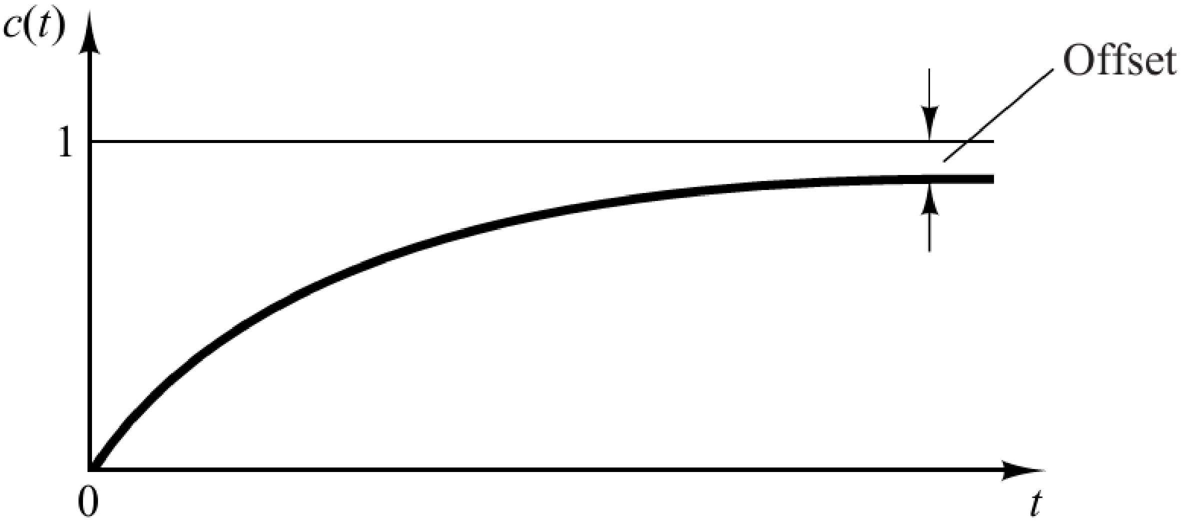
$$E(s) = \frac{1}{1+G(s)}R(s) = \frac{1+Ts}{1+Ts+K}R(s)$$

For the unit-step input $E(s) = \frac{1+Ts}{1+Ts+K}\frac{1}{s}$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{1+Ts}{1+Ts+K} = \frac{1}{1+K}$$

Proportional Control of Systems (2)

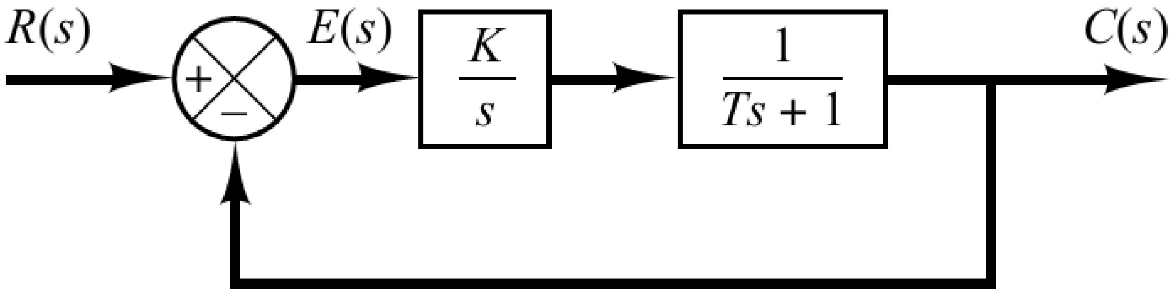
- Such a system without an integrator in the feed-forward path always has a steady-state error in the step response
- Such a steady-state error is called an offset



