MODERN CONTROL

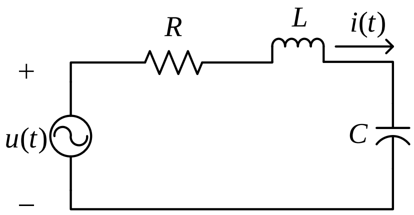
Faculty of Engineering
Alexandria University

Classical Control and Modern Control

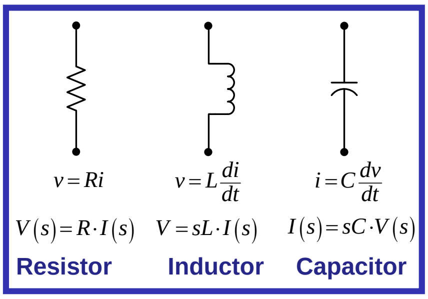
Classical Control	Modern Control
SISO (Single Input Single Output)	MIMO (Multiple Input Multiple Output)
Low order ODEs	High order ODEs, PDEs
Time-invariant	Time-invariant and time variant
Fixed parameters	Changing parameters
Linear	Linear and non-linear
Time-response approach	Time- and frequency response approach
Continuous, analog	Tends to be discrete, digital
Before 80s	80s and after

- The difference between classical control and modern control originates from the different modeling approach used by each control.
- The modeling approach used by modern control enables it to have new features not available for classical control.

Laplace Transform Approach



RLC Circuit



Input variables:
- Input voltage $u(t)$

Output variables:
- Current $i(t)$

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(\tau)d\tau + v_0 = u(t)$$

$$RI(s) + L(sI(s) - i_0) + \frac{1}{Cs}I(s) + \frac{v_0}{s} = U(s)$$

Laplace Transform Approach

$$RI(s) + L(sI(s) - i_0) + \frac{1}{Cs}I(s) + \frac{v_0}{s} = U(s)$$

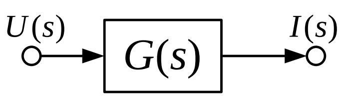
$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = U(s) + Li_0 - \frac{v_0}{s}$$

$$I(s) = \underbrace{\frac{Cs}{LCs^2 + RCs + 1}U(s)}_{\text{Current due to input}} + \underbrace{\frac{LCsi_0 - Cv_0}{LCs^2 + RCs + 1}}_{\text{Current due to initial condition}}$$

- For zero initial conditions ($v_0 = 0, i_0 = 0$),

$$I(s) = G(s)U(s)$$

where $$G(s) = \frac{Cs}{LCs^2 + RCs + 1}$$

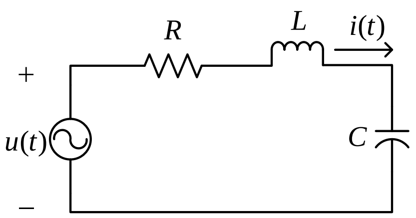


Transfer function

State Space Approach

- Laplace Transform method is not effective to model time-varying and non-linear systems.
- The state space approach to be studied in this course will be able to handle more general systems.
- The state space approach characterizes the properties of a system without solving for the exact output.
- Let us now consider the same RLC circuit and try to use state space to model it.

State Space Approach



RLC Circuit

State variables:
- Voltage across *C*
- Current through *L*

$$i_C = C\frac{dv_C}{dt} \Rightarrow \frac{dv_C}{dt} = \frac{1}{C}i_C$$

$$\boxed{\frac{dv_C}{dt} = \frac{1}{C}i_L}$$

$$v_L = L\frac{di_L}{dt} \Rightarrow \frac{di_L}{dt} = \frac{1}{L}v_L$$

$$\frac{di_L}{dt} = \frac{1}{L}(u - v_R - v_C)$$

$$\boxed{\frac{di_L}{dt} = \frac{1}{L}(u - Ri_L - v_C)}$$

- **We now have two first-order ODEs**
- **Their variables are the state variables and the input**

State Space Approach

- The two equations are called **state equations**, and can be rewritten in the form of:

$$\frac{dv_C}{dt} = \frac{1}{C} i_L$$

$$\frac{di_L}{dt} = \frac{1}{L}(u - Ri_L - v_C)$$

$$\begin{bmatrix} dv_C/dt \\ di_L/dt \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u$$

- The output is described by an **output equation**:

$$i_L = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

State Space Approach

- The state equations and output equation, combined together, form the **state space** description of the circuit.

$$\begin{bmatrix} dv_C/dt \\ di_L/dt \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u$$

$$i_L = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

- In a more compact form, the state space can be written as:

$$\underline{\dot{x}} = \underline{A}\underline{x} + \underline{B}u$$

$$y = \underline{C}\underline{x}$$

$$\underline{A} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \quad \underline{x} = \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \quad \underline{\dot{x}} = \begin{bmatrix} dv_C/dt \\ di_L/dt \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

State Space Approach

- The state of a system at t_0 is the information at t_0 that, together with the input u for $t_0 \leq t < \infty$, uniquely determines the behavior of the system for $t \geq t_0$.
- The number of state variables = the number of initial conditions needed to solve the problem.
- As we will learn in the future, there are infinite numbers of state space that can represent a system.

- The main features of state space approach are:
  - It describes the behaviors inside the system.
  - Stability and performance can be analyzed without solving for any differential equations.
  - Applicable to more general systems such as non-linear systems, time-varying system.
  - Modern control theory are developed using state space approach.

Classification of Systems

- Systems are classified based on:
  - *The number of inputs and outputs*: single-input single-output (SISO), multi-input multi-output (MIMO), MISO, SIMO.
  - *Existence of memory*: if the current output depends on the current input only, then the system is said to be memoryless, otherwise it has memory → purely resistive circuit vs. RLC-circuit.
  - *Causality*: a system is called causal or non-anticipatory if the output depends only on the present and past inputs and independent of the future unfed inputs.
  - *Dimensionality*: the dimension of system can be finite (lumped) or infinite (distributed).
  - *Linearity*: superposition of inputs yields the superposition of outputs.
  - *Time-Invariance*: the characteristics of a system with the change of time.

Linear System

- A system $\underline{y}(t) = f(\underline{x}(t),\underline{u}(t))$ is said to be linear if it follows the following conditions:
  - If $f\left(\underline{x}_1(t),\underline{u}_1(t)\right)=\underline{y}_1(t),$
    then $f\left(\alpha\underline{x}(t),\alpha\underline{u}_1(t)\right)=\alpha\underline{y}_1(t)$
  - If $f\left(\underline{x}_1(t),\underline{u}_1(t)\right)=\underline{y}_1(t)$
    and $f\left(\underline{x}_2(t),\underline{u}_2(t)\right)=\underline{y}_2(t)$
    then $f\left(\underline{x}_1(t)+\underline{x}_2(t),\underline{u}_1(t)+\underline{u}_2(t)\right)=\underline{y}_1(t)+\underline{y}_2(t)$
  - Then, it can also be implied that

$$f\left(\alpha\underline{x}_1(t)+\beta\underline{x}_2(t),\alpha\underline{u}_1(t)+\beta\underline{u}_2(t)\right)=\alpha\underline{y}_1(t)+\beta\underline{y}_2(t)$$

Linear Time-Invariant (LTI) System

- A system is said to be linear time-invariant if it is linear and its parameters do not change over time.

State Space Equations

- The state equations of a system can generally be written as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & \cdots & b_{1r} \\ \vdots & b_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{n1} & \cdots & \cdots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}$$

$x_1(t), x_2(t), \cdots, x_n(t)$ are the state variables

$u_1(t), u_2(t), \cdots, u_r(t)$ are the system inputs

- **State equations are built of *n* linearly-coupled first-order ordinary differential equations**

State Space Equations

- By defining:

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \qquad \underline{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix},$$

we can write $\boxed{\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)}$ **State Equations**

State Space Equations

- The outputs of the state space are the linear combinations of the state variables and the inputs:

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \vdots \\ \dot{y}_m(t) \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & \cdots & c_{1n} \\ \vdots & c_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ c_{m1} & \cdots & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} d_{11} & \cdots & \cdots & d_{1r} \\ \vdots & d_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ d_{m1} & \cdots & \cdots & d_{mr} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}$$

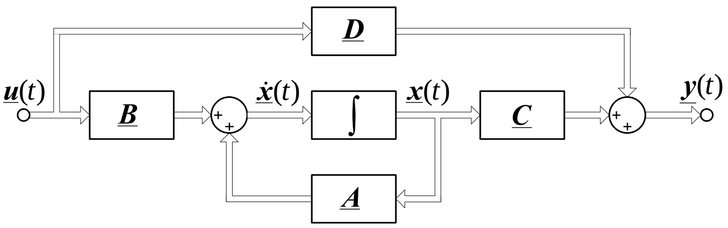
$y_1(t), y_2(t), \cdots, y_m(t)$ are the system outputs

State Space Equations

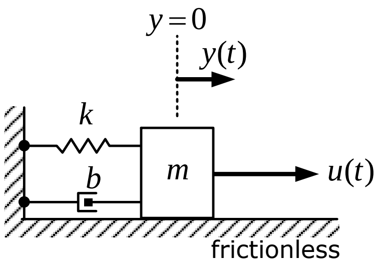
- By defining:

$$\underline{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix},$$

we can write $$\underline{y}(t) = \underline{C}\,\underline{x}(t) + \underline{D}\,\underline{u}(t)$$ **Output Equations**



Example: Mechanical System



$$u(t)-ky(t)-b\frac{dy(t)}{dt}=m\frac{d^2y(t)}{dt^2}$$

Input variables:

- Applied force $u(t)$

Output variables:

- Displacement $y(t)$

State variables:

$$x_1(t)=y(t) \quad \Rightarrow \dot{x}_1(t)=x_2(t)$$

$$x_2(t)=\frac{dy(t)}{dt} \quad \Rightarrow \dot{x}_2(t)=\frac{d^2y(t)}{dt^2}$$

State equations:

$$\dot{x}_1(t)=x_2(t)$$

$$\dot{x}_2(t)=-\frac{k}{m}x_1(t)-\frac{b}{m}x_2(t)+\frac{1}{m}u(t)$$

Example: Mechanical System

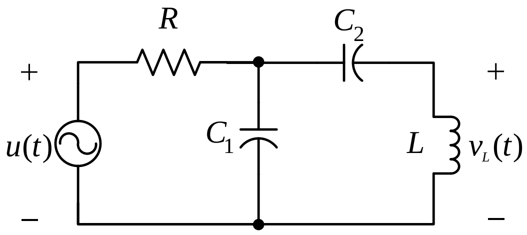
- The state space equations can now be constructed as below:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -k/m & -b/m\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1/m\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

Homework 1: Electrical System

- Derive the state space representation of the following electric circuit:

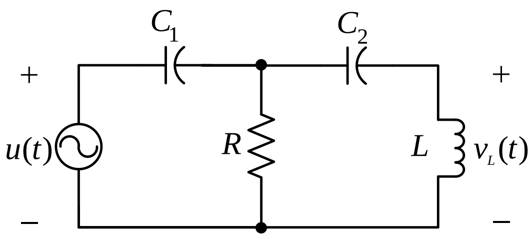


Input variables:
- Input voltage $u(t)$

Output variables:
- Inductor voltage $v_L(t)$

Homework 1A: Electrical System

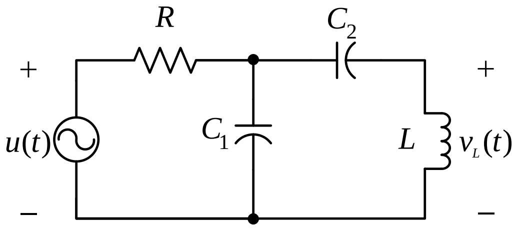
- Derive the state space representation of the following electric circuit:



Input variables:
- Input voltage $u(t)$

Output variables:
- Inductor voltage $v_L(t)$

Solution of Homework I: Electrical System



$$v_R = Ri_R$$
$$v_L = L\frac{di_L}{dt} = L\dot{i}_L$$
$$i_C = C\frac{dv_C}{dt} = C\dot{v}_C$$

State variables:

- x_1 is the voltage across C_1
- x_2 is the voltage across C_2
- x_3 is the current through L

$$(x_1 - u)/R + C_1\dot{x}_1 + C_2\dot{x}_2 = 0 \qquad \dot{x}_1 = -1/RC_1 \cdot x_1 - 1/C_1 \cdot x_3 + 1/RC_1 \cdot u$$
$$C_2\dot{x}_2 = x_3 \qquad \dot{x}_2 = 1/C_2 \cdot x_3$$
$$x_1 - x_2 = L\dot{x}_3 \qquad \dot{x}_3 = 1/L \cdot x_1 - 1/L \cdot x_2$$

Solution of Homework I: Electrical System

- The state space equation can now be written as:

$$\dot{x}_1 = -1/RC_1 \cdot x_1 - 1/C_1 \cdot x_3 + 1/RC_1 \cdot u$$
$$\dot{x}_2 = 1/C_2 \cdot x_3$$
$$\dot{x}_3 = 1/L \cdot x_1 - 1/L \cdot x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1/RC_1 & 0 & -1/C_1 \\ 0 & 0 & 1/C_2 \\ 1/L & -1/L & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/RC_1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0 \cdot u$$

Example: Transfer Function

- Given the following transfer function

$$Y(s) = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} U(s)$$

and assuming zero initial conditions, construct a state space equations that can represent the given transfer function.

$$s^3 Y(s) + a_2 s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = U(s)$$

$$\dddot{y}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = u(t)$$

$$\left.\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \ddot{y} \end{aligned}\right\rangle \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \dddot{y} = -a_0 x_1 - a_1 x_2 - a_2 x_3 + u(t) \end{aligned}$$

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Example: Transfer Function

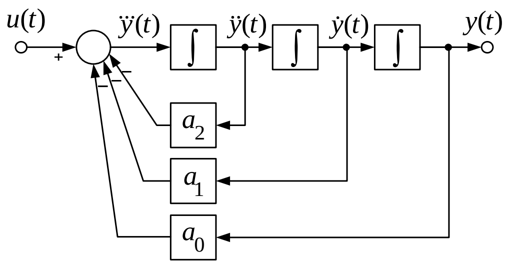
The state space equation can now be given as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

$$Y(s) = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} U(s)$$

The state space equation can also be given using block diagram:



Vector Case and Scalar Case

- The general form of state space in vector case, where there are multiple inputs and multiple outputs, is given as:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t)$$
$$\underline{y}(t) = \underline{C}\underline{x}(t) + \underline{D}\underline{u}(t)$$

- In scalar case, where the input and the output are scalar or single, the state space is usually written as:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{b}u(t)$$
$$y(t) = \underline{c}^{T}\underline{x}(t) + \underline{d}u(t)$$

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Solution of State Equations

- Consider the state equations in vector case.

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t)$$

- Multiplying each term with $e^{-\underline{A}t}$,

$$e^{-\underline{A}t}\dot{\underline{x}}(t) = e^{-\underline{A}t}\underline{A}\underline{x}(t) + e^{-\underline{A}t}\underline{B}\underline{u}(t)$$

$$e^{-\underline{A}t}\dot{\underline{x}}(t) - e^{-\underline{A}t}\underline{A}\underline{x}(t) = e^{-\underline{A}t}\underline{B}\underline{u}(t)$$

$$\frac{d}{dt}\left(e^{-\underline{A}t}\underline{x}(t)\right) = e^{-\underline{A}t}\underline{B}\underline{u}(t)$$

$$\frac{d}{dt}\left(e^{-\underline{A}t}\right) = -\underline{A}e^{-\underline{A}t}$$

- The last equation will be integrated from 0 to *t*:

$$e^{-\underline{A}t}\underline{x}(\tau)\Big]_0^t = \int_0^t e^{-\underline{A}\tau}\underline{B}\underline{u}(\tau)d\tau$$

Solution of State Equations

$$e^{-A\tau}\underline{x}(\tau)\Big]_0^t = \int_0^t e^{-A\tau}\underline{Bu}(\tau)d\tau$$

$$e^{-At}\underline{x}(t) - e^{-A0}\underline{x}(0) = \int_0^t e^{-A\tau}\underline{Bu}(\tau)d\tau$$

$$\underline{x}(t) = e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)}\underline{Bu}(\tau)d\tau$$

Solution of State Equations

- At t=0, $\underline{x}(t)$ = $\underline{x}(0)$ = $\underline{x}_0$, which are the initial conditions of the states.

