MATHEMATICAL MODELING OF CONTROL SYSTEMS

Dr. Mohammed Morsy

Faculty of Engineering
Alexandria University

Outline

- Introduction
- Transfer function and impulse response function
- Laplace Transform Review
- Automatic control systems
- Signal Flow Graph
- Modeling In state-space
- Transformation of mathematical models with Matlab
- Linearization of nonlinear mathematical models

Introduction

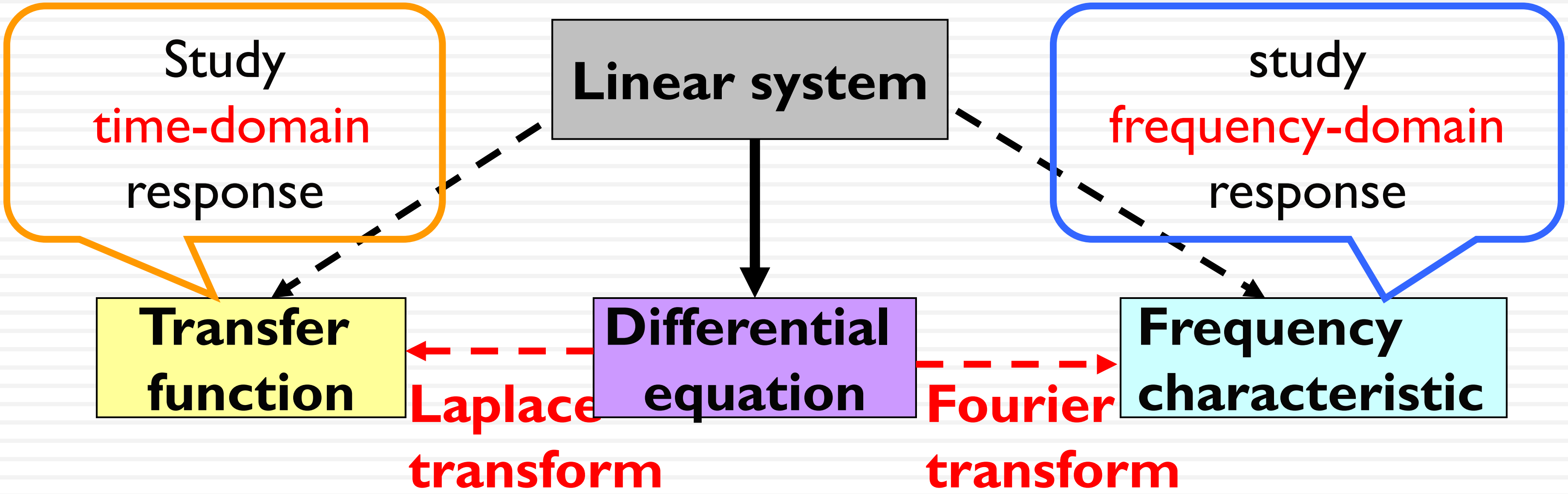
- A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the system accurately
- The dynamics of many systems may be described in terms of differential equations obtained from physical laws governing a particular system
- In obtaining a mathematical model, we must make a compromise between the simplicity of the model and the accuracy of the results of the analysis
- In general, in solving a new problem, it is desirable to build a simplified model so that we can get a general feeling for the solution

Introduction

- Linear Systems
  - A system is called linear if the principle of superposition applies.
  - The principle of superposition states that the response produced by the simultaneous application of two different forcing functions is the sum of the two individual responses.
  - Hence, for the linear system, the response to several inputs can be calculated by treating one input at a time and adding the results.
- Linear Time-Invariant (LTI) Systems and Linear Time-Varying Systems
  - A differential equation is linear if the coefficients are constants or functions only of the independent variable
  - Systems that are represented by differential equations whose coefficients are functions of time are called linear time-varying systems.

System Model

- Differential equation
- Transfer function
- Frequency characteristic



Transfer Function and Impulse Response

- Transfer Function:
  - The transfer function of an LTI system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y$$
$$= b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

$$\text{Transfer function} = G(s) = \left.\frac{\mathcal{L}[\text{output}]}{\mathcal{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$
$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Transfer Function

- The applicability of the transfer function concept is limited to LTI systems
  - The transfer function is a property of a system itself, independent of the magnitude and nature of the input or driving function.
  - If the transfer function of a system is known, the output or response can be studied for various forms of inputs
  - If the transfer function of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system.

Transfer Function (2)

- For an LTI system the transfer function G(s) is

$$G(s) = \frac{Y(s)}{X(s)} \Longrightarrow Y(s) = G(s)X(s)$$

- Multiplication in the complex domain is equivalent to convolution in the time domain

$$y(t) = \int_0^t x(\tau)g(t-\tau)\,d\tau = \int_0^t g(\tau)x(t-\tau)\,d\tau$$

Impulse Response

- Consider the output (response) of a linear time-invariant system to a unit-impulse input when the initial conditions are zero.
- Since the Laplace transform of the unit-impulse function is unity, the Laplace transform of the system output

$$Y(s) = G(s) \Longrightarrow \mathscr{L}^{-1}[G(s)] = g(t)$$

- The impulse-response function *g(t)* is thus the response of an LTI system to a unit-impulse input when the initial conditions are zero
- It is hence possible to obtain complete information about the dynamic characteristics of the system by exciting it with an impulse input and measuring the response

Laplace Transform Review

Note: This review is not included in the textbook (Slides 10 - 42)