Integral Control of Systems

- Let

$$G(s) = \frac{K/s}{1+Ts}$$



$$E(s) = \frac{1}{1+G(s)}R(s) = \frac{(1+Ts)s}{(1+Ts)s+K}R(s)$$

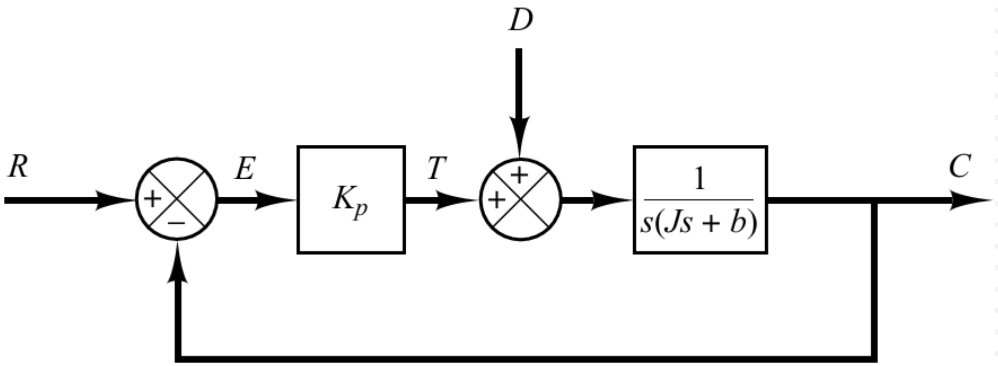
For the unit-step input:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s^2(1+Ts)}{s+Ts^2+K}\frac{1}{s} = 0$$

- Integral control of the system thus eliminates the steady-state error in the response to the step input.

Response to Torque Disturbances

- Let us investigate a control system with a torque disturbance
- Assuming that *R(s)=0*



$$\frac{C(s)}{R(s)} = \frac{1}{Js^2+bs+K}$$

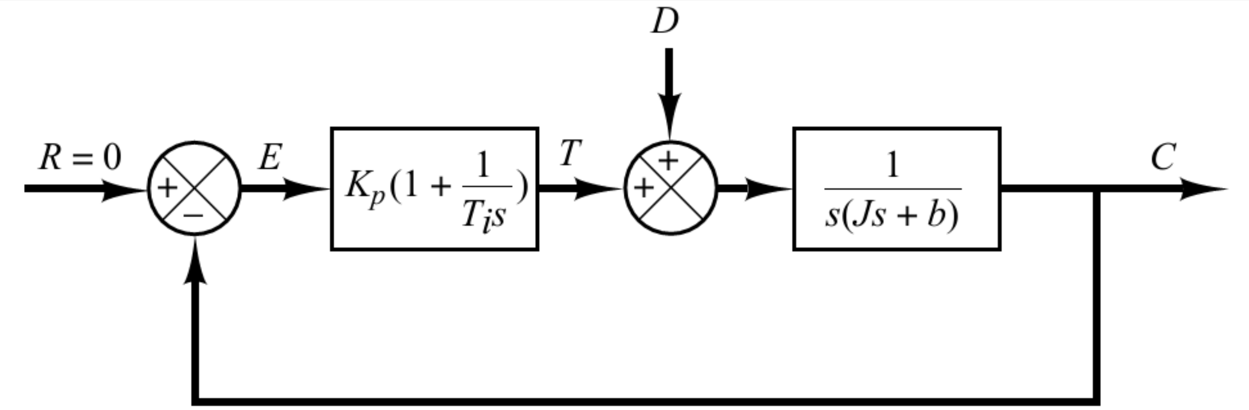
Hence: $$\frac{E(s)}{D(s)} = -\frac{C(s)}{D(s)} = -\frac{1}{Js^2+bs+K_P}$$

- The steady-state error due to a step disturbance torque of magnitude T_d is given by

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{-s}{Js^2+bs+K_P}\frac{T_d}{s} = \frac{-T_d}{K_P}$$

Response to Torque Disturbances (PI Control)

- ☐ To eliminate offset due to torque disturbance, the proportional controller may be replaced by a proportional-plus-integral controller



- ☐ The closed-loop transfer function between *C*(*s*) and *D*(*s*) is

$$\frac{C(s)}{D(s)} = \frac{s}{Js^3 + bs^2 + K_P s + \frac{K_P}{T_i}}$$

Response to Torque Disturbances (PI Control) (2)

- Assuming that *R(s)=0*

$$E(s) = -\frac{s}{Js^3 + bs^2 + K_P s + \frac{K_P}{T_i}} D(s)$$

- The steady-state error due to a step disturbance torque of magnitude is given by

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{-s^2}{Js^3 + bs^2 + K_P s + \frac{K_P}{T_i}} \frac{1}{s} = 0$$

- The steady-state error to the step disturbance torque can be eliminated if the controller is PI

Response to Torque Disturbances (PI Control) (3)

- Note that the integral control action added to the proportional controller has converted the originally second-order system to a third-order one
- Hence the control system may become unstable for a large value of K_p, since the roots of the characteristic equation may have positive real parts
- It is important to point out that if the controller were only an integral controller, then the system always becomes unstable because the characteristic equation will have roots with positive real parts

$$Js^3 + bs^2 + K = 0$$

- Such an unstable system cannot be used in practice

Derivative Control Action

- Derivative control action, when added to a proportional controller, provides a means of obtaining a controller with high sensitivity
- An advantage of using derivative control action is that it responds to the rate of change of the actuating error and can produce a significant correction before the magnitude of the actuating error becomes too large
- Derivative control thus anticipates the actuating error, initiates an early corrective action, and tends to increase the stability of the system