Solution of Output Equations

- We know substitute the solution of state equations into the output equations:

$$\underline{y}(t) = \underline{C}\,\underline{x}(t) + \underline{Du}(t)$$

$$\underline{y}(t) = \underline{C}\left\{e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)}\underline{Bu}(\tau)d\tau\right\} + \underline{Du}(t)$$

Solution of Output Equations

Solutions of State Space in Frequency Domain

- The solution of state equations and output equations can also be written in frequency domain:

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)$$

$$s\underline{X}(s) - \underline{x}(0) = \underline{A}\,\underline{X}(s) + \underline{B}\,\underline{U}(s)$$

$$(s\underline{I} - \underline{A})\underline{X}(s) = \underline{x}(0) + \underline{B}\,\underline{U}(s)$$

$$\boxed{\underline{X}(s) = (s\underline{I} - \underline{A})^{-1}\underline{x}(0) + (s\underline{I} - \underline{A})^{-1}\underline{B}\,\underline{U}(s)}$$

Solution of State Equations

$$\underline{y}(t) = \underline{C}\,\underline{x}(t) + \underline{D}\,\underline{u}(t)$$

$$\underline{Y}(s) = \underline{C}\,\underline{X}(s) + \underline{D}\,\underline{U}(s)$$

$$\boxed{\underline{Y}(s) = \underline{C}\left\{(s\underline{I} - \underline{A})^{-1}\underline{x}(0) + (s\underline{I} - \underline{A})^{-1}\underline{B}\,\underline{U}(s)\right\} + \underline{D}\,\underline{U}(s)}$$

Solution of Output Equations

Relation between $e^{\underline{A}t}$ and $(s\underline{I} - \underline{A})$

- Taylor series expansion of exponential function is given by:

$$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \ldots + \frac{\lambda^n t^n}{n!}$$

Scalar Function
- **Exact solution, around t = 0, infinite number of terms**

$$e^{\underline{A}t} = \underline{I} + t\underline{A} + \frac{t^2}{2!}\underline{A}^2 + \ldots + \frac{t^n}{n!}\underline{A}^n$$

Vector Function

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!}\underline{A}^k$$

- It can be shown that $\mathcal{L}\left[\frac{t^k}{k!}\right] = s^{-(k+1)}$ so that:

$$\mathcal{L}\left[e^{\underline{A}t}\right] = \mathcal{L}\left[\sum_{k=0}^{\infty} \frac{t^k}{k!}\underline{A}^k\right] = \sum_{k=0}^{\infty} s^{-(k+1)}\underline{A}^k$$

Relation between $e^{\underline{A}t}$ and $(s\underline{I}-\underline{A})$

- Deriving further,

$$\mathcal{L}\left[e^{\underline{A}t}\right]=\sum_{k=0}^{\infty}s^{-(k+1)}\underline{A}^{k}$$
$$=s^{-1}\underline{I}+s^{-2}\underline{A}+s^{-3}\underline{A}^{2}+\ldots$$
$$=\frac{s^{-1}\underline{I}}{\underline{I}-s^{-1}\underline{A}}$$
$$=s^{-1}(\underline{I}-s^{-1}\underline{A})^{-1}$$
$$=\left(s(\underline{I}-s^{-1}\underline{A})\right)^{-1}$$

$$\mathcal{L}\left[e^{\underline{A}t}\right]=(s\underline{I}-\underline{A})^{-1} \qquad e^{\underline{A}t}=\mathcal{L}^{-1}\left[(s\underline{I}-\underline{A})^{-1}\right]$$

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Example: Solution of State Equations

Compute $(s\underline{I}-\underline{A})^{-1}$ if $\underline{A}=\begin{bmatrix}0 & -1\\ 1 & -2\end{bmatrix}.$

$$(s\underline{I}-\underline{A})=\begin{bmatrix}s & 1\\ -1 & s+2\end{bmatrix}$$

$$(s\underline{I}-\underline{A})^{-1}=\frac{1}{(s)(s+2)-(1)(-1)}\begin{bmatrix}s+2 & -1\\ 1 & s\end{bmatrix}$$

$$=\begin{bmatrix}\frac{s+2}{s^2+2s+1} & \frac{-1}{s^2+2s+1}\\ \frac{1}{s^2+2s+1} & \frac{s}{s^2+2s+1}\end{bmatrix}$$

Example: Solution of State Equations

Given $\underline{\dot{x}}(t)=\begin{bmatrix}0 & -1\\ 1 & -2\end{bmatrix}\underline{x}(t)+\begin{bmatrix}0\\ 1\end{bmatrix}u(t)$, find the solution for $\underline{\mathbf{x}}(t)$.

$$\underline{\mathbf{x}}(t)=e^{\underline{A}t}\,\underline{\mathbf{x}}(0)+\int_0^t e^{\underline{A}(t-\tau)}\underline{\mathbf{B}}u(\tau)d\tau$$

$$e^{\underline{A}t}=\mathcal{L}^{-1}\left[(s\underline{I}-\underline{A})^{-1}\right]$$

$$=\mathcal{L}^{-1}\begin{bmatrix}\dfrac{s+2}{(s+1)^2} & \dfrac{-1}{(s+1)^2}\\ \dfrac{1}{(s+1)^2} & \dfrac{s}{(s+1)^2}\end{bmatrix}$$

$$=\begin{bmatrix}(1+t)e^{-t} & -te^{-t}\\ te^{-t} & (1-t)e^{-t}\end{bmatrix}$$

$f(t)$	$F(s)$
$\frac{t^{n-1}e^{-at}}{(n-1)!}1(t), n\geq 1$	$\frac{1}{(s+a)^n}$

Example: Solution of State Equations

Now, we substitute $e^{\underline{A}t}$ to obtain the solution for $\underline{\mathbf{x}}(t)$:

$$\underline{\mathbf{x}}(t)=\begin{bmatrix}(1+t)e^{-t} & -te^{-t}\\ te^{-t} & (1-t)e^{-t}\end{bmatrix}\underline{\mathbf{x}}(0)+$$

$$\int_0^t\begin{bmatrix}(1+(t-\tau))e^{-(t-\tau)} & -(t-\tau)e^{-(t-\tau)}\\ (t-\tau)e^{-(t-\tau)} & (1-(t-\tau))e^{-(t-\tau)}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}u(\tau)d\tau$$

$$=\begin{bmatrix}(1+t)e^{-t} & -te^{-t}\\ te^{-t} & (1-t)e^{-t}\end{bmatrix}\underline{\mathbf{x}}(0)+\begin{bmatrix}-\int_0^t(t-\tau)e^{-(t-\tau)}u(\tau)d\tau\\ \int_0^t(1-(t-\tau))e^{-(t-\tau)}u(\tau)d\tau\end{bmatrix}$$

Example: Solution of State Equations

If $\underline{\mathbf{x}}(0)=\underline{\mathbf{0}}$ and $u(t)$ is a step function, determine $\underline{\mathbf{x}}(t)$.

$$\underline{\boldsymbol{x}}(t)=\begin{bmatrix}(1+t)e^{-t} & -te^{-t}\\ te^{-t} & (1-t)e^{-t}\end{bmatrix}\underline{\mathbf{0}}+\begin{bmatrix}-\int_0^t (t-\tau)e^{-(t-\tau)}1(\tau)d\tau\\ \int_0^t (1-(t-\tau))e^{-(t-\tau)}1(\tau)d\tau\end{bmatrix}$$

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\begin{bmatrix}-\int_0^t (t-\tau)e^{-(t-\tau)}d\tau\\ \int_0^t (1-(t-\tau))e^{-(t-\tau)}d\tau\end{bmatrix}$$

Example: Solution of State Equations

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\begin{bmatrix}\int_0^t (t-\tau)e^{-(t-\tau)}d(t-\tau)\\ \int_0^t ((t-\tau)-1)e^{-(t-\tau)}d(t-\tau)\end{bmatrix}$$

$$\frac{d(t-\tau)}{d\tau}=-1 \qquad d(t-\tau)=-d\tau$$

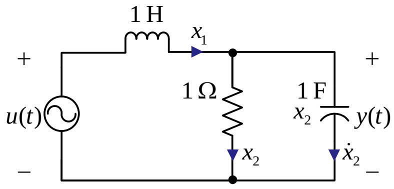
$$\int te^{-t}dt=-e^{-t}(1+t) \qquad \int e^{-t}dt=-e^{-t}$$

$$=\begin{bmatrix}-e^{-(t-\tau)}(1+(t-\tau))\Big]_0^t\\ -e^{-(t-\tau)}(t-\tau)\Big]_0^t\end{bmatrix}$$

$$=\begin{bmatrix}-1+e^{-t}(1+t)\\ e^{-t}t\end{bmatrix}$$

$$\Rightarrow \quad x_1(t)=-1+e^{-t}(1+t), \quad x_2(t)=e^{-t}t$$

Equivalent State Equations



State variables:

- x_1 : inductor current i_L
- x_2 : capacitor voltage v_C

$$v_L = L\frac{di_L}{dt} = \dot{x}_1$$

$$i_R = \frac{v_R}{R} = x_2$$

$$i_C = C\frac{dv_C}{dt} = \dot{x}_2$$

$$x_2 = u - \dot{x}_1$$

$$\dot{x}_2 = x_1 - x_2$$

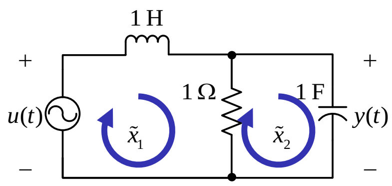
$$y = x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Homework 2: Equivalent State Equations

1. Prove that for the same system, with different definition of state variables, we can obtain a state space in the form of:



$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

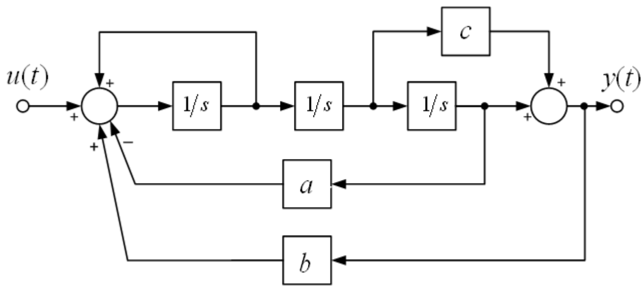
$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

State variables:

- $\tilde{x}_1$: current of left loop
- $\tilde{x}_2$: current of right loop

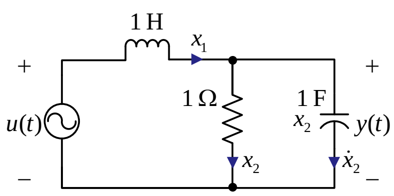
Homework 2: Equivalent State Equations

2. Derive a state-space description for the following diagram



FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Equivalent State Equations



$$v_L = L\frac{di_L}{dt} = \dot{x}_1$$

$$i_R = \frac{v_R}{R} = x_2$$

$$i_C = C\frac{dv_C}{dt} = \dot{x}_2$$

$$x_2 = u - \dot{x}_1$$

$$\dot{x}_2 = x_1 - x_2$$

$$y = x_2$$

State variables:

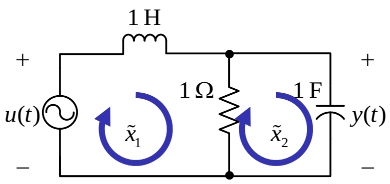
- x_1: inductor current i_L
- x_2: capacitor voltage v_C

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u(t)$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Homework 2: Equivalent State Equations

1. Prove that for the same system, with different definition of state variables, we can obtain a state space in the form of:



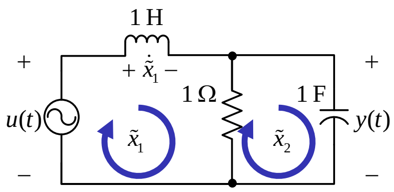
$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

State variables:

- $\tilde{x}_1$: current of left loop
- $\tilde{x}_2$: current of right loop

Homework 2: Equivalent State Equations



State variables:

- $\tilde{x}_1$: loop current left
- $\tilde{x}_2$: loop current right

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

$$-u + \dot{\tilde{x}}_1 + (\tilde{x}_1 - \tilde{x}_2) = 0$$

$$\boxed{\dot{\tilde{x}}_1 = u - \tilde{x}_1 + \tilde{x}_2}$$

$$\boxed{v_C = \tilde{x}_1 - \tilde{x}_2 = y}$$

$$i_C = C\frac{dv_C}{dt}$$

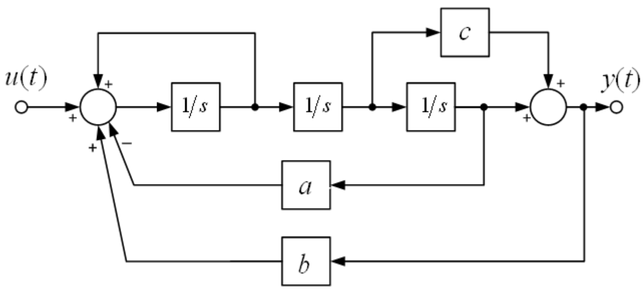
$$\tilde{x}_2 = \dot{\tilde{x}}_1 - \dot{\tilde{x}}_2$$

$$= (u - \tilde{x}_1 + \tilde{x}_2) - \dot{\tilde{x}}_2$$

$$\boxed{\dot{\tilde{x}}_2 = u - \tilde{x}_1}$$

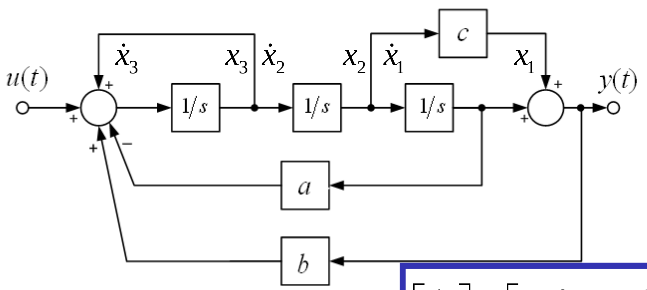
Homework 2: Equivalent State Equations

2. Derive a state-space description for the following diagram



FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Homework 2: Equivalent State Equations



$$y = x_1 + cx_2$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\begin{aligned}\dot{x}_3 &= -ax_1 + x_3 + by + u \\ &= -ax_1 + x_3 + b(x_1 + cx_2) + u \\ &= (-a+b)x_1 + bcx_2 + x_3 + u\end{aligned}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a+b & bc & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & c & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + 0u$$

Equivalent State Equations

- Consider an n-dimensional state space equations:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t)$$
$$\underline{y}(t) = \underline{C}\underline{x}(t) + \underline{D}\underline{u}(t)$$

- Let $\underline{P}$ be an $n \times n$ real nonsingular matrix, and let $\tilde{\underline{x}} = \underline{P}\,\underline{x}$. Then, the state space equations

$$\dot{\tilde{\underline{x}}}(t) = \tilde{\underline{A}}\tilde{\underline{x}}(t) + \tilde{\underline{B}}\underline{u}(t)$$
$$\underline{y}(t) = \tilde{\underline{C}}\tilde{\underline{x}}(t) + \tilde{\underline{D}}\underline{u}(t)$$

where

$$\tilde{\underline{A}} = \underline{P}\,\underline{A}\,\underline{P}^{-1}, \quad \tilde{\underline{B}} = \underline{P}\,\underline{B}, \quad \tilde{\underline{C}} = \underline{C}\,\underline{P}^{-1}, \quad \tilde{\underline{D}} = \underline{D}.$$

is said to be algebraically equivalent with the original state space equations.

- $\tilde{\underline{x}} = \underline{P}\,\underline{x}$ is called an equivalence transformation.

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Equivalent State Equations

- *Proof*:

Substituting $\underline{x}(t) = \underline{P}^{-1}\tilde{\underline{x}}(t)$

$$\underline{P}^{-1}\dot{\tilde{\underline{x}}}(t) = \underline{A}\underline{P}^{-1}\tilde{\underline{x}}(t) + \underline{B}\underline{u}(t)$$

$$\dot{\tilde{\underline{x}}}(t) = \underbrace{\underline{P}\,\underline{A}\,\underline{P}^{-1}}_{\tilde{\underline{A}}}\tilde{\underline{x}}(t) + \underbrace{\underline{P}\,\underline{B}}_{\tilde{\underline{B}}}\underline{u}(t)$$

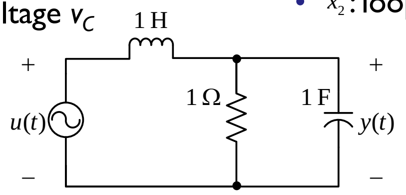
$$\underline{y}(t) = \underbrace{\underline{C}\,\underline{P}^{-1}}_{\tilde{\underline{C}}}\tilde{\underline{x}}(t) + \underbrace{\underline{D}}_{\tilde{\underline{D}}}\underline{u}(t)$$

Equivalent State Equations

- From the last electrical circuit,

  State variables:
  - x_1: inductor current i_L
  - x_2: capacitor voltage v_C

State variables:
- $\tilde{x}_1$: loop current left
- $\tilde{x}_2$: loop current right



- The two sets of states can be related in the way:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Transfer Function and Transfer Matrix

- Consider a state space equations for SISO systems:

$$\underline{\dot{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}u(t)$$

$$y(t) = \underline{C}\underline{x}(t) + Du(t)$$

- Using Laplace transform, we will obtain:

$$s\underline{X}(s) - \underline{x}(0) = \underline{A}\underline{X}(s) + \underline{B}U(s)$$

$$Y(s) = \underline{C}\underline{X}(s) + DU(s)$$

- For zero initial conditions, $\underline{x}(0) = \underline{0}$,

$$\underline{X}(s) = (s\underline{I} - \underline{A})^{-1}\underline{B}U(s)$$

$$Y(s) = \left(\underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} + D\right)U(s)$$

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} + D}$$

Transfer Function

Realization of State Space Equations

- Every linear time-invariant system can be described by the input-output description in the form of:

$$Y(s) = U(s)G(s)$$

- If the system is lumped (i.e., having concentrated parameters), it can also be described by the state space equations

$$\underline{\dot{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}u(t)$$

$$y(t) = \underline{C}\underline{x}(t) + Du(t)$$

- The problem concerning how to describe a system in state space equations, provided that the transfer function of a system, $G(s)$, is available, is called **Realization Problem**.

$G(s)$ ⇒ $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$.