Solving Differential Equation

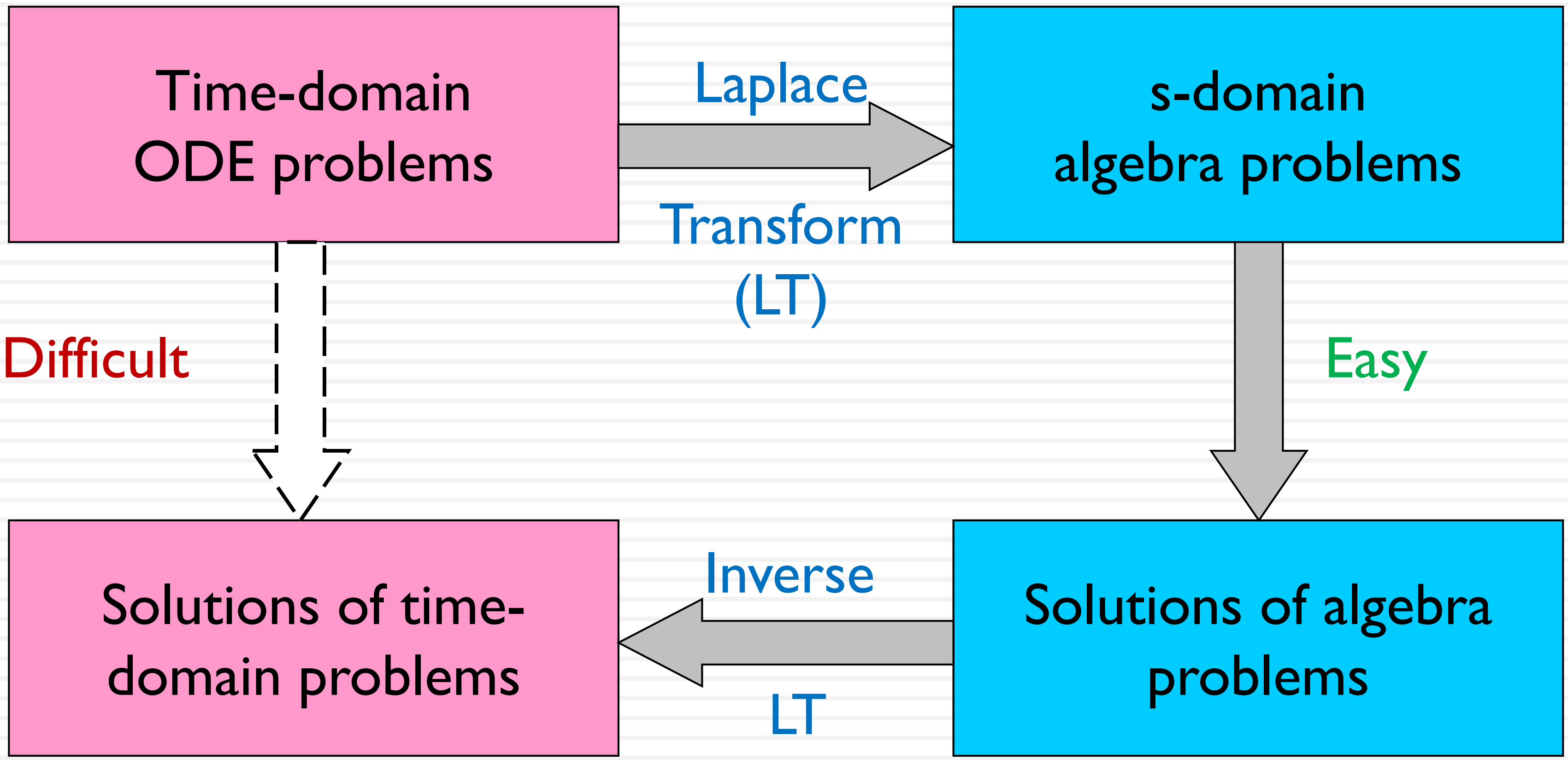
Example

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} = 2t + 1$$

Solving linear differential equations with constant coefficients:

- To find the general solution (involving solving the characteristic equation)
- To find a particular solution of the complete equation (involving constructing the family of a function)

Laplace Transform



Laplace Transform

Laplace, Pierre-Simon
1749-1827

The Laplace transform of a function *f*(*t*) is defined as

$$F(s) = \mathcal{L}[f(t)]$$

$$= \int_0^\infty f(t)e^{-st}dt$$

where $s = \sigma + j\omega$ is a complex variable.

Examples

- Step signal: *f(t)=A*

$$F(s)=\int_0^\infty f(t)e^{-st}dt=\int_0^\infty Ae^{-st}dt=-\frac{A}{s}e^{-st}\Big|_0^\infty=\frac{A}{s}$$

- Exponential signal *f(t)=* e^{-at}

$$F(s)=\int_0^\infty e^{-at}e^{-st}dt=-\frac{1}{s+a}e^{-(a+s)t}\Big|_0^\infty=\frac{1}{s+a}$$

Laplace Transform Table

f(t)	F(s)	f(t)	F(s)
$\delta(t)$	1	$\sin wt$	$\frac{w}{s^2 + w^2}$
1(t)	$\frac{1}{s}$	$\cos wt$	$\frac{s}{s^2 + w^2}$
t	$\frac{1}{s^2}$	$e^{-at} \sin wt$	$\frac{w}{(s+a)^2 + w^2}$
e^{-at}	$\frac{1}{s+a}$	$e^{-at} \cos wt$	$\frac{s+a}{(s+a)^2 + w^2}$

Properties of Laplace Transform

(1) Linearity

$$\mathcal{L}[af_1(t)+bf_2(t)] = a\mathcal{L}[f_1(t)] + b\mathcal{L}[f_2(t)]$$

(2) Differentiation

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

Using Integration By Parts method to prove

where f(0) is the initial value of f(t).