Derivative Control Action (2)

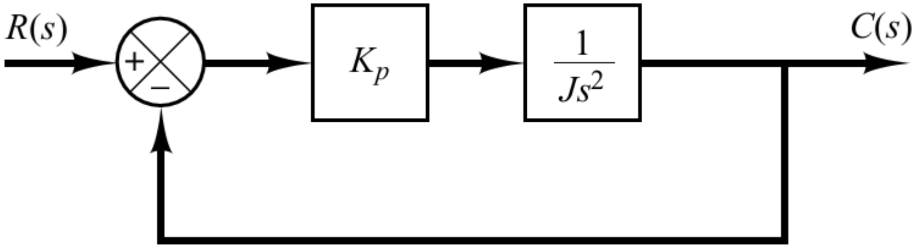
- Although derivative control does not affect the steady-state error directly, it adds damping to the system and thus permits the use of a larger value of the gain K, which will result in an improvement in the steady-state accuracy
- Because derivative control operates on the rate of change of the actuating error and not the actuating error itself, this mode is never used alone
- It is always used in combination with proportional or proportional-plus-integral control action

Proportional Control of Systems with Inertia Load

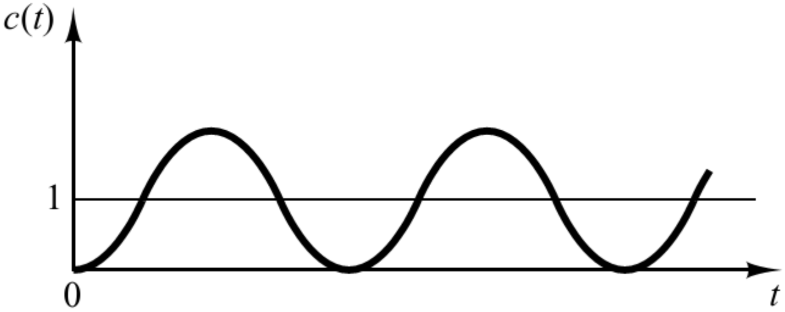
- Consider the system shown in Figure

$$\frac{C(s)}{R(s)} = \frac{K_P}{Js^2+K_P}$$

- the roots of the characteristic equation are imaginary
- the response to a unit-step input continues to oscillate indefinitely, as shown in Figure
- Such a system is not desirable



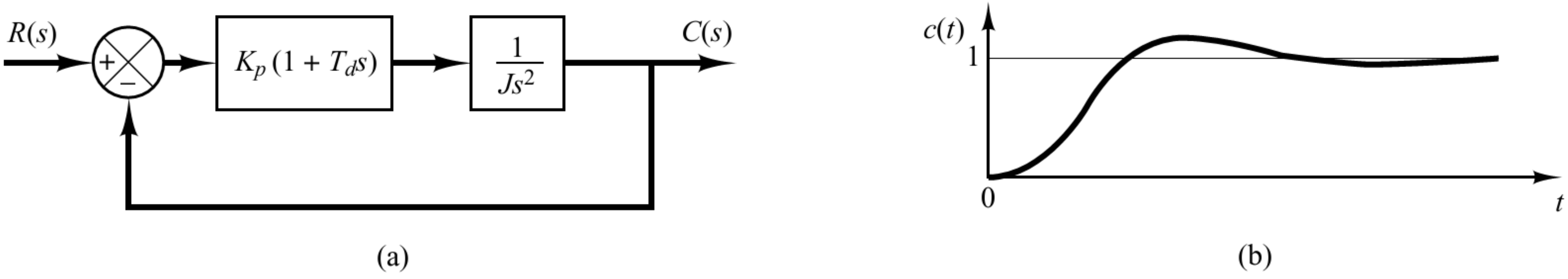
(a)



(b)

PD Control of a System with Inertia Load

- Let us consider the following system



(a) (b)