Realization of State Space Equations

- Three realization methods will be discussed now:
  - Frobenius Form
  - Observer Form
  - Canonical Form

Frobenius Form

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}$$

$$\frac{d^n y(t)}{dt^n}+a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}}+\cdots+a_1\frac{dy(t)}{dt}+a_0 y(t)=$$
$$b_m\frac{d^m u(t)}{dt^m}+b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}}+\cdots+b_1\frac{du(t)}{dt}+b_0 u(t)$$

- **Special Case**: No derivation of input

$$\frac{d^n y(t)}{dt^n}+a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}}+\cdots+a_1\frac{dy(t)}{dt}+a_0 y(t)=b_0 u(t)$$

Frobenius Form

- We now define:

$$x_1(t)=y(t)$$
$$x_2(t)=\dot{y}(t)=\dot{x}_1(t)$$
$$x_3(t)=\ddot{y}(t)=\dot{x}_2(t)$$
$$\vdots$$
$$x_n(t)=y^{(n-1)}(t)=\dot{x}_{n-1}(t)$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \vdots\\ \dot{x}_n(t)\end{bmatrix}=\begin{bmatrix}0 & 1 & \cdots & 0\\ 0 & 0 & 1\cdots & 0\\ \vdots & & & \vdots\\ -a_0 & -a_1 & \cdots & -a_{n-1}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \vdots\\ b_0\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}+0u(t)$$

Frobenius Form, Special Case

Frobenius Form

- **General Case:** With derivation of input

$$\frac{Y(s)}{U(s)}=\frac{b_m s^m + b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{N(s)}{D(s)} \qquad m<n$$

- If $m = n-1$ (largest possible value), then

$$Y(s)=\left(b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_1 s+b_0\right)\frac{U(s)}{D(s)}$$

$$Y(s)=b_0\underbrace{\frac{U(s)}{D(s)}}_{X_1(s)}+b_1\underbrace{s\frac{U(s)}{D(s)}}_{X_2(s)}+\cdots+b_{n-2}\underbrace{s^{n-2}\frac{U(s)}{D(s)}}_{X_{n-1}(s)}+b_{n-1}\underbrace{s^{n-1}\frac{U(s)}{D(s)}}_{X_n(s)}$$

Frobenius Form

- If $m = n-1$ (largest possible value), then

$$\begin{aligned}
x_1(t) &= \mathcal{L}^{-1}\left[X_1(s)\right]\\
x_2(t) &= \dot{x}_1(t)\\
&\vdots\\
x_{n-1}(t) &= \dot{x}_{n-2}(t)\\
x_n(t) &= \dot{x}_{n-1}(t)
\end{aligned}$$

- But $X_1(s)=\dfrac{U(s)}{D(s)}=\dfrac{U(s)}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}$

$$s^n X_1(s)+a_{n-1}s^{n-1}X_1(s)+\cdots+a_2 s^2 X_1(s)+a_1 s X_1(s)+a_0 X_1(s)=U(s)$$

$$s^n X_1(s)=U(s)-a_{n-1}s^{n-1}X_1(s)-\cdots-a_2 s^2 X_1(s)-a_1 s X_1(s)-a_0 X_1(s)$$

$$x_1^{(n)}(t)=u(t)-a_{n-1}x_1^{(n-1)}(t)-\cdots-a_2\ddot{x}_1(t)-a_1\dot{x}_1(t)-a_0 x_1(t)$$

$$\dot{x}_n(t)=u(t)-a_{n-1}x_n(t)-\cdots-a_2 x_3(t)-a_1 x_2(t)-a_0 x_1(t)$$

Frobenius Form

- The state space equations can now be written as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & & \vdots \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 & b_1 \cdots b_{n-2} & b_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + 0u(t)$$

Frobenius Form, General Case

Observer Form

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \qquad n = m+1$$

$$s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_1 sY(s) + a_0 Y(s) = b_{n-1}s^{n-1}U(s) + b_{n-2}s^{n-2}U(s) + \cdots + b_1 sU(s) + b_0 U(s)$$

$$Y(s) + a_{n-1}\frac{Y(s)}{s} + \cdots + a_1\frac{Y(s)}{s^{n-1}} + a_0\frac{Y(s)}{s^n} = b_{n-1}\frac{U(s)}{s} + b_{n-2}\frac{U(s)}{s^2} + \cdots + b_1\frac{U(s)}{s^{n-1}} + b_0\frac{U(s)}{s^n}$$

$$Y(s) = \frac{1}{s}\left\{\left(b_{n-1}U(s) - a_{n-1}Y(s)\right) + \frac{1}{s}\left\{\left(b_{n-2}U(s) - a_{n-2}Y(s)\right) + \frac{1}{s}(\cdots) + \underbrace{\frac{1}{s}\left\{b_0 U(s) - a_0 Y(s)\right\}}_{X_1(s)}\cdots\right\}\right.$$

Observer Form

$$X_1(s) = \frac{1}{s}\{b_0 U(s) - a_0 Y(s)\} \longrightarrow \dot{x}_1(t) = b_0 u(t) - a_0 y(t)$$

$$X_2(s) = \frac{1}{s}\{(b_1 U(s) - a_1 Y(s)) + X_1(s)\} \longrightarrow \dot{x}_2(t) = b_1 u(t) - a_1 y(t) + x_1(t)$$

$$\vdots \qquad\qquad \vdots$$

$$X_n(s) = \frac{1}{s}\{(b_{n-1} U(s) - a_{n-1} Y(s)) + X_{n-1}(s)\} \longrightarrow \dot{x}_n(t) = b_{n-1} u(t) - a_{n-1} y(t) + x_{n-1}(t)$$

$$Y(s) = X_n(s) \longrightarrow y(t) = x_n(t)$$

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Observer Form

- The state space equations in observer form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & -a_0 \\ 1 & 0 & \cdots & -a_1 \\ \vdots & \ddots & & \vdots \\ 0 & 0 \cdots 1 & & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + 0u(t)$$

Observer Form

Canonical Form

- To construct state space equations in canonical form, we need to perform partial fraction decomposition to the respective transfer function.

$$Y(s)=\frac{N(s)}{D(s)}U(s)=\left\{\sum_{i=1}^{n}\frac{r_i}{s-\lambda_i}+r_0\right\}U(s)$$

- In case **all poles are distinct**, we define:

$$X_1(s)=\frac{1}{s-\lambda_1}U(s) \longrightarrow \dot{x}_1(t)=\lambda_1 x_1(t)+u(t)$$

$$X_2(s)=\frac{1}{s-\lambda_2}U(s) \longrightarrow \dot{x}_2(t)=\lambda_2 x_2(t)+u(t)$$

$$\vdots$$

$$X_n(s)=\frac{1}{s-\lambda_n}U(s) \longrightarrow \dot{x}_n(t)=\lambda_n x_n(t)+u(t)$$

$$Y(s)=r_1X_1(s)+r_2X_2(s)+\cdots+r_nX_n(s)+r_0U(s) \longrightarrow y(t)=r_1x_1(t)+r_2x_2(t)+\cdots+r_nx_n(t)+r_0u(t)$$

Canonical Form

- The state space equations in case all poles are distinct:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \vdots\\ \dot{x}_n(t)\end{bmatrix}=\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}+\begin{bmatrix}1\\ 1\\ \vdots\\ 1\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}r_1 & r_2 & \cdots & r_n\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}+r_0u(t)$$

Canonical Form, Distinct Poles

- **The resulting matrix *A* is a diagonal matrix.**
- **The ODEs are decoupled, each of them can be solved independently.**

Canonical Form

- In case of **repeating poles**, for example λ_1 is repeated for p times, the decomposed equation will be:

$$Y(s)=\left\{r_0+\frac{r_{11}}{s-\lambda_1}+\frac{r_{12}}{(s-\lambda_1)^2}+\cdots+\frac{r_{1p}}{(s-\lambda_1)^p}+\frac{r_2}{s-\lambda_2}+\cdots+\frac{r_{n-p+1}}{s-\lambda_{n-p+1}}\right\}U(s)$$

- We define:

$$X_1(s)=\frac{1}{s-\lambda_1}U(s) \longrightarrow \dot{x}_1(t)=\lambda_1 x_1(t)+u(t)$$

$$X_2(s)=\frac{1}{(s-\lambda_1)^2}U(s)$$

$$=\frac{1}{s-\lambda_1}X_1(s) \longrightarrow \dot{x}_2(t)=\lambda_1 x_2(t)+x_1(t)$$

- $x_1(t)$ coupled with $x_2(t)$

$$\vdots$$

$$X_p(s)=\frac{1}{(s-\lambda_1)^p}U(s)$$

- $x_{p-1}(t)$ coupled with $x_p(t)$

$$=\frac{1}{s-\lambda_1}X_{p-1}(s) \longrightarrow \dot{x}_p(t)=\lambda_1 x_p(t)+x_{p-1}(t)$$

Canonical Form

$$X_{p+1}(s)=\frac{1}{s-\lambda_2}U(s) \longrightarrow \dot{x}_{p+1}(t)=\lambda_2 x_{p+1}(t)+u(t)$$

$$\vdots \qquad \vdots$$

$$X_n(s)=\frac{1}{s-\lambda_{n-p+1}}U(s) \longrightarrow \dot{x}_n(t)=\lambda_{n-p+1}x_n(t)+u(t)$$

$$\begin{aligned}Y(s)&=r_{11}X_1(s)+r_{12}X_2(s)+\\&\cdots+r_{1p}X_p(s)+r_2X_{p+1}(s)+\\&\cdots+r_{n-p+1}X_n(s)+r_0U(s)\end{aligned} \longrightarrow \begin{aligned}y(t)&=r_{11}x_1(t)+r_{12}x_2(t)+\\&\cdots+r_{1p}x_p(t)+r_2x_{p+1}(t)+\\&\cdots+r_{n-p+1}x_n(t)+r_0u(t)\end{aligned}$$

Canonical Form

- The state space equations in case of repeating poles:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_p(t) \\ \dot{x}_{p+1}(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & & \cdots & & & 0 \\ 1 & \lambda_1 & 0 & & \cdots & & 0 \\ 0 & \ddots & \lambda_1 & \ddots & & & \vdots \\ 0 & \cdots & 1 & \lambda_1 & & & \\ \vdots & & & 0 & \lambda_2 & & 0 \\ & & & & \ddots & \ddots & \vdots \\ 0 & & \cdots & & & 0 & \lambda_{n-p+1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_p(t) \\ x_{p+1}(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

Canonical Form, Repeating Poles

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Canonical Form

- The state space equations in case of repeating poles:

$$y(t) = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1p} & r_2 & \cdots & r_{n-p+1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_p(t) \\ x_{p+1}(t) \\ \vdots \\ x_n(t) \end{bmatrix} + r_0 u(t)$$

Canonical Form, Repeating Poles

Homework 3: Transfer Function - State Space

- Find the state-space realizations of the following transfer function in *Frobenius Form*, *Observer Form*, and *Canonical Form*.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^3 + 8s^2 + 19s + 12}$$

- **Hint**: Learn the following functions in Matlab and use the to solve this problem: **roots**, **residue**, **convolution**.

Homework 3A: Transfer Function - State Space

- Perform a step by step transformation (by calculation of transfer matrix) from the following state-space equations to result the corresponding transfer function.

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -2 \end{bmatrix} \cdot \underline{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot u(t)$$

$$y(t) = \begin{bmatrix} 5 & 1 & 0 \end{bmatrix} \cdot \underline{x}(t)$$

- Verify your calculation result using Matlab.

Homework 3: Transfer Function to State Space

- Find the state-space realizations of the following transfer function in *Frobenius Form*, *Observer Form*, and *Canonical Form*.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^3+8s^2+19s+12}$$

- Hint: Learn the following functions in Matlab and use the to solve this problem: **roots**, **residue**, **conv**.

Homework 3: Transfer Function to State Space

- Find the state-space realizations of the following transfer function in *Frobenius Form*, *Observer Form*, and *Canonical Form*.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^3+8s^2+19s+12} = \frac{b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

- ***Frobenius Form***

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -19 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + 0u(t)$$

Homework 3: Transfer Function to State Space

- Find the state-space realizations of the following transfer function in *Frobenius Form*, *Observer Form*, and *Canonical Form*.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^3+8s^2+19s+12} = \frac{b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

- ***Observer Form***

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -12 \\ 1 & 0 & -19 \\ 0 & 1 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + 0u(t)$$

Homework 3: Transfer Function to State Space

- Find the state-space realizations of the following transfer function in *Frobenius Form*, *Observer Form*, and *Canonical Form*.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^3+8s^2+19s+12} = \frac{b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

- Using Matlab function, [R,P,K] = residue(NUM,DEN),

$$\frac{s+2}{s^3+8s^2+19s+12} = \frac{-2/3}{s+4} + \frac{1/2}{s+3} + \frac{1/6}{s+1}$$

$$= \frac{r_1}{s-\lambda_1} + \frac{r_2}{s-\lambda_2} + \frac{r_3}{s-\lambda_3}$$

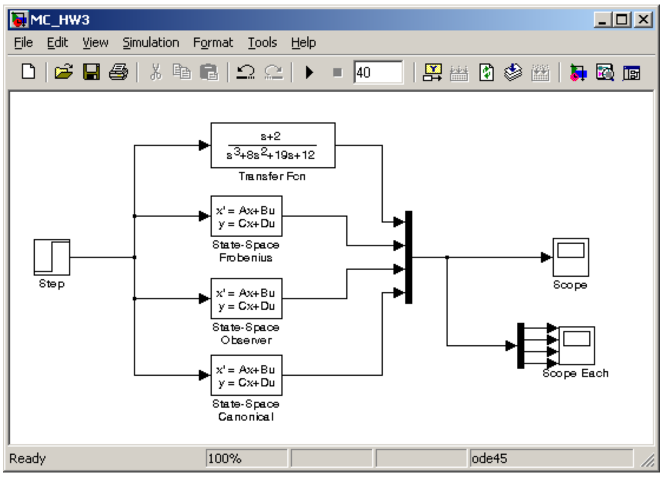
Homework 3: Transfer Function to State Space

- ***Canonical Form***

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -2/3 & 1/2 & 1/6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + 0u(t)$$

Homework 3: Transfer Function to State Space



Canonical Form

- The state space equations in case all poles are distinct:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + r_0 u(t)$$

Canonical Form, Distinct Poles

- **The resulting matrix $\underline{A}$ is a diagonal matrix.**
- **The ODEs are decoupled, each of them can be solved independently.**

Stability

- There are several ways to define the stability of a system. One of them is "BIBO (Bounded Input Bounded Output) Stability".
- A system is said to be **BIBO stable** if every bounded input excites a bounded output also.
- Bounded input means, there exists a constant u_m such that

$$|u(t)| < u_m < \infty, \quad \text{for all } t \geq 0$$

- Thus, a SISO system, described by a transfer function *G*(*s*) is said to be BIBO stable **if and only if** every pole of *G*(*s*) has a negative real part.
- Other way stated, a SISO system *G*(*s*) is stable if every pole of *G*(*s*) lies on the left half plane of *s*.

Stability

- A state space in the form of:

  $$\underline{\dot{x}}(t) = \underline{Ax}(t) + \underline{Bu}(t)$$

  $$\underline{y}(t) = \underline{Cx}(t) + \underline{Du}(t)$$

  is said to be marginally stable if for $\underline{\mathbf{u}}(t)=\underline{\mathbf{0}}$, every finite initial state $\underline{\mathbf{x}}_0$ will excite a bounded response.

- The state space is said to be asymptotically stable if for $\underline{\mathbf{u}}(t)=\underline{\mathbf{0}}$, every finite initial state $\underline{\mathbf{x}}_0$ will excite a bounded response and it approaches $\underline{\mathbf{0}}$ as $t\rightarrow\infty$.

Controllability

- Consider the n-dimensional state equations with r inputs:

  $$\underline{\dot{x}}(t) = \underline{Ax}(t) + \underline{Bu}(t)$$

- The state equations above are said to be "controllable" if for any initial state $\underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0$ and any final state $\underline{\mathbf{x}}(t_1) = \underline{\mathbf{x}}_1$, there exists an input that transfers $\underline{\mathbf{x}}_0$ to $\underline{\mathbf{x}}_1$ in a finite time.
- Otherwise, the state equations are said to be "uncontrollable".

Controllability Matrix

- The controllability of state equations can be checked using the $[n\times nr]$ controllability matrix:

$$\underline{\mathcal{C}} = \left[\underline{B} \;\vdots\; \underline{A}\underline{B} \;\vdots\; \underline{A}^2\underline{B} \;\vdots\; \cdots \;\vdots\; \underline{A}^{n-1}\underline{B}\right]$$

- A state space described by the pair ($\underline{\boldsymbol{A}}$,$\underline{\boldsymbol{B}}$) is controllable if the column rank of $\underline{\mathcal{C}} = n$, or equivalently, if matrix $\underline{\mathcal{C}}$ has n linearly independent columns.

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Observability

- Consider the n-dimensional state space equations with r inputs and m outputs:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t)$$
$$\underline{y}(t) = \underline{C}\underline{x}(t) + \underline{D}\underline{u}(t)$$

- The state space equations above are said to be "observable" if for any unknown initial state $\underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0$, there exists a finite $t_1>0$ such that the knowledge of the input $\underline{\mathbf{u}}(t)$ and the output $\underline{\mathbf{y}}(t)$ over the time interval $[t_0,t_1]$ suffices to determine uniquely the initial state $\underline{\mathbf{x}}(t_0)$.
- Otherwise, the state space equations are said to be "unobservable".

EE391 Control Systems and Components

Observability Matrix

- The observability of state space equations can be checked using the [*nm*×*n*] observability matrix:

$$\underline{\mathcal{O}} = \begin{bmatrix} \underline{C} \\ \underline{C}\,\underline{A} \\ \underline{C}\,\underline{A}^2 \\ \vdots \\ \underline{C}\,\underline{A}^{n-1} \end{bmatrix}$$

- A state space described by the pair ($\underline{A}$,$\underline{C}$) is observable if the row rank of $\underline{\mathcal{O}}$ = *n*, or equivalently, if matrix $\underline{\mathcal{O}}$ has *n* linearly independent rows.

Example

A state space is given as

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}(t)$$

Check its controllability and observability.

$$n = 2$$

$$\underline{\mathcal{C}} = [\underline{B} \mid \underline{A}\underline{B}] = \left[\begin{bmatrix} 1 \\ 3 \end{bmatrix} \middle| \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right] = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}$$

- **Column rank of $\underline{\mathcal{C}}$ = 1 ≠ *n***
- **The state space is "uncontrollable"**

$$\underline{\mathcal{O}} = \begin{bmatrix} \underline{C} \\ \underline{C}\,\underline{A} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

- **Row rank of $\underline{\mathcal{O}}$ = 2 = *n***
- **The state space is "observable"**

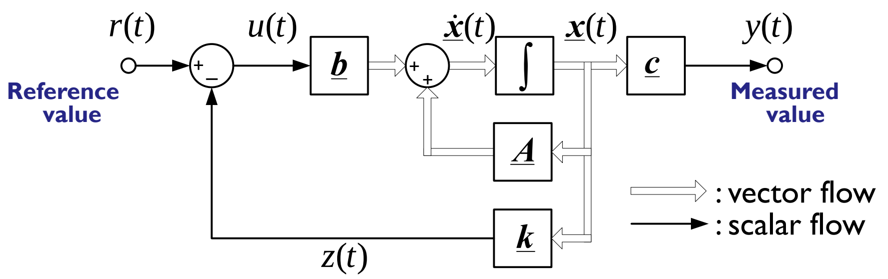
State Feedback

- Consider the *n*-dimensional single-variable state space equations:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{b}u(t)$$
$$y(t) = \underline{c}\underline{x}(t)$$

- **Main idea**: Using measurements of state variables $\underline{x}(t)$, determine an input $u(t)=f(\underline{x}(t))$ such that the dynamic properties of the system can be changed to fulfill a certain criteria.