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)$$

Properties of Laplace Transform (2)

(3) Integration

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$$

Using Integration By Parts method to prove)

$$\mathcal{L}\left[\int_o^{t_1}\int_o^{t_2}\cdots\int_o^{t_n} f(\tau)d\tau dt_1 dt_2 \cdots dt_{n-1}\right] = \frac{F(s)}{s^n}$$

Properties of Laplace Transform (3)

（4） Final-value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

The final-value theorem relates the steady-state behavior of f(t) to the behavior of sF(s) in the neighborhood of s=0

(5) Initial-value Theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Properties of Laplace Transform (4)

(6) Shifting Theorem:

a. shift in time (real domain)

$$\mathcal{L}[f(t-\tau)] = e^{-\tau \cdot s} F(s)$$

b. shift in complex domain

$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$

(7) Real convolution (Complex multiplication) Theorem

$$\mathcal{L}[\int_0^t f_1(t-\tau) f_2(\tau) d\tau] = F_1(s) \cdot F_2(s)$$

Inverse Laplace Transform

Definition: Inverse Laplace transform, denoted by is given by $\mathcal{L}^{-1}[F(s)]$

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi \cdot j} \int_{C-j\infty}^{C+j\infty} F(s) e^{st} ds (t > 0)$$

where C is a real constant。

Note: The inverse Laplace transform operation involving rational functions can be carried out using Laplace transform table and partial-fraction expansion.

Partial Fraction Expansion

$$F(s)=\frac{N(s)}{D(s)}=\frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \quad (m<n)$$

If F(s) is broken up into components

$$F(s)=F_1(s)+F_2(s)+\ldots+F_n(s)$$

If the inverse Laplace transforms of components are readily available, then

$$\mathcal{L}^{-1}[F(s)]=\mathcal{L}^{-1}[F_1(s)]+\mathcal{L}^{-1}[F_2(s)]+\ldots+\mathcal{L}^{-1}[F_n(s)]$$

$$= f_1(t)+f_2(t)+\ldots+f_n(t)$$

Poles and Zeros

- Poles
  - A complex number s_0 is said to be a pole of a complex variable function $F(s)$ if $F(s_0)=\infty$.

- Zeros
  - A complex number s_0 is said to be a zero of a complex variable function $F(s)$ if $F(s_0)=0$.

Examples:

$$\frac{(s-1)(s+2)}{(s+3)(s+4)}$$ poles: -3, -4; zeros: 1, -2

$$\frac{s+1}{s^2+2s+2}$$ poles: -1+j, -1-j; zeros: -1

Case 1: F(s) has simple real poles

$$F(s)=\frac{N(s)}{D(s)}=\frac{b_0 s^m+b_1 s^{m-1}+\cdots+b_{m-1}s+b_m}{s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n}$$

Partial-Fraction Expansion

$$=\frac{c_1}{s-p_1}+\frac{c_2}{s-p_2}+\cdots+\frac{c_n}{s-p_n}$$

where $p_i (i=1,2,\cdots,n)$ are eigenvalues of $D(s)=0$, and

$$c_i=\left[\frac{N(s)}{D(s)}(s-p_i)\right]\Bigg|_{s=p_i}$$

Inverse LT

$$f(t)=c_1 e^{-p_1 t}+c_2 e^{-p_2 t}+\ldots+c_n e^{-p_n t}$$

Parameters p_k give shape and numbers c_k give magnitudes.

Example 1

Partial-Fraction Expansion

$$F(s)=\frac{1}{(s+1)(s-2)(s+3)}=\frac{c_1}{s+1}+\frac{c_2}{s-2}+\frac{c_3}{s+3}$$

$$c_1=\left[\frac{1}{(s+1)(s-2)(s+3)}\cdot(s+1)\right]\Bigg|_{s=-1}=-\frac{1}{6}$$

$$c_2=\left[\frac{1}{(s+1)(s-2)(s+3)}\cdot(s-2)\right]\Bigg|_{s=2}=\frac{1}{15}$$

$$c_3=\left[\frac{1}{(s+1)(s-2)(s+3)}\cdot(s+3)\right]\Bigg|_{s=-3}=\frac{1}{10}$$

$$\therefore F(s)=-\frac{1}{6}\cdot\frac{1}{s+1}+\frac{1}{15}\cdot\frac{1}{s-2}+\frac{1}{10}\cdot\frac{1}{s+3}$$

$$\therefore f(t)=-\frac{1}{6}e^{-t}+\frac{1}{15}e^{2t}+\frac{1}{10}e^{-3t}$$

Case 2: F(s) has simple complex-conjugate poles

Example 2

$$\ddot{y}(t)+4\dot{y}(t)+5y(t)=0,\ y(0)=\dot{y}(0)=1$$

Laplace transform

$$s^2Y(s)-sy(0)-\dot{y}(0)+4sY(s)-4y(0)+5Y(s)=0$$

Applying initial conditions

$$\therefore Y(s)=\frac{s+5}{s^2+4s+5}=\frac{s+5}{(s+2)^2+1}=\frac{s+2+3}{(s+2)^2+1}$$

Partial-Fraction Expansion

$$=\frac{s+2}{(s+2)^2+1}+\frac{3}{(s+2)^2+1}$$

Inverse Laplace transform

$$y(t)=e^{-2t}\cos t+3e^{-2t}\sin t$$

Case 3: F(s) has multiple-order poles

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s-p_1)(s-p_2)\cdots(s-p_{n-r})(s-p_i)^l}$$

$$=\frac{c_1}{s-p_1}+\cdots+\frac{c_{n-l}}{s-p_{n-l}}+\frac{b_l}{(s-p_i)^l}+\frac{b_{l-1}}{(s-p_i)^{l-1}}+\cdots+\frac{b_1}{s-p_i}$$

Simple poles Multi-order poles

The coefficients $c_1,\cdots,c_{n-l}$ of simple poles can be calculated as Case 1;