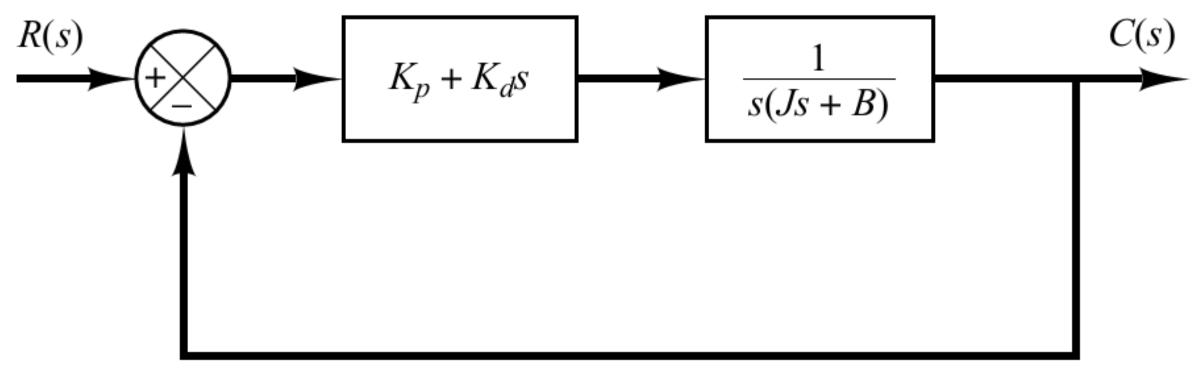
- The closed-loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{K_P(1+T_d s)}{Js^2+K_P T_d s+K_P}$$

- The Transfer function has two poles with negative real parts for positive values of J ,K_p , and T_d
- Thus derivative control introduces a damping effect

PD Control of Second-Order Systems

- Consider the system shown in Figure



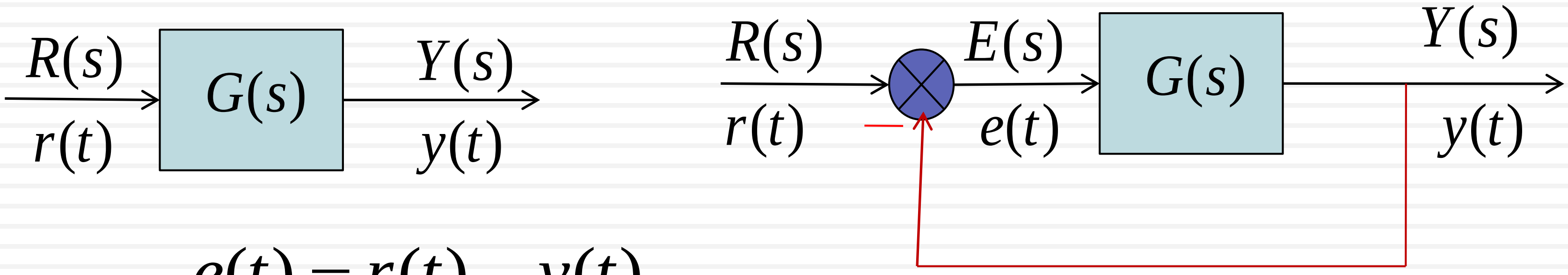
- The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K_P + K_d s}{Js^2 + (B + K_d)s + K_P}$$

- The characteristic equation is: $Js^2 + (B + K_d)s + K_P = 0$
- The effective damping coefficient of this system is: $\zeta = \frac{B+K_d}{2\sqrt{K_P J}}$
- It is possible to control ζ by adjusting B, K_P , K_d.

Steady-state Error

Error and steady-state error


Open-loop control system

Closed-loop control system

Error: $e(t) = r(t) - y(t)$

Steady-state error: $e_{ss} = \lim_{t \to \infty} e(t)$

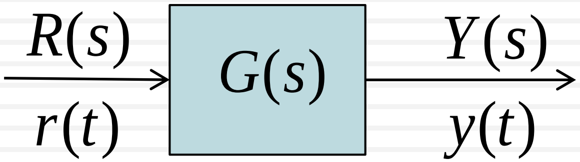
Utilizing the final value theorem: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

Assuming r(t)=l(t) is a unit-step input, according to the above definition, could you calculate the steady-state error of the open-loop and closed-loop control systems?

Error and steady-state error for a unit-step input

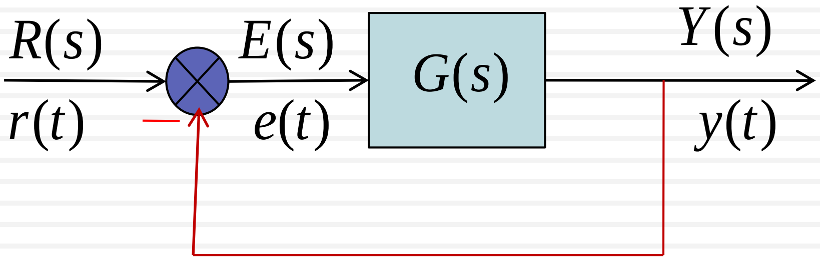
- Open-loop control system



$$E(s) = R(s) - Y(s)$$
$$= R(s) - G(s)R(s)$$
$$= [1 - G(s)]R(s)$$

$$e_{ss} = \lim_{s \to 0} sE(s)$$
$$= \lim_{s \to 0} s[1 - G(s)]\frac{1}{s}$$
$$= \lim_{s \to 0}[1 - G(s)]$$
$$= 1 - G(0)$$