State Feedback



- The states $\underline{x}(t)$ are fed back through a feedback gain $\underline{k}$.
- The input $u(t)$ is given by:

$$u(t) = r(t) - \underline{k}\,\underline{x}(t) = r(t) - \sum_{i=1}^{n} k_i x_i(t)$$

$$\underline{k} = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$$
$$\underline{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix}^{\mathrm{T}}$$

State Feedback

- Substituting $u(t)$ to the original state space equations,

$$\dot{\underline{x}}(t) = \left(\underline{A} - \underline{b}\underline{k}\right)\underline{x}(t) + \underline{b}r(t)$$

$$y(t) = \underline{c}\underline{x}(t)$$

Example

Consider a state space

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix}\underline{x}(t)$$

The controllability and observability matrices are:

$$\underline{\mathcal{C}} = \left[\underline{B} \;\vdots\; \underline{AB}\right] = \left[\begin{array}{c|c} 0 & 2 \\ 1 & 1 \end{array}\right]$$

- **Column rank of $\underline{\mathcal{C}}$ = 2 → "controllable"**

$$\underline{\mathcal{O}} = \left[\begin{array}{c} \underline{C} \\ \hline \underline{C}\,\underline{A} \end{array}\right] = \left[\begin{array}{cc} 1 & 2 \\ \hline 7 & 4 \end{array}\right]$$

- **Row rank of $\underline{\mathcal{O}}$ = 2 → "observable"**

Example

Let us now introduce a state feedback:

$$u(t) = r(t) - \begin{bmatrix} 3 & 1 \end{bmatrix} \underline{x}(t)$$

The state space is now:

$$\dot{\underline{x}}(t) = (\underline{A} - \underline{b}\underline{k})\underline{x}(t) + \underline{b}r(t)$$

$$= \left(\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \end{bmatrix} \right) \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

$$= \left(\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix} \right) \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \underline{x}(t)$$

$$\underline{\mathcal{C}} = \left[\begin{array}{c|c} 0 & 2 \\ 1 & 0 \end{array} \right] \qquad \underline{\mathcal{O}} = \left[\begin{array}{cc} 1 & 2 \\ \hline 1 & 2 \end{array} \right]$$

- **Column rank of $\underline{\mathcal{C}}$ = 2 → "controllable"**
- **Row rank of $\underline{\mathcal{O}}$ = 1 → "not observable"**
- **State feedback may make a state space become "not observable"**

Example

Consider a SISO system with the following state equations:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

The transfer function of the system is:

$$G(s) = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} + D$$

The characteristic equation, or the denominator of *G*(*s*), is given by:

$$a(s) = \det(s\underline{I} - \underline{A})$$

$$= \det\left(\begin{bmatrix} s-1 & -3 \\ -3 & s-1 \end{bmatrix} \right)$$

$$= (s-1)^2 - (-3)(-3)$$

$$= s^2 - 2s - 8$$

$$= (s-4)(s+2)$$

- **λ = 4, positive**
- **Unstable eigenvalues or unstable pole**

Example

Let us now introduce a state feedback:

$$u(t) = r(t) - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \underline{x}(t)$$

The state space is now:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1-k_1 & 3-k_2 \\ 3 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r(t)$$

The characteristic equation becomes:

$$a(s) = \det\left(\begin{bmatrix} s-(1-k_1) & -(3-k_2) \\ -3 & s-1 \end{bmatrix} \right)$$

$$= (s-1+k_1)(s-1) - (-3+k_2)(-3)$$

$$= s^2 + \underbrace{(k_1-2)}s + \underbrace{(3k_2-k_1-8)}$$

- **The roots of the new characteristic equation can be placed in any location by assigning appropriate value of k_1 and k_2**
- **Condition: complex eigenvalues must be given in pairs**

Homework 4

Again, consider a SISO system with the state equations:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \underline{x}(t)$$

a. If the state feedback in the form of:

$$u(t) = r(t) - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \underline{x}(t)$$

is implemented to the system and it is wished that the poles of the system will be –3 and –4, determine the value of k_1 and k_2.

b. Find the transfer function of the system and again, check the location of the poles of the transfer function.

Homework 4A

Consider a SISO system with the state equations:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} \underline{x}(t)$$

a. If the state feedback in the form of:

$$u(t) = r(t) - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \underline{x}(t)$$

is implemented to the system and it is wished that the damping factor ζ of the system is equal to 0.8 while keeping the system stable. Determine the required value of k_1 and k_2.

b. Find the transfer function of the system and again, check the location of the poles of the transfer function.

Homework 4

Again, consider a SISO system with the state equations:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \underline{x}(t)$$

a. If the state feedback in the form of:

$$u(t) = r(t) - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \underline{x}(t)$$

is implemented to the system and it is wished that the poles of the system will be –3 and –4, determine the value of k_1 and k_2.

b. Find the transfer function of the system and again, check the location of the poles of the transfer function.

Solution of Homework 4

a. With the state feedback:

$$u(t) = r(t) - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \underline{x}(t)$$

The state equations become:

$$\dot{\underline{x}}(t) = (\underline{A} - \underline{b}\underline{k})\underline{x}(t) + \underline{b}r(t)$$

$$= \left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

$$= \left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \right) \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

$$= \left(\begin{bmatrix} 1 & 3 \\ 3 - k_1 & 1 - k_2 \end{bmatrix} \right) \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Solution of Homework 4

The characteristic equation is:

$$a(s) = \det(s\underline{I} - (\underline{A} - \underline{b}\underline{k}))$$

$$= \det\left(\begin{bmatrix} s-1 & -3 \\ -(3-k_1) & s-(1-k_2) \end{bmatrix} \right)$$

$$= (s-1)(s-1+k_2) - (-3)(-(3-k_1))$$

$$= s^2 + (k_2 - 2)s + (3k_1 - k_2 - 8)$$

The wished poles are –3 and –4, corresponding with the wished characteristic equation of:

$$\breve{a}(s) = (s+3)(s+4)$$

$$= s^2 + 7s + 12$$

Comparing $a(s)$ and $\breve{a}(s)$, we obtain:

$$(k_2 - 2) \equiv 7 \qquad \Rightarrow k_2 = \underline{\underline{9}}$$

$$(3k_1 - k_2 - 8) \equiv 12 \qquad \Rightarrow k_1 = 29/3 = \underline{\underline{9.67}}$$

Solution of Homework 4

b. The transfer function of the system can be found as:

$$G(s)=\frac{Y(s)}{R(s)}=\underline{c}(s\underline{I}-(\underline{A}-\underline{b}\underline{k}))^{-1}\underline{b}$$

$$=\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}s-1 & -3\\ -(3-k_1) & s-(1-k_2)\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}s-1 & -3\\ 6.67 & s+8\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}$$

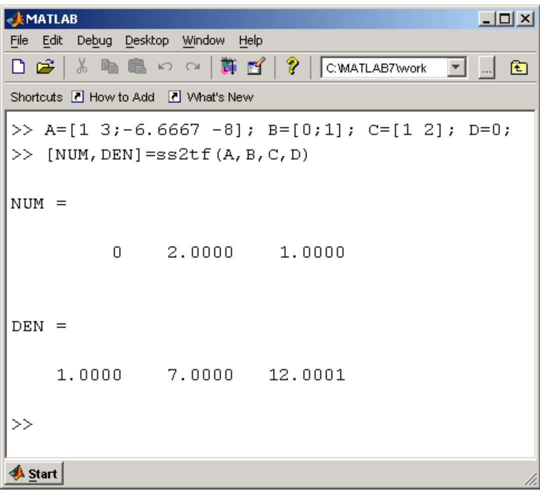
$$=\frac{\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}s+8 & 3\\ -6.67 & s-1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}}{s^2+7s+12}$$

$$=\frac{2s+1}{s^2+7s+12}$$

$$=\frac{2s+1}{(s+3)(s+4)}$$

Solution of Homework 4

Using Matlab, the following function can be utilized:



State Feedback

- Consider the *n*-dimensional single-variable state space equations:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{b}u(t)$$
$$y(t) = \underline{c}\underline{x}(t)$$

- For this SISO system, if the pair (***A***,***b***) is ***controllable***, there exists a nonsingular transformation matrix ***Q*** such that:

$$\underline{x}(t) = \underline{Q}\underline{z}(t)$$

and:

$$\dot{\underline{z}}(t) = \hat{\underline{A}}\underline{z}(t) + \hat{\underline{b}}u(t)$$
$$y(t) = \hat{\underline{c}}\underline{z}(t)$$

with the matrices $\hat{\underline{A}}$ and $\hat{\underline{b}}$ given by:

$$\hat{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad \hat{\underline{b}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

State Feedback

- The coefficients a_i are the coefficients of the characteristic equation of ***A***, that is:

$$a(s) = \det(s\underline{I} - \underline{A}) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

- The state feedback for the transformed system is given by:

$$u(t) = r(t) - \hat{\underline{k}}\underline{z}(t)$$

with:

$$\hat{\underline{k}} = \begin{bmatrix} \hat{k}_1 & \hat{k}_2 & \cdots & \hat{k}_n \end{bmatrix}$$

- Substituting $u(t)$ into the transformed system:

$$\dot{\underline{z}}(t) = \hat{\underline{A}}\underline{z}(t) + \hat{\underline{b}}(r(t) - \hat{\underline{k}}\underline{z}(t))$$
$$\dot{\underline{z}}(t) = \left(\hat{\underline{A}} - \hat{\underline{b}}\hat{\underline{k}}\right)\underline{z}(t) + \hat{\underline{b}}r(t)$$

State Feedback

$$\dot{\underline{z}}(t)=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -(a_0+\hat{k}_1) & -(a_1+\hat{k}_2) & -(a_2+\hat{k}_3) & \cdots & -(a_{n-1}+\hat{k}_n) \end{bmatrix}\underline{z}(t)+\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} r(t)$$

- The characteristic equation of the closed-loop system is:

$$a(s)=s^n+(a_{n-1}+\hat{k}_n)s^{n-1}+(a_{n-2}+\hat{k}_{n-1})s^{n-2}+ \cdots+(a_1+\hat{k}_2)s+(a_0+\hat{k}_1)$$

- If the desired closed-loop poles are specified by $p_1, p_2, \ldots, p_n$ then:

$$\breve{a}(s)=(s-p_1)(s-p_2)\cdots(s-p_n) = s^n+\breve{a}_{n-1}s^{n-1}+\cdots+\breve{a}_1 s+\breve{a}_0$$

State Feedback

- By comparing the coefficients of the previous two polynomials, it is clear, that in order to obtain the desired characteristic equation, the feedback gain must satisfy:

$$a_0+\hat{k}_1=\breve{a}_0 \longrightarrow \hat{k}_1=\breve{a}_0-a_0$$

$$a_1+\hat{k}_2=\breve{a}_1 \longrightarrow \hat{k}_2=\breve{a}_1-a_1$$

$$\vdots \qquad\qquad \vdots$$

$$a_{n-1}+\hat{k}_n=\breve{a}_{n-1} \longrightarrow \hat{k}_n=\breve{a}_{n-1}-a_{n-1}$$

Example: State Feedback

Consider the following system given in Frobenius Form

$$\underline{\dot{z}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -10 \end{bmatrix} \underline{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

It is required that the closed-loop system has the eigenvalues located at $s=-1\pm j$ and $s=-5$.

Find the feedback gain vector $\underline{\hat{k}}$.

$$\breve{a}(s) = (s+(1-j))\cdot(s+(1+j))\cdot(s+5)$$
$$= ((s+1)-j)\cdot((s+1)+j)\cdot(s+5)$$
$$= ((s+1)^2+1)\cdot(s+5)$$
$$= (s^2+2s+2)\cdot(s+5)$$
$$= s^3 + \underbrace{7}_{\breve{a}_2}s^2 + \underbrace{12}_{\breve{a}_1}s + \underbrace{10}_{\breve{a}_0}$$

From the state equations we can find out that:

$$a_0 = 2 \rightarrow \hat{k}_1 = \breve{a}_0 - a_0 = 8$$
$$a_1 = 5 \rightarrow \hat{k}_2 = \breve{a}_1 - a_1 = 7$$
$$a_2 = 10 \rightarrow \hat{k}_3 = \breve{a}_2 - a_2 = -3$$

$$\underline{\hat{k}} = \underline{\underline{[8 \quad 7 \quad -3]}}$$

Transformation to Frobenius Form

- By performing the procedure presented previously, we are be able to place the poles of a controllable SISO system in any location so easily.
- **The condition**: The system is written in Frobenius Form.
- In order to be able to apply this procedure to any controllable SISO systems easily, we need to transform the systems to Frobenius Form first.
- That means, we need to know the nonsingular transformation matrix $\underline{Q}$.

Transformation to Frobenius Form

- If the controllability matrix of the open-loop system is given by:

$$\underline{\mathcal{C}} = \left[\underline{b} \;\vdots\; \underline{A}\underline{b} \;\vdots\; \underline{A}^2\underline{b} \;\vdots\; \cdots \;\vdots\; \underline{A}^{n-1}\underline{b} \right]$$

It can be shown that the required transformation matrix **Q** to transform the system to a Frobenius Form is given by:

$$\underline{Q} = \left[\underline{q}_1 \;\vdots\; \underline{q}_2 \;\vdots\; \underline{q}_3 \;\vdots\; \cdots \;\vdots\; \underline{q}_n \right]$$

where

$$\begin{aligned} \underline{q}_n &= \underline{b} \\ \underline{q}_{n-1} &= \underline{A}\underline{q}_n + a_{n-1}\underline{q}_n = \underline{A}\underline{b} + a_{n-1}\underline{b} \\ \underline{q}_{n-2} &= \underline{A}\underline{q}_{n-1} + a_{n-2}\underline{q}_n = \underline{A}^2\underline{b} + a_{n-1}\underline{A}\underline{b} + a_{n-2}\underline{b} \\ &\vdots \qquad\qquad\qquad \vdots \\ \underline{q}_1 &= \underline{A}\underline{q}_2 + a_1\underline{q}_n = \underline{A}^{n-1}\underline{b} + a_{n-1}\underline{A}^{n-2}\underline{b} + \cdots + a_2\underline{A}\underline{b} + a_1\underline{b} \end{aligned}$$

- ***a_i* are the coefficients of the characteristic polynomial *a*(*s*) of matrix *A***

Transformation to Frobenius Form

- In matrix form, the set of equations can be formulated as:

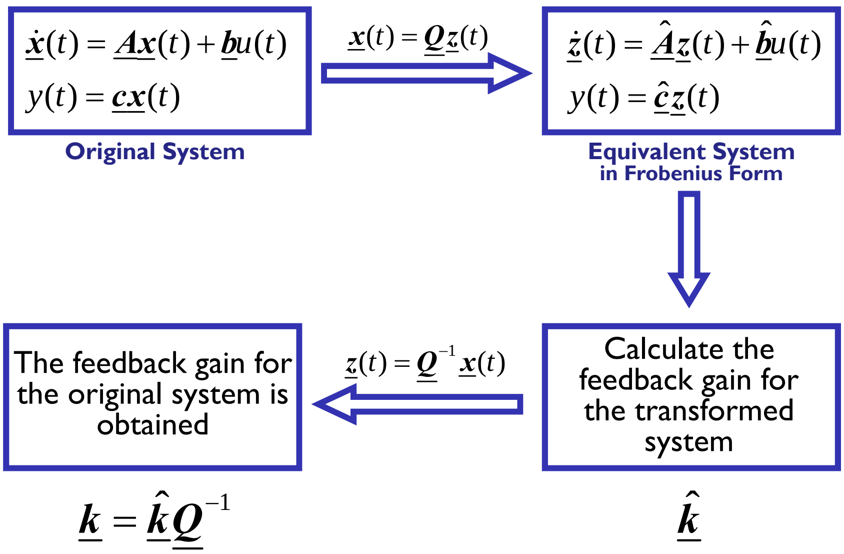
$$\underline{Q} = \left[\underline{q}_1 \;\vdots\; \underline{q}_2 \;\vdots\; \underline{q}_3 \;\vdots\; \cdots \;\vdots\; \underline{q}_n \right]$$

$$\underline{Q} = \underbrace{\left[\underline{b} \;\vdots\; \underline{A}\underline{b} \;\vdots\; \underline{A}^2\underline{b} \;\vdots\; \cdots \;\vdots\; \underline{A}^{n-1}\underline{b} \right]}_{\underline{\mathcal{C}}} \cdot \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & & 1 & 0 \\ a_3 & & & \iddots & & \vdots \\ \vdots & & \iddots & & & \\ a_{n-1} & 1 & 0 & \cdots & & 0 \\ 1 & 0 & 0 & \cdots & & 0 \end{bmatrix}}_{\underline{\mathcal{T}}}$$

$$\underline{Q} = \underline{\mathcal{C}}\underline{\mathcal{T}}$$

- ***a_i* are the coefficients of the characteristic polynomial *a*(*s*) of matrix *A***

Transformation to Frobenius Form



Example: Transformation

Two poles at –1 are wished for the following system:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

Calculate the required $\underline{k}$.