The coefficients corresponding to the multi-order poles are determined as

$$b_l=\left[F(s)\cdot(s-p_i)^l\right]\Big|_{s=p_1},\; b_{l-1}=\left\{\frac{d}{ds}\left[F(s)\cdot(s-p_i)^l\right]\right\}\Bigg|_{s=pi},\cdots,$$

$$b_{l-m}=\frac{1}{m!}\left\{\frac{d^m}{ds}\left[\frac{N(s)}{D(s)}(s-p_i)^l\right]\right\}\Bigg|_{s=p_1},\; b_1=\frac{1}{(l-1)!}\left\{\frac{d^{l-1}}{ds}\left[\frac{N(s)}{D(s)}(s-p_i)^l\right]\right\}\Bigg|_{s=p_i}$$

Example 3

Solve the following differential equation

$$y^{(3)} + 3\ddot{y} + 3\dot{y} + y = 1, y(0) = \dot{y}(0) = \ddot{y}(0) = 0$$

Laplace transform:

$$s^3Y(s) - s^2y(0) - s\dot{y}(0) - \ddot{y}(0) + 3s^2Y(s) - 3sy(0) - 3\dot{y}(0)$$

$$+ 3sY(s) - 3y(0) + Y(s) = \frac{1}{s}$$

Applying initial conditions:

$$Y(s) = \frac{1}{s(s+1)^3}$$

s= -1 is a *3*rd order pole

Partial-Fraction Expansion

$$Y(s) = \frac{c_1}{s} + \frac{b_3}{(s+1)^3} + \frac{b_2}{(s+1)^2} + \frac{b_1}{s+1}$$

Determining coefficients:

$$c_1 = \frac{1}{s(s+1)^3} s \bigg|_{s=0} = 1$$

$$b_3 = [\frac{1}{s(s+1)^3}(s+1)^3]_{s=-1} = -1 \qquad b_1 = \frac{1}{2!}(2s^{-3})\bigg|_{s=-1} = -1$$

$$b_2 = \left\{\frac{d}{ds}[\frac{1}{s(s+1)^3}(s+1)^3]\right\}_{s=-1} = [\frac{d}{ds}(\frac{1}{s})]_{s=-1} = (-s^{-2})\big|_{s=-1} = -1$$

$$\therefore Y(s) = \frac{1}{s} - \frac{1}{(s+1)^3} - \frac{1}{(s+1)^2} - \frac{1}{s+1}$$

Inverse Laplace transform:

$$y(t) = 1 - \frac{1}{2}t^2 e^{-t} - te^{-t} - e^{-t}$$

Matlab Application

1. Laplace Transform

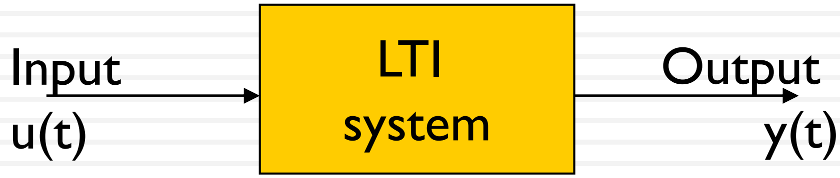
L= laplace(f)

2. Inverse Laplace Transform

F= ilaplace(L)

```
>> syms t
>> L=laplace(t)
L=
1/s^2
>> L=laplace(sin(t))
L=
1/(s^2+1)
>> F=ilaplace(L)
F=
sin(t)
```

Transfer Function



Consider a linear system described by differential equation

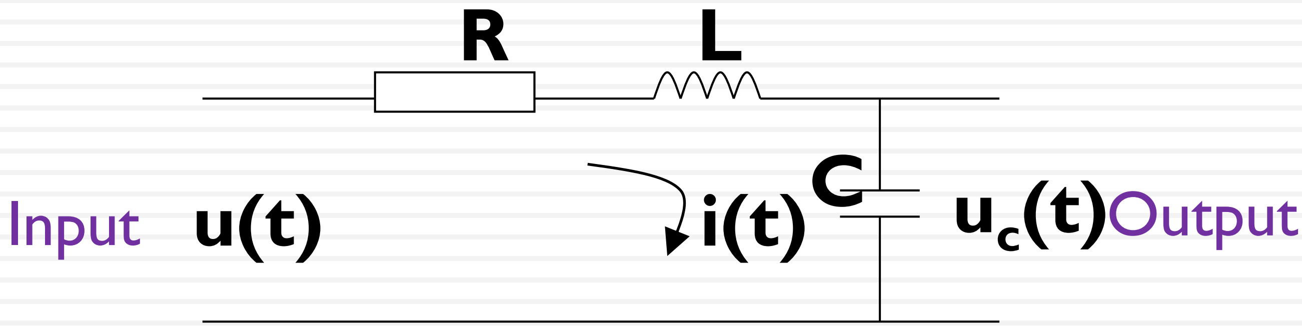
$$y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\cdots+a_0 y(t)=\ b_m u^{(m)}(t)+b_{m-1}u^{(m-1)}(t)+\cdots+bu^{(1)}(t)+b_0 u(t)$$

Assume all initial conditions are zero, we get the transfer function(TF) of the system as

$$TF=G(s)=\left.\frac{\mathcal{L}\left[output\ \ y(t)\right]}{\mathcal{L}\left[input\ u(t)\right]}\right|_{zero\ \ initial\ \ condition}$$

$$=\frac{Y(s)}{U(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}$$

Example 1. Find the transfer function of the RLC



Solution:

1) Writing the differential equation of the system according to physical law:

$$LC\ddot{u}_C(t) + RC\dot{u}_C(t) + u_C(t) = u(t)$$

2) Assuming all initial conditions are zero and applying Laplace transform

$$LCs^2U_c(s) + RCsU_c(s) + U_c(s) = U(s)$$

3) Calculating the transfer function $G(s)$ as

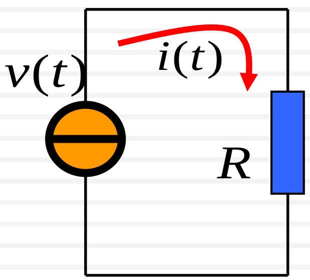
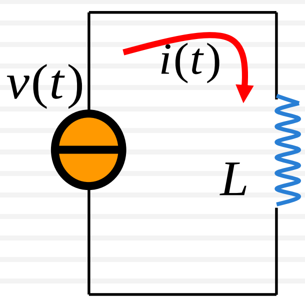
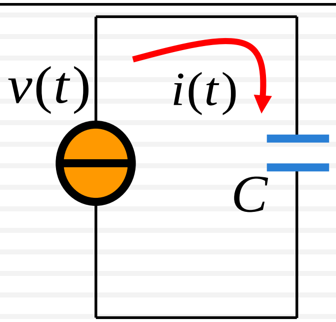
$$G(s) = \frac{U_c(s)}{U(s)} = \frac{1}{LCs^2 + RCs + 1}$$