- Closed-loop control system



$$E(s) = R(s) - Y(s)$$
$$= R(s) - \frac{G(s)}{1 + G(s)}R(s)$$
$$= \frac{1}{1 + G(s)}R(s)$$
$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s\frac{1}{1 + G(s)}\frac{1}{s}$$
$$= \lim_{s \to 0}\frac{1}{1 + G(s)} = \frac{1}{1 + G(0)}$$

The forward-path transfer function G(s) can be formulated as

$$G(s)=\frac{k(\tau_1 s+1)\cdots(\tau_2^2 s^2+2\xi_1\tau_2 s+1)}{s^{v}(T_1 s+1)\cdots(T_2^2 s^2+2\xi_2 T_2 s+1)}$$

ν is the order of the pole of G(s) at s=0

$$=\frac{k}{s^{v}}G_0(s) \qquad \text{when } s\to 0,\quad G_0(s)\to 1$$

System Type: the order of the pole of G(s) at s=0.

When ν=0,1,2, the system is called type 0, type 1, type 2; k is called open-loop gain.

$$E(s)=\frac{1}{1+G(s)}R(s)$$

$$e_{ss}=\lim_{s\to 0} sE(s)=\lim_{s\to 0} s\frac{1}{1+\frac{k}{s^{v}}G_0(s)}R(s)$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+\frac{k}{s^0}G_0(s)}R(s)$$

ν=0, type 0 system

Step input:

$r(t) = 1(t) \quad R(s) = \frac{1}{s}$

$$e_{ss} = \frac{1}{1+k}$$

Steady-state error exists and is finite.

Ramp input:

$r(t) = t \quad R(s) = \frac{1}{s^2}$

$$e_{ss} = \infty$$

Unstable

Parabolic input:

$r(t) = \frac{1}{2}t^2 \quad R(s) = \frac{1}{s^3}$

$$e_{ss} = \infty$$

Unstable

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{1}{1+\frac{k}{s^1}G_0(s)} R(s)$$

ν=1, type 1 system

Step input:

$r(t) = 1(t) \quad R(s) = \frac{1}{s}$

$$e_{ss} = \frac{1}{1+\infty} = 0$$

No steady-state error

Ramp input:

$r(t) = t \quad R(s) = \frac{1}{s^2}$

$$e_{ss} = \frac{1}{k}$$

Steady-state error exists

Parabolic input:

$r(t) = \frac{1}{2}t^2 \quad R(s) = \frac{1}{s^3}$

$$e_{ss} = \infty$$

Unstable

Type-1 system can track step signal accurately.

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{1}{1+\frac{k}{s^2}G_0(s)} R(s)$$

ν=2, type 2 system

Step input:

$r(t) = 1(t) \quad R(s) = \frac{1}{s}$

$$e_{ss} = \frac{1}{1+\infty} = 0$$

No steady-state error

Ramp input:

$r(t) = t \quad R(s) = \frac{1}{s^2}$

$$e_{ss} = 0$$

No steady-state error

Parabolic input:

$r(t) = \frac{1}{2}t^2 \quad R(s) = \frac{1}{s^3}$

$$e_{ss} = \frac{1}{k}$$

Steady-state error exists

Type-2 system can track step and ramp signals accurately.

Steady-state error constants

with step input

$$k_p = \lim_{s\to 0} G(s)$$

- step-error constant (static position error constant)

with ramp input

$$k_v = \lim_{s\to 0} sG(s)$$

- ramp-error constant (static velocity error constant)

with parabolic input

$$k_a = \lim_{s\to 0} s^2G(s)$$

- parabolic-error constant (static acceleration error constant)

Type of System	**Error constants**			**Steady-state error** e_{ss}		
j	k_p	k_v	k_a	$r(t)=R_0\cdot 1(t)$	$r(t)=V_0 t$	$r(t)=A_0 t^2/2$
0	k	0	0	$\frac{R_0}{1+k}$	∞	∞
I	∞	k	0	0	$\frac{V_0}{k}$	∞
II	∞	∞	k	0	0	$\frac{A_0}{k}$

✵ **Summary of steady-state error and error constants for unit-feedback systems (H(s)=1)**