- Find the characteristic equation

$$a(s) = \det(s\underline{I} - \underline{A}) = \det\left(\begin{bmatrix} s-1 & -1 \\ -1 & s-1 \end{bmatrix}\right)$$
$$= (s-1)(s-1) - (-1)(-1)$$
$$= s(s-2)$$

- **unstable**

$$= s^2 - 2s \qquad a_0 = 0, \quad a_1 = -2$$

- Calculate $\underline{\mathcal{C}}, \underline{\mathcal{T}}$

$$\underline{\mathcal{C}} = [\underline{b} \;\vdots\; \underline{A}\underline{b}] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$\underline{\mathcal{T}} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$

Transformation to Frobenius Form

- Calculate $\underline{Q}$ $\quad \underline{Q} = \mathcal{C}\mathcal{T} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

$$\underline{Q}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

- Perform transformation

$$\underline{\hat{A}} = \underline{Q}^{-1}\underline{A}\underline{Q} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\underline{\hat{b}} = \underline{Q}^{-1}\underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Calculate $\underline{\hat{k}}$ $\quad \breve{a}(s) = (s+1)\cdot(s+1)$

$$= s^2 + 2s + 1 \quad \breve{a}_0 = 1 \quad \hat{k}_1 = \breve{a}_0 - a_0 = 1 \quad \underline{\hat{k}} = \begin{bmatrix} 1 & 4 \end{bmatrix}$$

$$\breve{a}_1 = 2 \quad \hat{k}_2 = \breve{a}_1 - a_1 = 4$$

- Calculate $\underline{k}$ $\quad \underline{k} = \underline{\hat{k}}\underline{Q}^{-1} = \begin{bmatrix} 1 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \end{bmatrix}$

- **Check the new characteristic equation**

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Homework 5

A state-space equation of a third-order system is given as:

$$\dot{\underline{\tilde{x}}}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\underline{\tilde{x}}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} 6 & -6 & 1 \end{bmatrix}\underline{\tilde{x}}(t)$$

a. Perform a step-by-step transformation of the given model to Frobenius Form.

b. Calculate the required feedback gain $\underline{k}$ so that the system may have two conjugate poles at $-2\pm j1$ and -4.

Homework 5A

A state-space equation of a third-order system is given as:

$$\dot{\underline{x}}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \tilde{\underline{x}}(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 6 & 6 & -1 \end{bmatrix} \tilde{\underline{x}}(t)$$

a. Perform a step-by-step transformation of the given model to Frobenius Form.

b. Calculate the required feedback gain ***k*** so that the system may have two conjugate poles at $-1 \pm j3$ and -2.

Homework 5

A state-space equation of a third-order system is given as:

$$\dot{\underline{x}}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \tilde{\underline{x}}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 6 & -6 & 1 \end{bmatrix} \tilde{\underline{x}}(t)$$

a. Perform a step-by-step transformation of the given model to Frobenius Form.

b. Calculate the required feedback gain ***k*** so that the system may have two conjugate poles at $-2 \pm j1$ and -4.

Solution of Homework 5

a. Transformation to Frobenius Form.

- Find the characteristic equation

$$a(s) = \det(s\underline{I} - \underline{A}) = \det\left(\begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+2 & 0 \\ 0 & 0 & s+3 \end{bmatrix}\right)$$

$$= (s+1)(s+2)(s+3)$$
$$= s^3 + 6s^2 + 11s + 6$$

- Calculate $\underline{\mathcal{C}}, \underline{\mathcal{T}}$

$$\underline{\mathcal{C}} = \left[\underline{b} \;\vdots\; \underline{Ab} \;\vdots\; \underline{A}^2\underline{b}\right] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix}$$

$$\underline{\mathcal{T}} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Solution of Homework 5

- Calculate $\underline{Q}$

$$\underline{Q} = \underline{\mathcal{C}}\underline{\mathcal{T}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix}\begin{bmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 6 & 5 & 1 \\ 3 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\underline{Q}^{-1} = \begin{bmatrix} 0.5 & -1 & 0.5 \\ -0.5 & 2 & -1.5 \\ 0.5 & -4 & 4.5 \end{bmatrix}$$

- Perform transformation

$$\hat{\underline{A}} = \underline{Q}^{-1}\underline{A}\underline{Q} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$\hat{\underline{b}} = \underline{Q}^{-1}\underline{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- **Transformation accomplished**

Solution of Homework 5

b. Finding $\underline{k}$.

- Calculate $\hat{\underline{k}}$

$$\breve{a}(s) = (s+2-j)\cdot(s+2+j)\cdot(s+4)$$
$$\breve{a}(s) = s^3 + 8s^2 + 21s + 20$$
$$a(s) = s^3 + 6s^2 + 11s + 6$$

$$\left.\begin{matrix} \hat{k}_1 = \breve{a}_0 - a_0 = 14 \\ \hat{k}_2 = \breve{a}_1 - a_1 = 10 \\ \hat{k}_3 = \breve{a}_2 - a_2 = 2 \end{matrix}\right\} \hat{\underline{k}} = [14 \quad 10 \quad 2]$$

- Calculate $\underline{k}$

$$\underline{k} = \hat{\underline{k}}\underline{Q}^{-1}$$
$$= [14 \quad 10 \quad 2]\begin{bmatrix} 0.5 & -1 & 0.5 \\ -0.5 & 2 & -1.5 \\ 0.5 & -4 & 4.5 \end{bmatrix}$$
$$= \underline{\underline{[3 \quad -2 \quad 1]}}$$

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Solution of Homework 5

c. Direct calculation of $\underline{k}$ without transformation.

$$a(s) = \det(s\underline{I} - (\underline{A} - \underline{bk}))$$

$$= \det\left(s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}[k_1 \quad k_2 \quad k_3]\right)\right)$$

$$= \det\left(s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -(1+k_1) & -k_2 & -k_3 \\ -k_1 & -(2+k_2) & -k_3 \\ -k_1 & -k_2 & -(3+k_3) \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} s+(1+k_1) & k_2 & k_3 \\ k_1 & s+(2+k_2) & k_3 \\ k_1 & k_2 & s+(3+k_3) \end{bmatrix}\right)$$

- **Complicated to be done**

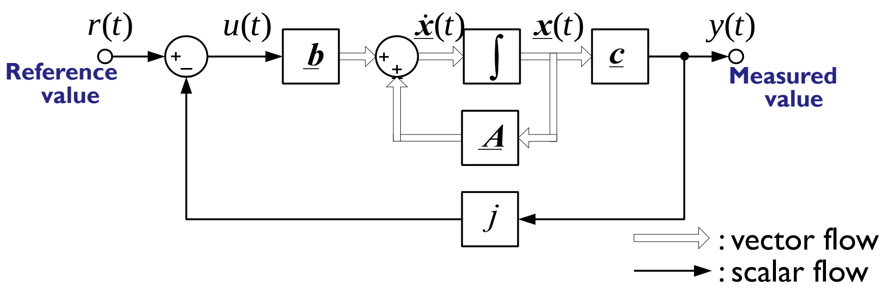
Output Feedback

- Consider the n-dimensional ***controllable*** single-variable state space equations:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{b}u(t)$$
$$y(t) = \underline{c}\underline{x}(t)$$

- **Main idea**: Using measurement of output variable *y*(*t*), determine an input *u*(*t*)=*f*(*r*(*t*),*y*(*t*)) such that the dynamic properties of the system can be changed to fulfill a certain criteria.
- In contrast to state feedback, output feedback has less degree of freedom in the controller parameter.
- However, the output feedback method is superior to the state feedback method from the practical point of view, because the output *y*(*t*) is known and measurable.
- On the contrary, it is almost always difficult, if not impossible, to measure the entire state vector *x*(*t*) due to practical limitations. This can be encompassed by using state observer which will be discussed later.

Output Feedback



- The output *y*(*t*) is fed back through a feedback gain *j*.
- The input *u*(*t*) is given by:

$$u(t) = r(t) - jy(t)$$
$$= r(t) - j\underline{c}\underline{x}(t)$$

- By inspection the state feedback gain $\underline{k}$ and output feedback gain *j* are actually related via the following equation: $\underline{k} \Leftrightarrow j\underline{c}$

Output Feedback

- Substituting $u(t)$ to the original state space equations,

$$\dot{\underline{x}}(t) = (\underline{A} - j\underline{b}\underline{c})\underline{x}(t) + \underline{b}r(t)$$
$$y(t) = \underline{c}\underline{x}(t)$$

- The roots of the characteristic equation can now be repositioned through

$$a(s) = \det(s\underline{I} - (\underline{A} - j\underline{b}\underline{c}))$$

- The output feedback has less degree of freedom in placing the poles, thus it may happen that the location to place the poles is limited.

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Example: Output Feedback

Let us redo the previous example with the following state equations:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x}(t)$$

The characteristic equation of the given system is:

$$a(s) = \det(s\underline{I} - \underline{A})$$
$$= (s-4)(s+2)$$

• **Unstable eigenvalues or unstable pole**

Introducing the output feedback,

$$u(t) = r(t) - jy(t) = r(t) - j\underline{b}\underline{c}\underline{x}(t)$$

The state space is now:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 3 \\ 3-j & 1-j \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

Example: Output Feedback

The characteristic equation becomes:

$$a(s) = \det\left(\begin{bmatrix} s-1 & -3 \\ -(3-j) & s-(1-j) \end{bmatrix}\right)$$

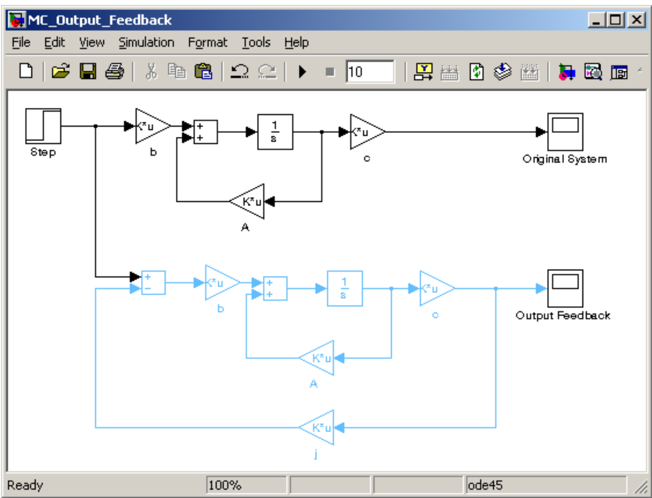
$$= (s-1)(s-(1-j)) - (3-j)(3)$$

$$= s^2 + (j-2)s + (2j-8)$$

- **The roots of the new characteristic equation can be moved to a new stable position**
- **However, there are cases where the system cannot be stabilized, *i.e.*, if c=[1 0]**

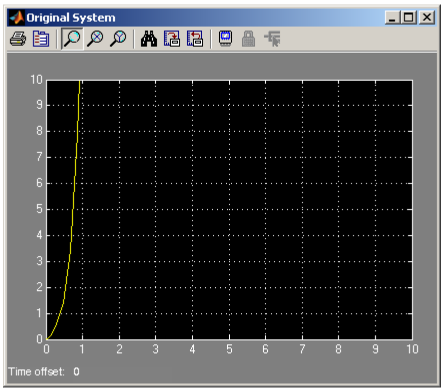
Example: Output Feedback

Unit step input

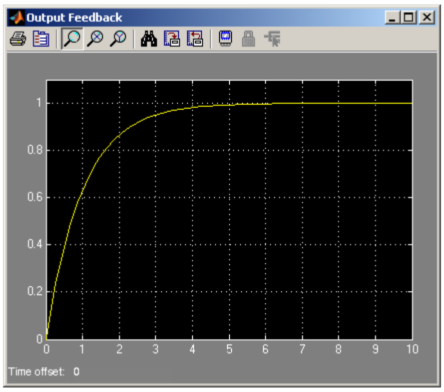


- **Matlab Simulink realization**

Example: Output Feedback



- **Output of original system**



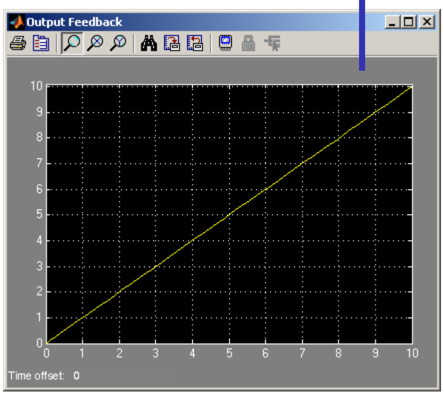
- **Output of system with output feedback, *j* = 5**

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

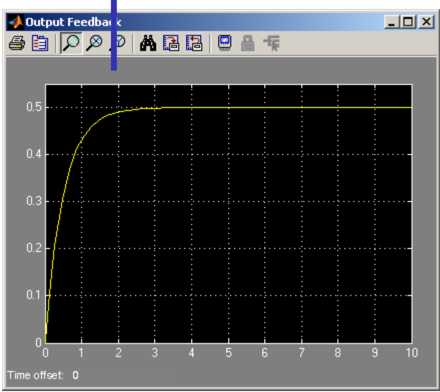
Example: Output Feedback

Integrator property

Steady-state error



- **Output of system with output feedback, *j* = 4**



- **Output of system with output feedback, *j* = 6**

State Estimator

- In previous section, we have discussed the state feedback, based on the assumption that all state variables are available for feedback.
- Also, we have discussed the output feedback, provided that the output is available for feedback.
- On the purpose of state feedback, practically, the state variables might be not accessible for direct connection. The sensing devices or transducers might be not available or very expensive.
- In this case, we need a "**state estimator**" or a "**state observer**". Their output will be the "estimate of the state", provided that the system under consideration is ***observable***.

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

State Estimator

- Consider the *n*-dimensional single-variable state space equations:

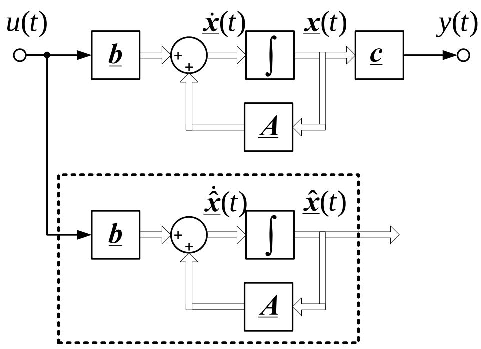
$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{b}u(t)$$
$$y(t) = \underline{c}\underline{x}(t)$$

where $\underline{A}$, $\underline{b}$, $\underline{c}$ are given, $u(t)$ and $y(t)$ are available, and the states $\underline{x}(t)$ are not available.

- **Problem**: How to estimate $\underline{x}(t)$?

Open-Loop State Estimator

- The block diagram of an open-loop state estimator can be seen below:



- The open-loop state estimator duplicates the original system and deliver:

$$\dot{\hat{\underline{x}}}(t) = \underline{A}\hat{\underline{x}}(t) + \underline{b}u(t)$$

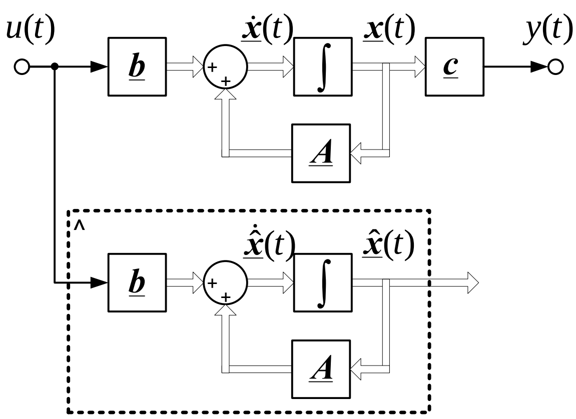
Open-Loop State Estimator

- Several conclusions can be drawn by comparing both state equations

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{b}u(t)$$
$$\dot{\hat{\underline{x}}}(t) = \underline{A}\hat{\underline{x}}(t) + \underline{b}u(t)$$

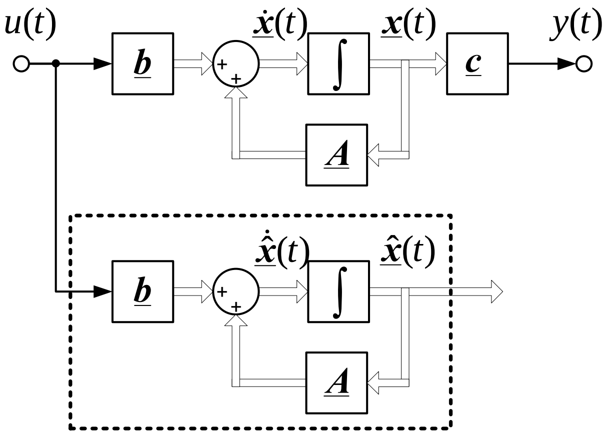
- If the initial states of both equations are the same, $\hat{\underline{x}}(t_0) = \underline{x}(t_0)$, then for any $t \geq 0$, $\hat{\underline{x}}(t) = \underline{x}(t)$.

- If the pair $(\underline{A},\underline{c})$ is observable, the initial state can be computed over any time interval $[0,t_0]$, and after setting $\hat{\underline{x}}(t_0)=\underline{x}(t_0)$, then $\hat{\underline{x}}(t)=\underline{x}(t)$ for $t \geq t_0$.



Open-Loop State Estimator

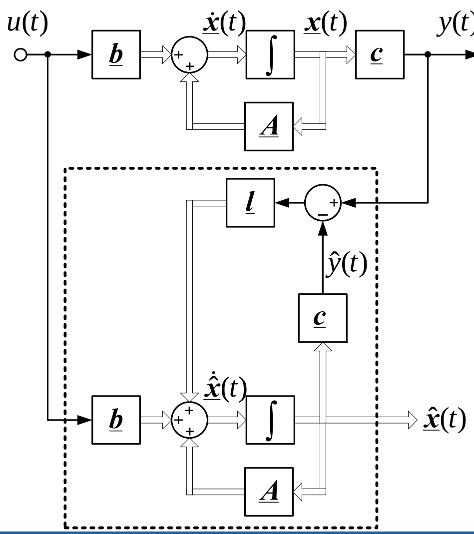
- The disadvantages of open-loop estimator are:
  - Initial state must be computed and appointed each time the estimator is used.
  - If the system is unstable, any small difference between $\underline{\mathbf{x}}(t_0)$ and $\hat{\underline{\mathbf{x}}}(t_0)$ will lead to even bigger difference between $\underline{\mathbf{x}}(t)$ and $\hat{\underline{\mathbf{x}}}(t)$, making $\hat{\underline{\mathbf{x}}}(t)$ unusable.



FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Closed-Loop State Estimator

- The block diagram of a closed-loop state estimator can be seen below:



- In closed-loop estimator, $\hat{y}(t) = \mathbf{c}\hat{\underline{\mathbf{x}}}(t)$ is compared with $y(t) = \mathbf{c}\underline{\mathbf{x}}(t)$.
- Their difference, after multiplied by the matrix $\underline{\mathbf{l}}$, is used as a correcting term in the calculation of $\hat{\underline{\mathbf{x}}}(t)$.
- If $\underline{\mathbf{l}}$ is properly assigned, the difference will drive $\hat{\underline{\mathbf{x}}}(t)$ to $\underline{\mathbf{x}}(t)$.

Closed-Loop State Estimator

- Following the previous figure,

$$\dot{\hat{\underline{x}}}(t) = \underline{A}\hat{\underline{x}}(t) + \underline{b}u(t) + \underline{l}\left(y(t) - \underline{c}\hat{\underline{x}}(t)\right)$$

$$\dot{\hat{\underline{x}}}(t) = \left(\underline{A} - \underline{l}\underline{c}\right)\hat{\underline{x}}(t) + \underbrace{\underline{b}u(t) + \underline{l}y(t)}$$

- **"New" inputs**

- We now define

$$\underline{e}(t) = \underline{x}(t) - \hat{\underline{x}}(t)$$

$$\dot{\underline{e}}(t) = \dot{\underline{x}}(t) - \dot{\hat{\underline{x}}}(t)$$

$$= \left\{\underline{A}\underline{x}(t) + \underline{b}u(t)\right\} - \left\{\left(\underline{A} - \underline{l}\underline{c}\right)\hat{\underline{x}}(t) + \underline{b}u(t) + \underline{l}y(t)\right\}$$

$$= \left(\underline{A} - \underline{l}\underline{c}\right)\underline{x}(t) - \left(\underline{A} - \underline{l}\underline{c}\right)\hat{\underline{x}}(t)$$

$$= \left(\underline{A} - \underline{l}\underline{c}\right)\left(\underline{x}(t) - \hat{\underline{x}}(t)\right)$$

$$\boxed{\dot{\underline{e}}(t) = \left(\underline{A} - \underline{l}\underline{c}\right)\underline{e}(t)}$$

Closed-Loop State Estimator

$$\dot{\underline{e}}(t) = \left(\underline{A} - \underline{l}\underline{c}\right)\underline{e}(t) \;\Longrightarrow\; \underline{e}(t) = e^{\left(\underline{A} - \underline{l}\underline{c}\right)t}\,\underline{e}(0)$$

- **The time domain solution of estimation error $\underline{e}(t)$ depends on $\underline{e}(0)$, but not $u(t)$**

Canonical Transformation (↓)

$$\dot{\underline{\mathcal{E}}}(t) = \underline{\Lambda}\underline{\mathcal{E}}(t) \;\Longrightarrow\; \underline{\mathcal{E}}(t) = e^{\underline{\Lambda}t}\,\underline{\mathcal{E}}(0)$$

$$\underline{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \qquad e^{\underline{\Lambda}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix}$$

Closed-Loop State Estimator

- From the fact that $\underline{\mathcal{E}}(t) = e^{\Lambda t}\underline{\mathcal{E}}(0)$, we can conclude that:
  - If all eigenvalues of $\underline{\Lambda} = (\underline{A}\text{-}\underline{Ic})$ are negative, then the estimation error $\underline{e}(t)$ will approach zero as t increases.
  - There is no need to calculated the initial states each time the closed-loop estimator will be used.
  - After a certain time, the estimation error $\underline{e}(t)$ will approach zero and the state estimates $\underline{x}(t)$ will approach the system's state $\underline{x}(t)$.

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Example: State Estimators

A system is given in state space form as below:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x}(t)$$

(a) Find a state feedback gain $\underline{k}$, so that the closed-loop system has –1 and –2 as its eigenvalues.

(b) Design a closed-loop state estimator for the system, with eigenvalues $-2 \pm j2$.

Example: State Estimators

(a) Find a state feedback gain $\underline{k}$, so that the closed-loop system has –1 and –2 as its eigenvalues.

$$a(s) = \det(s\underline{I} - \underline{A} + \underline{bk})$$

$$= \det\left(\begin{bmatrix} s-2+k_1 & -1+k_2 \\ 1+2k_1 & s-1+2k_2 \end{bmatrix}\right)$$

$$= s^2 + (k_1 + 2k_2 - 3)s + (k_1 - 5k_2 + 3)$$

$$\equiv (s+1)(s+2)$$

$$k_1 = 4 \qquad k_2 = 1 \qquad \underline{k} = \begin{bmatrix} 4 & 1 \end{bmatrix}$$

(b) Design a closed-loop state estimator for the system, with eigenvalues $-2 \pm j2$.

$$\alpha(s) = \det(s\underline{I} - \underline{A} + \underline{lc})$$

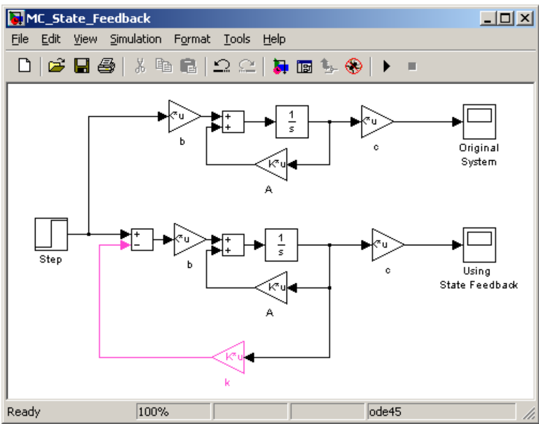
$$= \det\left(\begin{bmatrix} s-2+l_1 & -1+l_1 \\ 1+l_2 & s-1+l_2 \end{bmatrix}\right)$$

$$= s^2 + (l_1 + l_2 - 3)s + (-2l_1 - l_2 + 3)$$

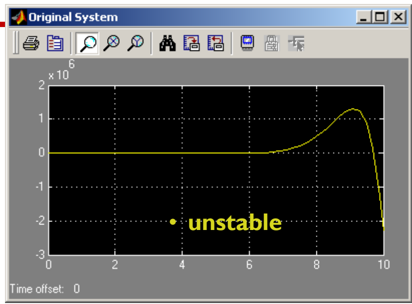
$$\equiv (s+2+j2)(s+2-j2)$$

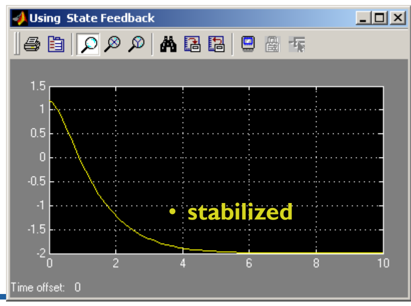
$$l_1 = -12 \qquad l_2 = 19 \qquad \underline{l} = \begin{bmatrix} -12 \\ 19 \end{bmatrix}$$

Example: State Estimators

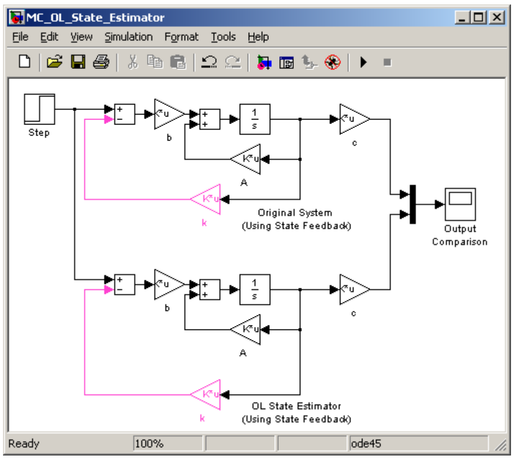


$$\underline{x}_0 = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$$

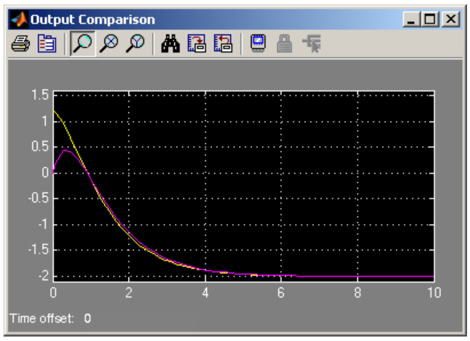




Example: State Estimators



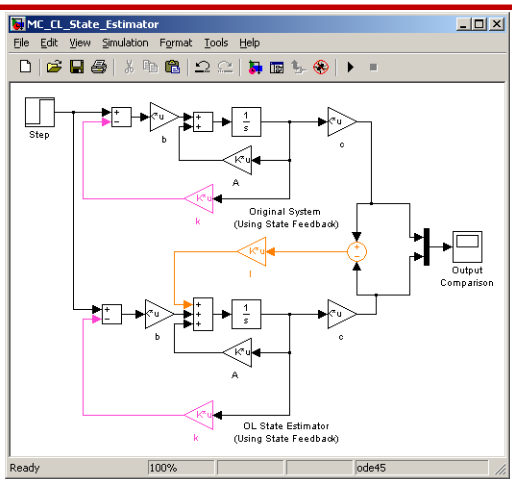
$$\underline{x}_0 = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}, \quad \hat{\underline{x}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



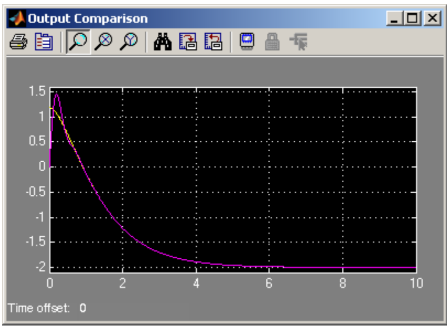
- **If $\hat{\underline{x}}_0 \neq \underline{x}_0$, then for a certain amount of time there will be estimation error e = $\underline{x} - \hat{\underline{x}}$, which in the end will decay to zero**
- **For unstable system, e → ∞, no decay**

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Example: State Estimators



$$\underline{x}_0 = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}, \quad \hat{\underline{x}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



- **Although $\hat{\underline{x}}_0 \neq \underline{x}_0$, in a very short time the estimation error decays to zero**
- **Estimation error e will also decay in case of unstable system**

Homework 6: State Estimators

(a) For the same system as discussed in previous slides, design another closed-loop state estimator, with eigenvalues at -3 and -4.

(b) Compare the performance of the estimator in the previous slides and the one you have designed through simulation using Matlab Simulink.

(c) Give some explanations of the comparison results.

Homework 6A: State Estimators

(a) For the same system as discussed in previous slides, design another closed-loop state estimator, with eigenvalues at
$-0.5 \pm j1$. This means, the eigenvalues of the estimator is to the right of those of the system, which is -1 and -2.

(b) Compare the performance of the estimator in the previous slides and the one you have designed through simulation using Matlab Simulink.

(c) Give some explanations of the comparison results.

Homework 6: State Estimators

(a) For the same system as discussed in previous slides, design another closed-loop state estimator, with eigenvalues at –3 and –4.

(b) Compare the performance of the estimator in the previous slides and the one you have designed through simulation using Matlab Simulink.

(c) Give some explanations of the comparison results.

Solution of Homework 6: State Estimators

The system is rewritten as:

$$\underline{\dot{x}}(t) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x}(t)$$

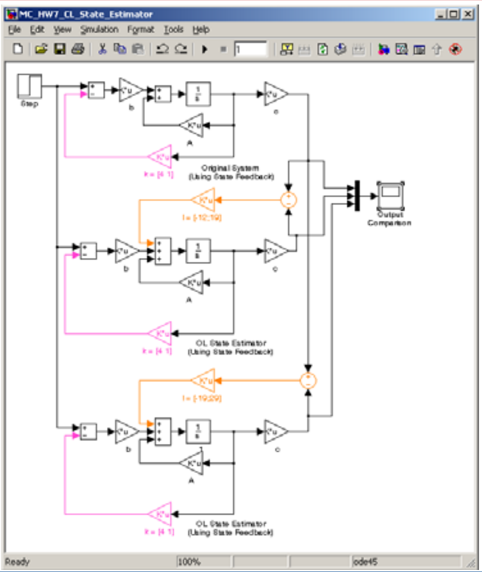
(a) Design another closed-loop state estimator, with eigenvalues at –3 and –4.

$$\begin{aligned}\alpha(s) &= \det(s\underline{I} - \underline{A} + \underline{l}\underline{c}) \\ &= \det\left(\begin{bmatrix} s-2+l_1 & -1+l_1 \\ 1+l_2 & s-1+l_2 \end{bmatrix}\right) \\ &= s^2 + (l_1 + l_2 - 3)s + (-2l_1 - l_2 + 3) \\ &\equiv (s+3)(s+4)\end{aligned}$$

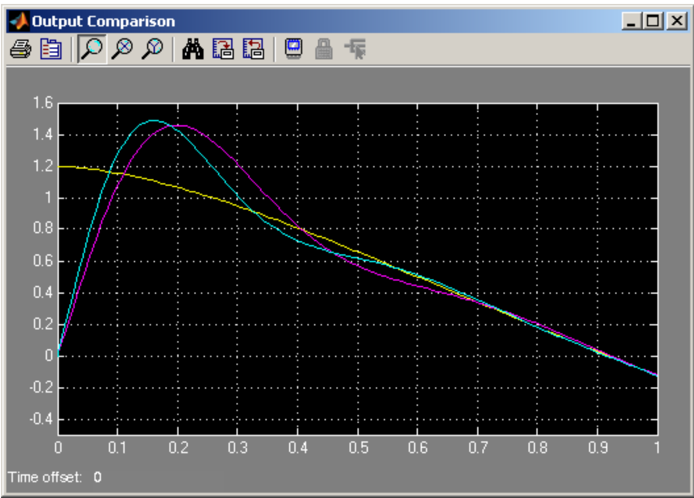
$$\begin{aligned} l_1 + l_2 - 3 &= 7 \\ -2l_1 - l_2 + 3 &= 12 \end{aligned} \qquad \underline{l} = \begin{bmatrix} -19 \\ 29 \end{bmatrix}$$

Solution of Homework 6: State Estimators

(b) Compare the performance of both closed-loop state estimators.



Solution of Homework 6: State Estimators



: $y(t)$

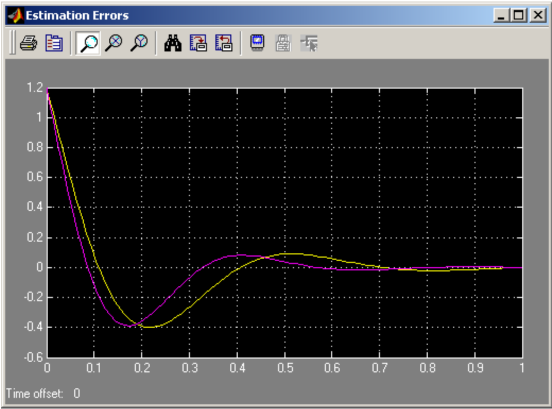
: $\hat{y}(t)$, $\underline{l} = [-12;19]$

: $\hat{y}(t)$, $\underline{l} = [-19;29]$

- **What is the difference?**

Solution of Homework 6: State Estimators

(c) Give some explanations of the comparison results.



Yellow line: $y(t) - \hat{y}(t)$, $\underline{l} = [-12;19]$

Purple line: $y(t) - \hat{y}(t)$, $\underline{l} = [-19;29]$

- **If the eigenvalues of a closed-loop estimator lie further to the left of imaginary axis, then the estimation error decays faster**
- **A faster estimator requires larger estimator gain, larger energy, more sensitive to disturbance**

Transformation to Observer Form

- The calculation of state estimator gain $\underline{l}$ can only be done for ***observable*** SISO systems.
- The procedure presented previously can be performed easily if the system is written in Observer Form.
- The original system needs to be transformed using a nonsingular transformation matrix $\underline{\boldsymbol{R}}$.