Exercise

- Find the transfer function of the following system :

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = u(t)$$

Transfer Function of Typical Components

Component	ODE	TF
	$v(t) = Ri(t)$	$G(s) = \dfrac{V(s)}{I(s)} = R$
	$v(t) = L\dfrac{di(t)}{dt}$	$G(s) = \dfrac{V(s)}{I(s)} = sL$
	$v(t) = \dfrac{1}{C}\int_0^t i(\tau)d\tau$	$G(s) = \dfrac{V(s)}{I(s)} = \dfrac{1}{sC}$

Properties of Transfer Functions

- The transfer function is defined only for a linear time-invariant system, not for nonlinear system
- All initial conditions of the system are set to zero
- The transfer function is independent of the input of the system
- The transfer function G(s) is the Laplace transform of the unit impulse response g(t)

How poles and zeros relate to system response

- Why we strive to obtain TF models?
- Why control engineers prefer to use TF models?
- How to use a TF model to analyze and design control systems?

- we start from the relationship between the locations of zeros and poles of TF and the output responses of a system

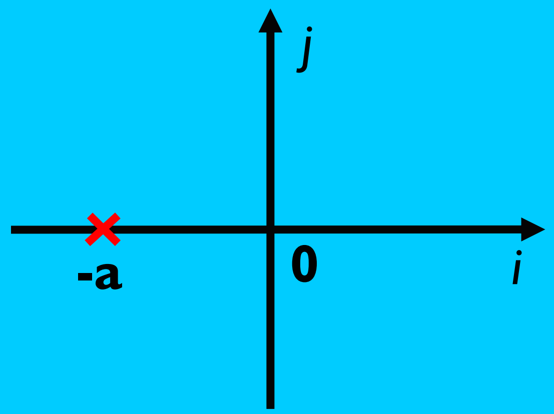
Transfer function

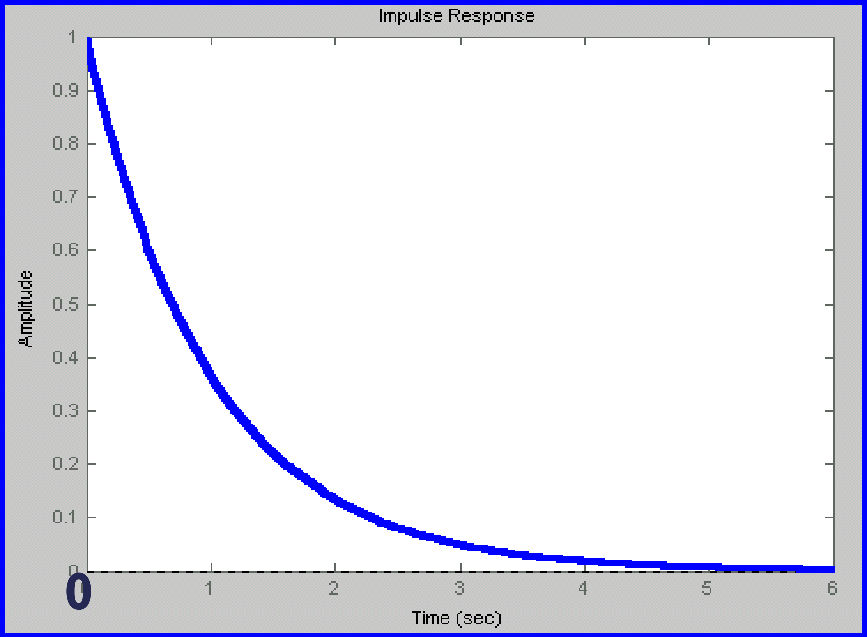
$$X(s) = \frac{A}{s+a}$$

Time-domain impulse response

$$x(t) = Ae^{-at}$$

Position of poles and zeros





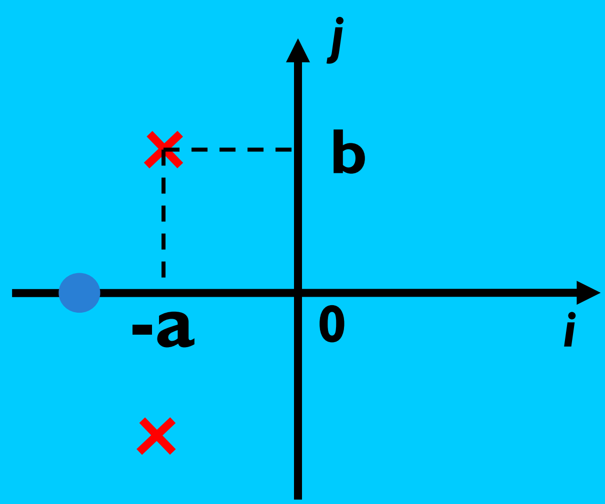
Transfer function

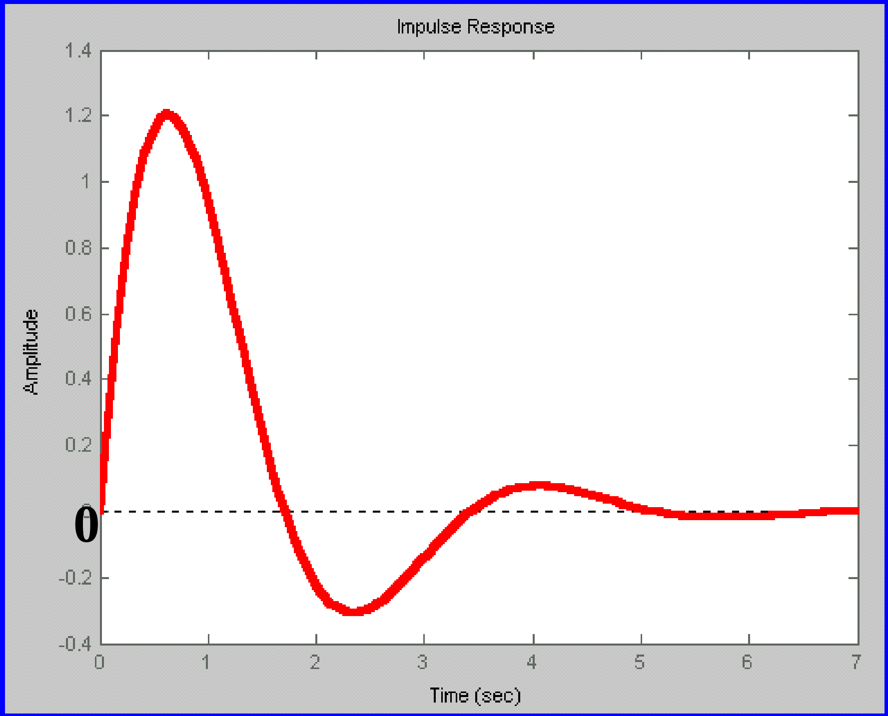
$$X(s) = \frac{A_1 s + B_1}{(s+a)^2 + b^2}$$

Time-domain impulse response

$$x(t) = Ae^{-at}\sin(bt + \phi)$$

Position of poles and zeros





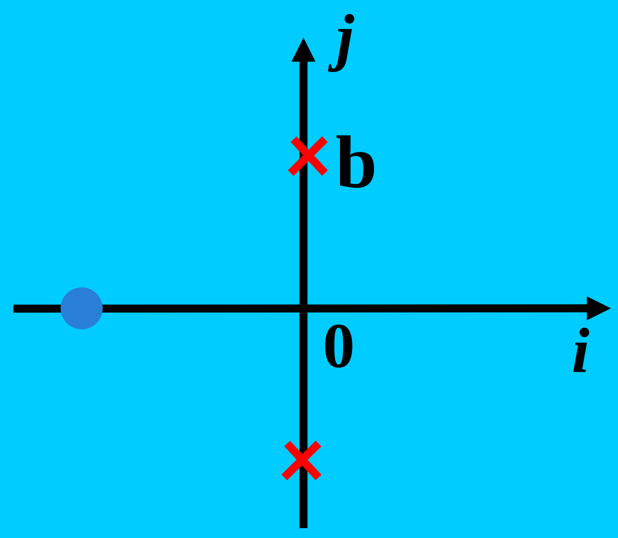
Transfer function

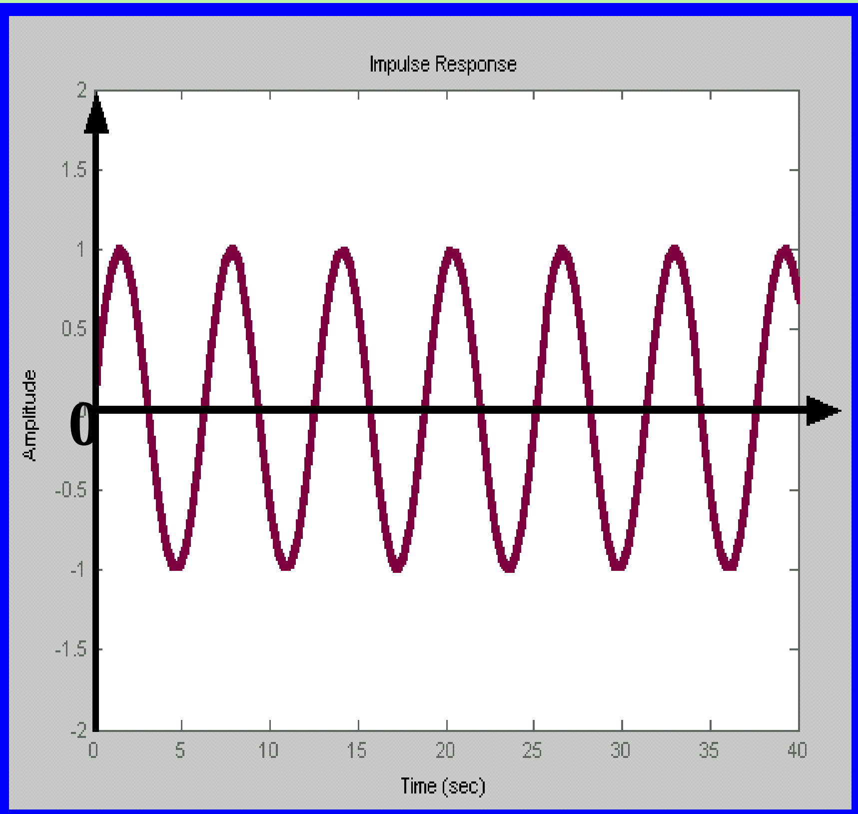
$$X(s) = \frac{A_1 s + B_1}{s^2 + b^2}$$

Time-domain impulse response

$$x(t) = A\sin(bt + \phi)$$

Position of poles and zeros





Transfer function

$$X(s) = \frac{A}{s-a}$$

Time-domain impulse response

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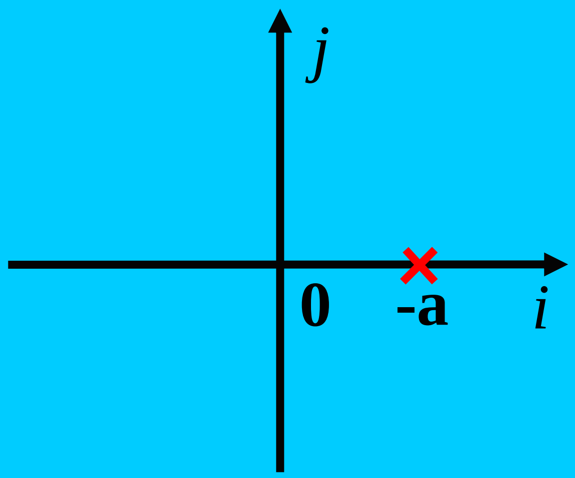
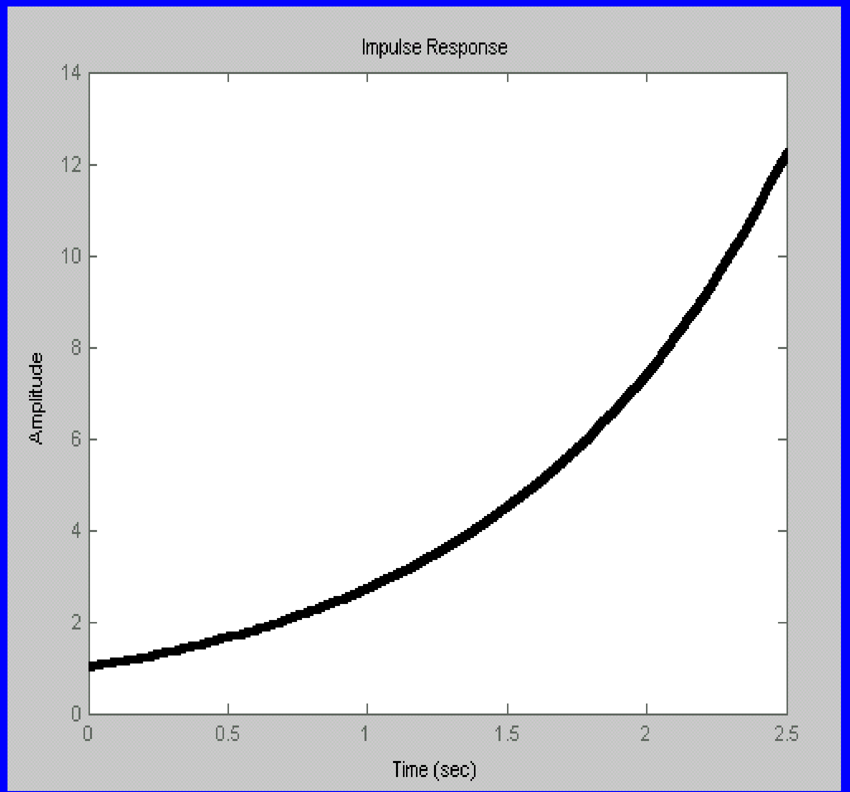
Position of poles and zeros