Transformation to Observer Form

- If the observability matrix of the open-loop system is given by:

$$\underline{\mathcal{O}} = \begin{bmatrix} \underline{C} \\ \underline{C}\,\underline{A} \\ \underline{C}\,\underline{A}^2 \\ \vdots \\ \underline{C}\,\underline{A}^{n-1} \end{bmatrix}$$

It can be shown that the required transformation matrix $\underline{R}^{-1}$ to transform a given system to an Observer Form is given by:

$$\underline{R}^{-1} = \underline{\mathcal{T}}\,\underline{\mathcal{O}}$$

$$\underline{R}^{-1} = \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & & 1 & 0 \\ a_3 & & & \iddots & & \vdots \\ \vdots & & \iddots & & & \\ a_{n-1} & 1 & 0 & \cdots & & 0 \\ 1 & 0 & 0 & \cdots & & 0 \end{bmatrix}}_{\underline{\mathcal{T}}} \cdot \underbrace{\begin{bmatrix} \underline{C} \\ \underline{C}\,\underline{A} \\ \underline{C}\,\underline{A}^2 \\ \vdots \\ \underline{C}\,\underline{A}^{n-1} \end{bmatrix}}_{\underline{\mathcal{O}}}$$

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Transformation to Observer Form

- With the pair ($\underline{A}$,$\underline{c}$) being observable, the transformation follows the equation:

$$\underline{x}(t) = \underline{R}\,\underline{z}(t)$$

so that:

$$\begin{bmatrix} \dot z_1(t) \\ \dot z_2(t) \\ \vdots \\ \dot z_n(t) \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 0 & \cdots & -a_0 \\ 1 & 0 & \cdots & -a_1 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & 1 & -a_{n-1} \end{bmatrix}}^{\hat{\underline{A}}} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} + \overbrace{\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}}^{\hat{\underline{b}}} u(t)$$

$$y(t) = \overbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}}^{\hat{\underline{c}}} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix}$$

$$\hat{\underline{A}} = \underline{R}^{-1}\underline{A}\,\underline{R}$$
$$\hat{\underline{b}} = \underline{R}^{-1}\underline{b}$$
$$\hat{\underline{c}} = \underline{c}\,\underline{R}$$

Transformation to Observer Form

- The characteristic equation of the transformed system can easily be found as:

$$a(s) = \det(s\underline{I} - \hat{\underline{A}}) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

- After connecting the closed-loop state estimator, its output can be written as:

$$\dot{\hat{\underline{z}}}(t) = \left(\hat{\underline{A}} - \hat{\underline{l}}\,\hat{\underline{c}}\right)\hat{\underline{z}}(t) + \hat{\underline{b}}u(t) + \hat{\underline{l}}\,y(t)$$

with:

$$\hat{\underline{l}} = \begin{bmatrix} \hat{l}_1 & \hat{l}_2 & \cdots & \hat{l}_n \end{bmatrix}^{\mathrm{T}}$$

Transformation to Observer Form

- Further matrix operations yield:

$$\dot{\hat{\underline{z}}}(t) = \begin{bmatrix} 0 & 0 & \cdots & -(a_0 + \hat{l}_1) \\ 1 & 0 & \cdots & -(a_1 + \hat{l}_2) \\ \vdots & \ddots & & \vdots \\ 0 & 0 & 1 & -(a_{n-1} + \hat{l}_n) \end{bmatrix} \hat{\underline{z}}(t) + \hat{\underline{b}}u(t) + \hat{\underline{l}}\,y(t)$$

- The characteristic equation of the closed-loop estimator is now:

$$a(s) = s^n + (a_{n-1} + \hat{l}_n)s^{n-1} + (a_{n-2} + \hat{l}_{n-1})s^{n-2} + \cdots + (a_1 + \hat{l}_2)s + (a_0 + \hat{l}_1)$$

- If the desired poles of the closed-loop estimator are specified by $p_1, p_2, \ldots, p_n$ then:

$$\begin{aligned}\breve{a}(s) &= (s - p_1)(s - p_2)\cdots(s - p_n) \\ &= s^n + \breve{a}_{n-1}s^{n-1} + \cdots + \breve{a}_1 s + \breve{a}_0\end{aligned}$$

Transformation to Observer Form

- By comparing the coefficients of the previous two polynomials, it is clear, that in order to obtain the desired characteristic equation, the feedback gain must satisfy:

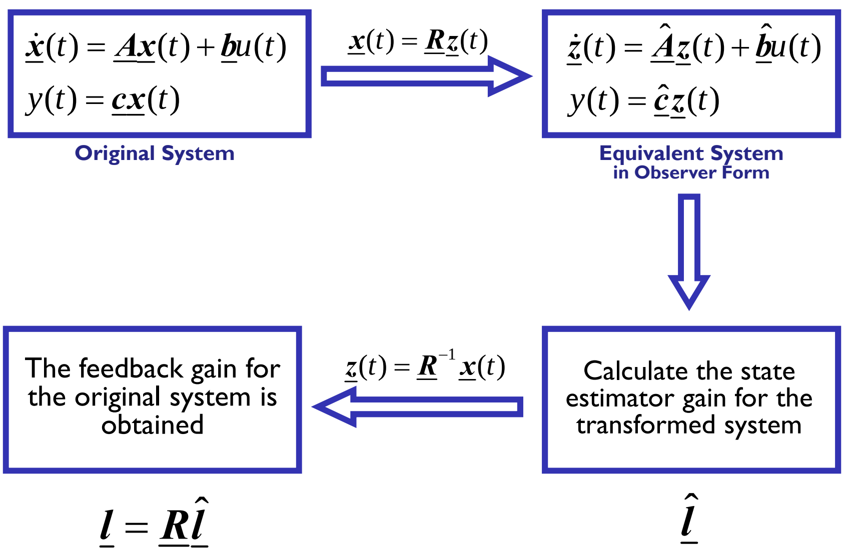
$$a_0 + \hat{l}_1 = \breve{a}_0 \longrightarrow \hat{l}_1 = \breve{a}_0 - a_0$$

$$a_1 + \hat{l}_2 = \breve{a}_1 \longrightarrow \hat{l}_2 = \breve{a}_1 - a_1$$

$$\vdots \qquad\qquad \vdots$$

$$a_{n-1} + \hat{l}_n = \breve{a}_{n-1} \longrightarrow \hat{l}_n = \breve{a}_{n-1} - a_{n-1}$$

Transformation to Observer Form



Example: State Estimators

Let us go back to the last example.

$$\dot{\underline{x}}(t) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x}(t)$$

Previously in part (b), the desired closed-loop state estimator for the system should have eigenvalues at $-2 \pm j2$.

Redo part (b), now by using the transformation to the Observer Form.

Example: State Estimators

$$a(s) = \det(s\underline{I} - \underline{A}) = \det\left(\begin{bmatrix} s-2 & -1 \\ 1 & s-1 \end{bmatrix}\right) = s^2 \underbrace{-3}_{a_1} s + \underbrace{3}_{a_0}$$

$$\underline{\mathcal{O}} = \begin{bmatrix} \underline{C} \\ \underline{C}\,\underline{A} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\underline{\mathcal{T}} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\underline{R}^{-1} = \underline{\mathcal{T}}\,\underline{\mathcal{O}} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

Checking the Observer Form,

$$\hat{\underline{A}} = \underline{R}^{-1}\underline{A}\,\underline{R} = \begin{bmatrix} 0 & -3 \\ 1 & 3 \end{bmatrix} \checkmark$$

$$\hat{\underline{c}} = \underline{c}\,\underline{R} = \begin{bmatrix} 0 & 1 \end{bmatrix} \checkmark$$

Example: State Estimators

The desired characteristic equation of the state observer is:

$$\breve{a}(s) = (s+2+j2)(s+2-j2) = s^2 + \underbrace{4}_{\breve{a}_1}s + \underbrace{8}_{\breve{a}_0}$$

$$\hat{l}_1 = \breve{a}_0 - a_0 = 8 - 3 = 5$$
$$\hat{l}_2 = \breve{a}_1 - a_1 = 4 - (-3) = 7$$

$$\underline{\hat{l}} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

For the transformed system

$$\underline{l} = \underline{R}\underline{\hat{l}} = \underline{\underline{\begin{bmatrix} -12 \\ 19 \end{bmatrix}}}$$

For the original system

Now, if the desired poles are –3 and –4, we can repeat the calculation as follows:

$$\breve{a}(s) = (s+3)(s+4) = s^2 + \underbrace{7}_{\breve{a}_1}s + \underbrace{12}_{\breve{a}_0}$$

$$\hat{l}_1 = \breve{a}_0 - a_0 = 12 - 3 = 9$$
$$\hat{l}_2 = \breve{a}_1 - a_1 = 7 - (-3) = 10$$

$$\underline{\hat{l}} = \begin{bmatrix} 9 \\ 10 \end{bmatrix}$$

For the transformed system

$$\underline{l} = \underline{R}\underline{\hat{l}} = \underline{\underline{\begin{bmatrix} -19 \\ 29 \end{bmatrix}}}$$

For the original system

Homework 7

Consider the following linear system given by:

$$\underline{\dot{x}}(t) = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -3 & 4 \\ -1 & 1 & -9 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \underline{x}(t)$$

(a) Using the transformation to the Observer Form, find the gain vector $\underline{l}$ of the closed-loop state estimator if the desired poles are –3 and –4 ± $j2$.

(b) Recall again the output feedback. In observer form, its effect on the characteristic equation of the system can be calculated much easier. By calculation, prove that the poles of the system cannot be assigned to any arbitrary location by only setting the value of output feedback j.

Homework 7A

Consider the following linear system given by:

$$\underline{\dot{x}}(t) = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \underline{x}(t)$$

(a) Calculate the eigenvalues and eigenvectors of the system. Is it stable or unstable?

(b) Using the transformation to the Observer Form, find the gain vector $\underline{l}$ of the closed-loop state estimator if the desired poles are $-1 \pm j2.5$ and -3.

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Homework 7

Consider the following linear system given by:

$$\underline{\dot{x}}(t) = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -3 & 4 \\ -1 & 1 & -9 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \underline{x}(t)$$

(a) Using the transformation to the Observer Form, find the gain vector $\underline{l}$ of the closed-loop state estimator if the desired poles are -3 and $-4 \pm j2$.

(b) Recall again the output feedback. In observer form, its effect on the characteristic equation of the system can be calculated much easier. By calculation, prove that the poles of the system cannot be assigned to any arbitrary location by only setting the value of output feedback *j*.

EE391 Control Systems and Components

Solution of Homework 7

$$a(s) = \det(s\underline{I} - \underline{A}) = s^3 + \underbrace{13}_{a_2}s^2 + \underbrace{33}_{a_1}s + \underbrace{13}_{a_0}$$

$$\underline{\mathcal{O}} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & -9 \\ 14 & -22 & 93 \end{bmatrix}$$

$$\underline{\mathcal{T}} = \begin{bmatrix} 33 & 13 & 1 \\ 13 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\underline{R}^{-1} = \begin{bmatrix} 21 & 17 & 9 \\ 11 & 3 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} -0.0361 & 0.2048 & -0.4940 \\ 0.0843 & -0.1446 & -0.1807 \\ 0.0361 & -0.2048 & 1.4940 \end{bmatrix}$$

Checking the Observer Form:

$$\hat{\underline{A}} = \begin{bmatrix} 0 & 0 & -13 \\ 1 & 0 & -33 \\ 0 & 1 & -13 \end{bmatrix} \checkmark$$

$$\hat{\underline{b}} = \begin{bmatrix} 46 \\ 13 \\ 0 \end{bmatrix}$$

$$\hat{\underline{c}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \checkmark$$

Solution of Homework 7

(a) Find the gain vector $\underline{l}$ of the closed-loop state estimator if the desired poles are –3 and –4 ± *j*2.

The desired characteristic equation of the state observer is:

$$\breve{a}(s) = (s+3)(s+4+j2)(s+4-j2) = s^3 + \underbrace{11}_{\breve{a}_2}s^2 + \underbrace{44}_{\breve{a}_1}s + \underbrace{60}_{\breve{a}_0}$$

$$\hat{l}_1 = \breve{a}_0 - a_0 = 60 - 13 = 47$$

$$\hat{l}_2 = \breve{a}_1 - a_1 = 44 - 33 = 11$$

$$\hat{l}_3 = \breve{a}_2 - a_2 = 11 - 13 = -2$$

$$\hat{\underline{l}} = \begin{bmatrix} 47 \\ 11 \\ -2 \end{bmatrix}$$

For the transformed system

$$\underline{l} = \underline{R}\hat{\underline{l}} = \begin{bmatrix} 1.5422 \\ 2.7349 \\ -3.5422 \end{bmatrix}$$

For the original system

Solution of Homework 7

(b) Prove that the poles of the system cannot be placed freely by only setting a single variable *j* of output feedback.

$$\dot{\hat{\underline{x}}}(t) = \begin{bmatrix} 0 & 0 & -13 \\ 1 & 0 & -33 \\ 0 & 1 & -13 \end{bmatrix} \hat{\underline{x}}(t) + \begin{bmatrix} 46 \\ 13 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \hat{\underline{x}}(t)$$

- **Transformation Result, Observer Form**

$$\dot{\underline{x}}(t) = (\underline{A} - j\underline{b}\underline{c})\underline{x}(t) + \underline{b}r(t)$$
$$y(t) = \underline{c}\underline{x}(t)$$

- **Output Feedback**

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Solution of Homework 7

$$\dot{\underline{x}}(t) = (\underline{A} - j\underline{b}\underline{c})\underline{x}(t) + \underline{b}r(t)$$

$$\dot{\underline{x}}(t) = \left(\begin{bmatrix} 0 & 0 & -13 \\ 1 & 0 & -33 \\ 0 & 1 & -13 \end{bmatrix} - j \begin{bmatrix} 46 \\ 13 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right) \underline{x}(t) + \underline{b}r(t)$$

$$= \left(\begin{bmatrix} 0 & 0 & -13 \\ 1 & 0 & -33 \\ 0 & 1 & -13 \end{bmatrix} - j \begin{bmatrix} 0 & 0 & 46 \\ 0 & 0 & 13 \\ 0 & 0 & 0 \end{bmatrix} \right) \underline{x}(t) + \underline{b}r(t)$$

$$= \begin{bmatrix} 0 & 0 & -(13+46j) \\ 1 & 0 & -(33+13j) \\ 0 & 1 & -13 \end{bmatrix} \underline{x}(t) + \underline{b}r(t)$$

- **Only a_0 and a_1 can be adjusted, both dependent to each other**
- **The poles of the system cannot be placed in any wished position**

$$\Rightarrow \breve{a}(s) = s^3 + 13s^2 + (33+13j)s + (13+46j)$$

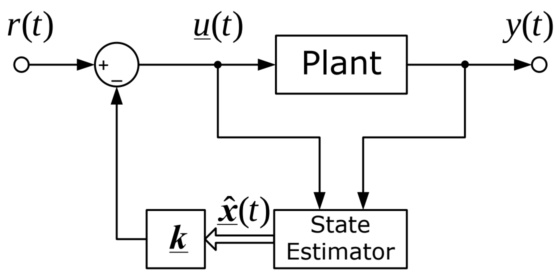
Feedback of Estimated States

- The estimated states will now be used to change the behavior of the system, through state feedback.
- Consider the *n*-dimensional single-variable state space equations:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{b}u(t)$$
$$y(t) = \underline{c}\underline{x}(t)$$

- If the pair ($\underline{A}$,$\underline{b}$) is ***controllable***, then the state feedback $u(t) = r(t) - \underline{k}\underline{x}(t)$ can place the eigenvalues of ($\underline{A}-\underline{b}\underline{k}$) in any desired positions.
- If the state variables are not available for feedback, then a state estimator with arbitrary eigenvalues can be designed for the system, provided that the pair ($\underline{A}$,$\underline{c}$) is ***observable***.

Feedback of Estimated States



"Controller-Estimator Configuration"

Feedback of Estimated States

- We recall again the state equation of the state estimator:

$$\dot{\hat{\underline{x}}}(t) = (\underline{A} - \underline{l}\,\underline{c})\hat{\underline{x}}(t) + \underline{b}u(t) + \underline{l}y(t)$$

- The rate of how the estimated states $\hat{\underline{x}}(t)$ approach the actual states $\underline{x}(t)$ can be adjusted by selecting an appropriate value for matrix $\underline{l}$.

- Because $\underline{x}(t)$ is not available in the configuration, it is replaced by $\hat{\underline{x}}(t)$ for feedback:

$$u(t) = r(t) - \underline{k}\,\hat{\underline{x}}(t)$$

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Feedback of Estimated States

- Substituting the last equation to the original system and the state estimator, will yield:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) - \underline{b}\underline{k}\hat{\underline{x}}(t) + \underline{b}r(t)$$
$$\dot{\hat{\underline{x}}}(t) = \left(\underline{A} - \underline{l}\,\underline{c} - \underline{b}\underline{k}\right)\hat{\underline{x}}(t) + \underline{b}r(t) + \underline{l}y(t)$$

- The two equations above can be combined in a new state space equation in the form of:

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \dot{\hat{\underline{x}}}(t) \end{bmatrix} = \begin{bmatrix} \underline{A} & -\underline{b}\underline{k} \\ \underline{l}\,\underline{c} & \underline{A} - \underline{l}\,\underline{c} - \underline{b}\underline{k} \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \hat{\underline{x}}(t) \end{bmatrix} + \begin{bmatrix} \underline{b} \\ \underline{b} \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} \underline{c} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \hat{\underline{x}}(t) \end{bmatrix}$$

Feedback of Estimated States

- To analyze these closed-loop systems, it is convenient to change the state variables by using the following transformation:

$$\begin{bmatrix} \underline{x}(t) \\ \underline{e}(t) \end{bmatrix} = \begin{bmatrix} \underline{x}(t) \\ \underline{x}(t) - \hat{\underline{x}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{I} & \underline{0} \\ \underline{I} & -\underline{I} \end{bmatrix}}_{\underline{P},\quad \underline{P}^{-1} = \underline{P}} \begin{bmatrix} \underline{x}(t) \\ \hat{\underline{x}}(t) \end{bmatrix}$$

- After performing the equivalence transformation, the following state equations can be obtained:

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \dot{\underline{e}}(t) \end{bmatrix} = \begin{bmatrix} \underline{A} - \underline{bk} & \underline{bk} \\ \underline{0} & \underline{A} - \underline{lc} \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{e}(t) \end{bmatrix} + \begin{bmatrix} \underline{b} \\ \underline{0} \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} \underline{c} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{e}(t) \end{bmatrix}$$

Feedback of Estimated States

- The eigenvalues of the new system in the "controller-estimator configuration" is the union of those of (**A**–**bk**) and (**A**–**lc**).
- This fact means, that the implementation of the state estimator does not affect the eigenvalues of the system with state feedback, and vice versa.
- The design of state feedback and state estimator are separated from each other. This is known as "**separation principle**" or "**separation property**."

Feedback of Estimated States

- [Franklin, Powell, Emami-Naeini] recommends that the real parts of the state estimator poles be a **factor of 2 to 6 deeper** in the left-half plane than the real parts of the state feedback poles.

Estimator poles deeper to the left → Faster decay of estimation error

Estimator poles deeper to the left → Larger magnitude of estimator gain

Larger magnitude of estimator gain → Saturation and unpredictable non-linear effects

Larger magnitude of estimator gain → High sensitivity to noise and disturbance

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Reference Input in State Feedback

- The state feedback has been proven to be able to place the poles of closed-loop system in arbitrary locations, and therefore can be used to design the **transient response** of a system.
- However, the **steady-state response** is still neglected until now, and the system will almost surely have a nonzero error to a step input.

$$u(t) = r(t) - \underline{k}\,\underline{x}(t)$$

$r(t)$ → **Reference Value**

- Now, two ways to incorporate the tracing of reference input while using state feedback will be introduced:
  - Pre-scaling/ Pre-amplifying
  - Integral Control

EE391 Control Systems and Components

Tracing of Reference Input: Pre-Scaling

- If the desired value of the states and the required process input to reach them are $\underline{x}_r(t)$ and $u_r(t)$, then the new feedback formula should be:

$$u(t) = u_r(t) - \underline{k}\left(\underline{x}(t) - \underline{x}_r(t)\right)$$

If $\underline{x}(t) \to \underline{x}_r(t)$, then $u(t) \to u_r(t)$

- Consider again the n-dimensional state space equations:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{b}u(t)$$

$$y(t) = \underline{c}\underline{x}(t)$$

- In steady-state condition, these equations reduce to:

$$\underline{0} = \underline{A}\underline{x}_{ss}(t) + \underline{b}u_{ss}(t)$$

$$y_{ss}(t) = \underline{c}\underline{x}_{ss}(t)$$

Tracing of Reference Input: Pre-Scaling

- Now, comparing the values from the above equations and the desired values, we obtain:

$$y_{ss}(t) = r(t)$$

$$\underline{x}_{ss}(t) = \underline{x}_r(t)$$

$$u_{ss}(t) = u_r(t)$$

- Let us now define:

$$\underline{x}_{ss}(t) = \underline{N}r(t)$$

$$u_{ss}(t) = Mr(t)$$

  - How? Why?

- The equations in steady-state condition can now be written as:

$$\begin{bmatrix} \underline{0} \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{b} \\ \underline{c} & 0 \end{bmatrix} \begin{bmatrix} \underline{N} \\ M \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \underline{N} \\ M \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{b} \\ \underline{c} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \underline{0} \\ 1 \end{bmatrix}$$

Tracing of Reference Input: Pre-Scaling

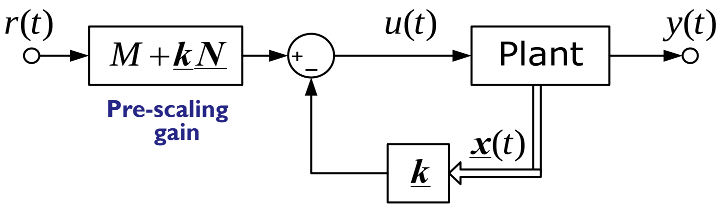
- After finding $\underline{N}$ and M, the required input to the system, $u(t)$, that guarantees zero steady-state error to a step input can be calculated as:

$$u(t) = u_r(t) - \underline{k}\left(\underline{x}(t) - \underline{x}_r(t)\right)$$

$$u(t) = Mr(t) - \underline{k}\left(\underline{x}(t) - \underline{N}r(t)\right)$$

$$= \underbrace{\left(M + \underline{k}\,\underline{N}\right)}_{E}r(t) - \underline{k}\,\underline{x}(t)$$

E • **New scalar gain for $r(t)$**



Example: Pre-Scaling

Referring again to the state-space equation that has been used before,

$$\dot{\underline{x}}(t) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\underline{x}(t)$$

For the desired eigenvalues of –1 and –2, it is already calculated that the required feedback gain is $\underline{k}$ = [4 1].

Now, it is desired that the output $y(t)$ should follow $r(t)$ = 1.5(t). Calculate the gain E for the reference value $r(t)$

Example: Pre-Scaling

$$\begin{bmatrix} \underline{N} \\ M \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{b} \\ \underline{c} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \underline{0} \\ 1 \end{bmatrix}$$

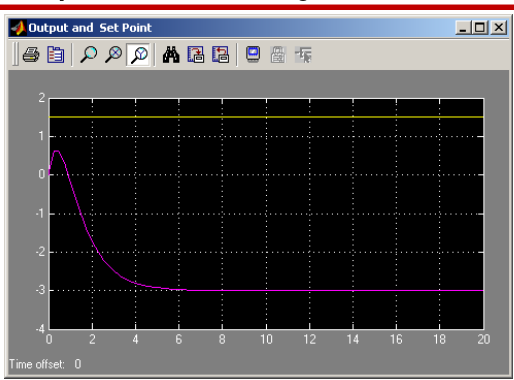
$$\begin{bmatrix} \underline{N} \\ M \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 1.25 \\ -0.75 \end{bmatrix}$$

$$\Rightarrow \underline{N} = \begin{bmatrix} -0.25 \\ 1.25 \end{bmatrix}, \quad M = -0.75$$

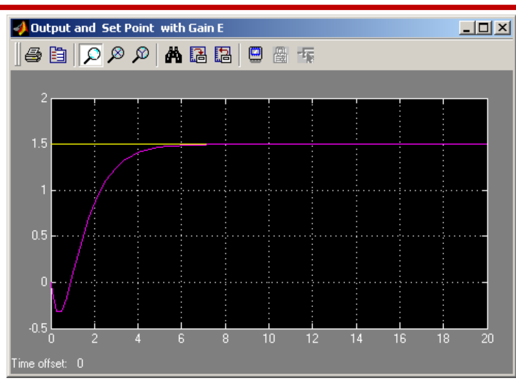
$$E = \left(M + \underline{k}\,\underline{N}\right) = \left(-0.75 + \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} -0.25 \\ 1.25 \end{bmatrix}\right) = \underline{\underline{-0.5}}$$

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Example: Pre-Scaling



Step Response Without Reference Gain *E*



Step Response With Reference Gain *E*

- **The previous steady-state value of the system is $y(\infty) = -3$, see left scope.**
- **The reference gain ($E = -0.5$) invert $y(\infty)$ to the desired value of $r(t) = 1.5(t)$, see right scope.**

Tracing of Reference Input: Integral Control

- The integral control is included by augmenting the state vector $\underline{x}(t)$ with the desired dynamics, such that the states of the system is increased, but still with the same form of:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{b}u(t)$$
$$y(t) = \underline{c}\underline{x}(t)$$

- The feedback is set to contain the integral of the error, e = $r-y$, as well as the state of the system, $x(t)$.
- We add the existing state with an extra integral state x_{int}, given by the following equation:

$$\dot{x}_{int}(t) = r(t) - \underline{c}\underline{x}(t) = e(t)$$

This implies that

$$x_{int}(t) = \int_0^{\tau} e(t)dt$$

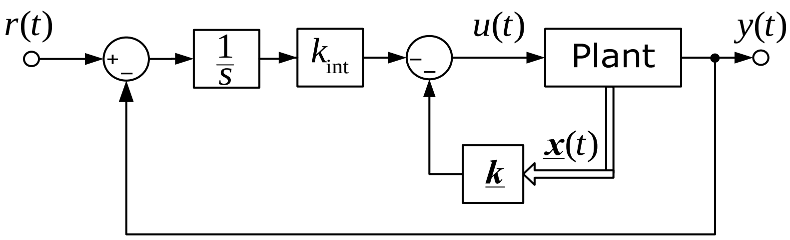
Tracing of Reference Input: Integral Control

- The augmented state space equations become

$$\begin{bmatrix} \dot{x}_{int} \\ \dot{\underline{x}} \end{bmatrix} = \begin{bmatrix} 0 & -\underline{c} \\ \mathbf{0} & \underline{A} \end{bmatrix} \begin{bmatrix} x_{int} \\ \underline{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \underline{b} \end{bmatrix} u + \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} r$$

with the feedback law –to incorporate the feedback $\underline{k}$ gain and integrator gain k_{int}– is chosen as

$$u(t) = -\underline{k}\,\underline{x}(t) - k_{int}x_{int}(t) = -\begin{bmatrix} k_{int} & \underline{k} \end{bmatrix} \begin{bmatrix} x_{int} \\ \underline{x} \end{bmatrix}$$



Tracing of Reference Input: Integral Control

- Substituting $u(t)$ to the augmented state space equations,

$$\begin{bmatrix} \dot{x}_{\text{int}} \\ \dot{\underline{x}} \end{bmatrix} = \left(\begin{bmatrix} 0 & -\underline{c} \\ \mathbf{0} & \underline{A} \end{bmatrix} - \begin{bmatrix} 0 \\ \underline{b} \end{bmatrix} \begin{bmatrix} k_{\text{int}} & \underline{k} \end{bmatrix} \right) \begin{bmatrix} x_{\text{int}} \\ \underline{x} \end{bmatrix} + \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} r$$

$$\begin{bmatrix} \dot{x}_{\text{int}} \\ \dot{\underline{x}} \end{bmatrix} = \begin{bmatrix} 0 & -\underline{c} \\ -k_{\text{int}} \underline{b} & \underline{A} - \underline{b}\underline{k} \end{bmatrix} \begin{bmatrix} x_{\text{int}} \\ \underline{x} \end{bmatrix} + \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} r$$

- The characteristic equation of the augmented system is now given as

$$a(s) = \det\left(s\underline{I} - \begin{bmatrix} 0 & -\underline{c} \\ -k_{\text{int}} \underline{b} & \underline{A} - \underline{b}\underline{k} \end{bmatrix} \right)$$

with the possibility to place the poles by means of $\underline{k}$ and k_{int}.

Example: Integral Control

The scheme should now be implemented on the state-space equations that has been used before,

$$\dot{\underline{x}}(t) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x}(t)$$

with the desired eigenvalues of –1 and –2, and $r(t)$ = 1.5(t).

The integrator increases the order of the system **by one** to become a third-order system. The third eigenvalues is assumed to be–3.

The augmented state-space equations is given by:

$$\begin{bmatrix} \dot{x}_{\text{int}} \\ \dot{\underline{x}} \end{bmatrix} = \begin{bmatrix} 0 & -\underline{c} \\ -k_{\text{int}} \underline{b} & \underline{A} - \underline{b}\underline{k} \end{bmatrix} \begin{bmatrix} x_{\text{int}} \\ \underline{x} \end{bmatrix} + \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} r$$

$$\begin{bmatrix} \dot{x}_{\text{int}} \\ \dot{\underline{x}} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ -k_{\text{int}} & 2-k_1 & 1-k_2 \\ -2k_{\text{int}} & -1-2k_1 & 1-2k_2 \end{bmatrix} \begin{bmatrix} x_{\text{int}} \\ \underline{x} \end{bmatrix} + \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} r$$

Example: Integral Control

$$a(s)=\det\left(s\underline{\boldsymbol{I}}-\begin{bmatrix}0 & -1 & -1\\ -k_{\text{int}} & 2-k_1 & 1-k_2\\ -2k_{\text{int}} & -1-2k_1 & 1-2k_2\end{bmatrix}\right)$$

$$a(s)=\det\left(\begin{bmatrix}s & 1 & 1\\ k_{\text{int}} & s-(2-k_1) & -(1-k_2)\\ 2k_{\text{int}} & 1+2k_1 & s-(1-2k_2)\end{bmatrix}\right)$$

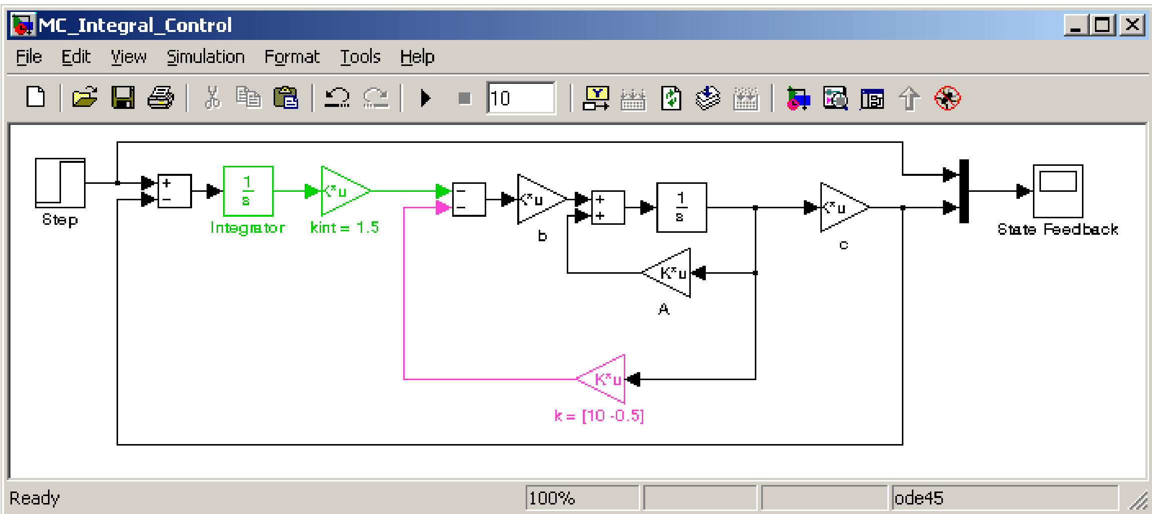
$$=s^3+(k_1+2k_2-3)s^2+(k_1-5k_2-3k_{\text{int}}+3)s+4k_{\text{int}}$$

$$\equiv s^3+6s^2+11s+6$$

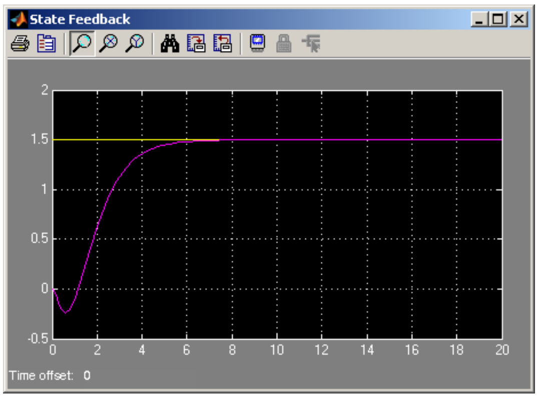
$$\equiv (s+1)(s+2)(s+3)$$

$$\therefore k_1=10, k_2=-0.5, k_{\text{int}}=1.5$$

Example: Integral Control

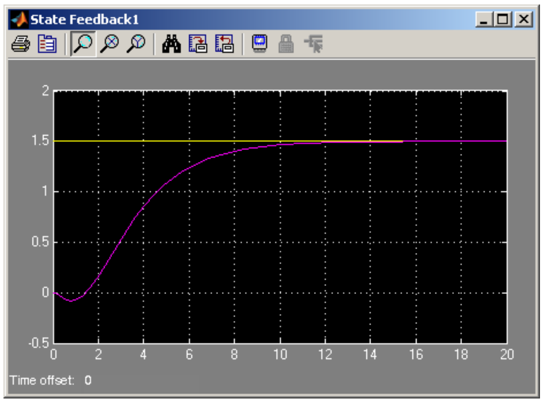


Example: Integral Control



Third pole at *s* = –3

$$k_1 = 10, k_2 = -0.5, k_{\text{int}} = 1.5$$



Third pole at *s* = –0.5

$$k_1 = 5, k_2 = 0.75, k_{\text{int}} = 0.25$$

- **What conclusion can be taken?**

FACULTY OF ENGINEERING - ALEXANDRIA UNIVERSITY 2013

Homework 8

Refer to the last example.

(a) Calculate the transfer function *G*(*s*) of the system.

(b) Calculate the steady-state value of the system to a unit step input, using the Final Value Theorem of Laplace Transform.

(c) Determine the gain *K* so that the steady-state response of *K*·*G*(*s*) has zero error to a unit step input.

(d) Find out the relation between the transfer function gain *K* and the reference gain *E*.

EE391 Control Systems and Components

Homework 8A

It is desired that the following linear system has zero steady state error to a unit step input. Find the solution by using:

$$\dot{\underline{x}}(t) = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -3 & 4 \\ -1 & 1 & -9 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \underline{x}(t)$$

(a) **Pre-scaling method**, by calculating the gain E.

(b) **Integral control method**, by calculate the gain $[k_{int}\ \boldsymbol{k}]$.
Hint: Assume the additional pole to be –1 and do not move the original poles of the system.

(c) Implement the original system, the system at (a) and the system at (b) in one Matlab Simulink file and compare the outputs.
Hint: For the matrix calculations, you may use Matlab. Write down or print the result on your homework papers.

Homework 8

Refer to the last example.

(a) Calculate the transfer function $G(s)$ of the system.

(b) Calculate the steady-state value of the system to a step input, using the Final Value Theorem of Laplace Transform.

(c) Determine the gain K so that the steady-state response of $K \cdot G(s)$ has zero error to a step input.

(d) Find out the relation between the transfer function gain K and the reference gain E.

Solution of Homework 8

(a) Calculate the transfer function of the system in s-Domain.

$$G(s) = \underline{c}(s\underline{I} - \underline{A} + \underline{bk})^{-1}\underline{b} = \frac{Y(s)}{U(s)}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & 1 \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 9 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ -9 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{s^2 + 3s + 2}$$

$$\underline{\underline{G(s) = \frac{3s-4}{s^2+3s+2}}}$$

Solution of Homework 8

(b) Calculate the steady-state value of the step response of the system, using the Final Value Theorem of Laplace Transform.

$$y(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0} s\cdot Y(s) = \lim_{s\to 0} s\cdot G(s)U(s)$$

$$= \lim_{s\to 0} s \cdot \frac{3s-4}{s^2+3s+2} \cdot \frac{1}{s}$$

$$= \frac{-4}{2}$$

$$= \underline{\underline{-2}}$$

Solution of Homework 8

(c) Determine the gain *K* so that the steady-state response of *K*·*G*(*s*) has zero error to a step input.

$$y(\infty) = \lim_{s \to 0} s \cdot G(s)U(s) = -2$$

$$y(\infty) = \lim_{s \to 0} s \cdot KG(s)U(s) \equiv u(\infty) = 1(t)$$

$$\Rightarrow K = -0.5$$

(d) Find out the relation between the transfer function gain *K* and the reference gain *E*.

$$K = E$$

$$K = \frac{a_0}{b_0}$$