Transfer function

$$X(s) = \frac{A_1 s + B_1}{(s-a)^2 + b^2}$$

Time-domain impulse response

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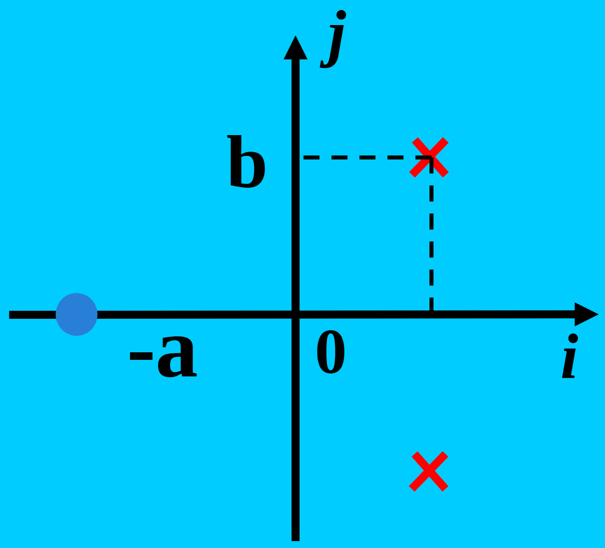
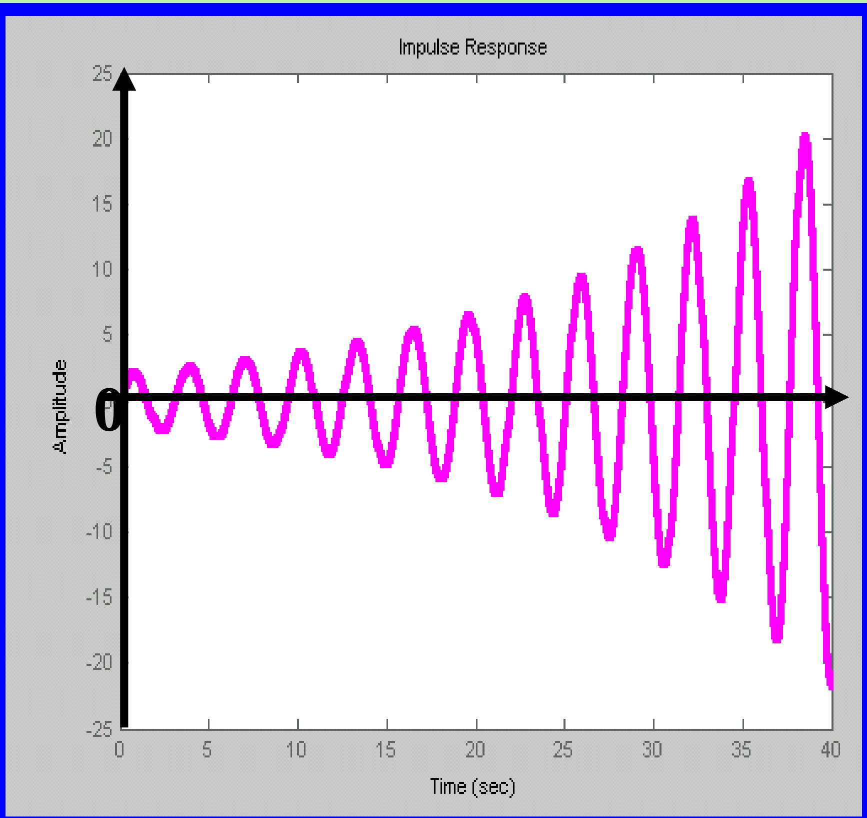
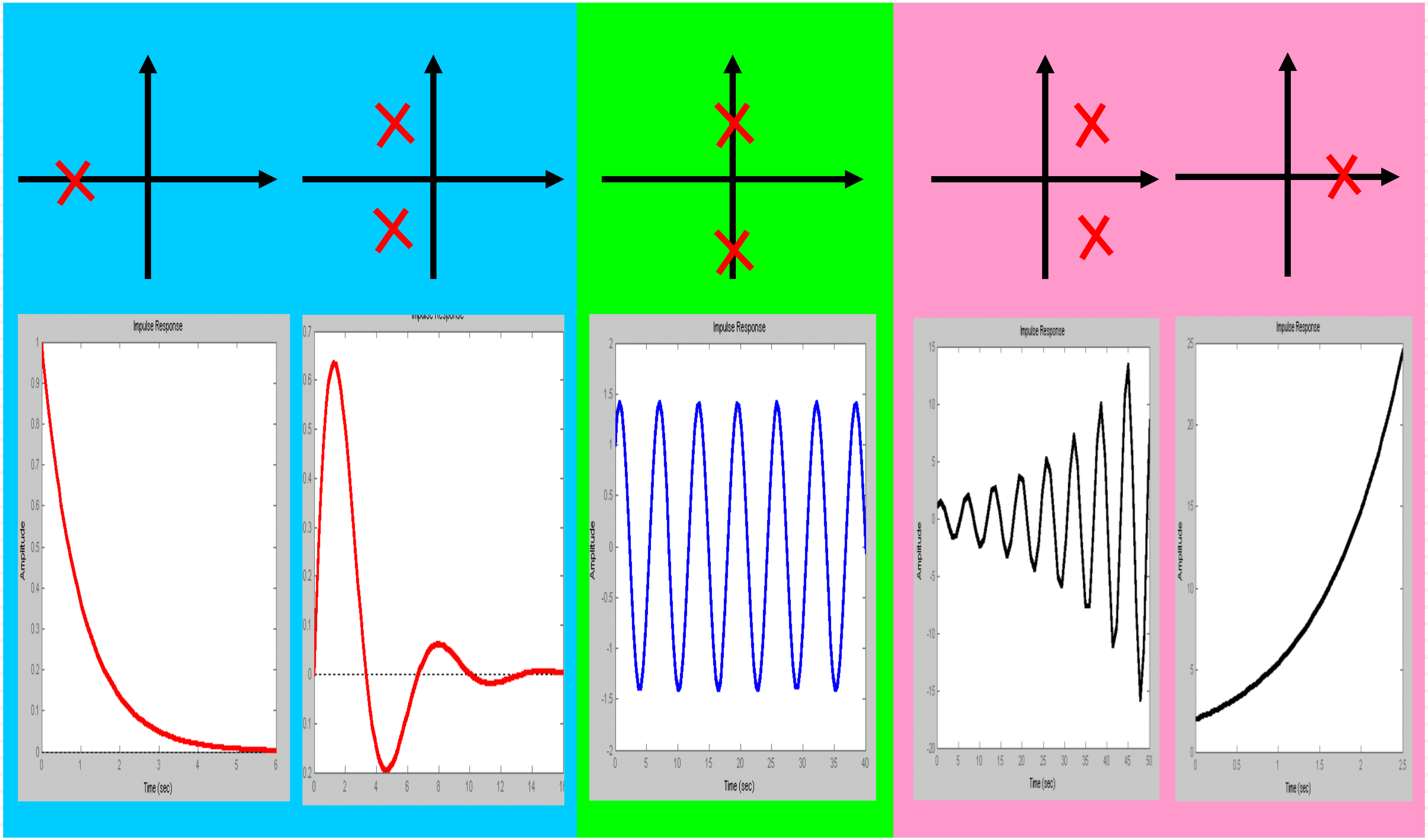
Position of poles and zeros





Summary of Pole Position & System Dynamics



Characteristic equation

- obtained by setting the denominator polynomial of the transfer function to zero

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

Note: stability of linear single-input, single-output systems is completely governed by the roots of the characteristics equation.

Transfer Function (TF) models in Matlab

Suppose a linear SISO system with input u(t), output y(t), the transfer function of the system is

$$G(s)=\frac{Y(S)}{U(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}$$

$$num=\left[b_m, b_{m-1},\ldots,b_0\right]$$

$$den=\left[1, a_{n-1},\ldots,a_0\right]$$

Descending power of s

TF in polynomial form

```
>> Sys = tf（num,  den）
>> [num, den] = tfdata (sys)
```

TF in zero-pole form

```
>> sys = zpk (z, p, k)
>> [z, p,k] = tfdata (sys)
```

Transform TS from zero-pole form into polynomial form

```
>> [z, p, k] = tf2zp(num, den)
```

Review Questions

- What is the definition of "transfer function"?
- When defining the transfer function, what happens to initial conditions of the system?
- Does a nonlinear system have a transfer function?
- How does a transfer function of a LTI system relate to its impulse response?
- Define the characteristic equation of a linear system in terms of the transfer function

Automatic Control Systems

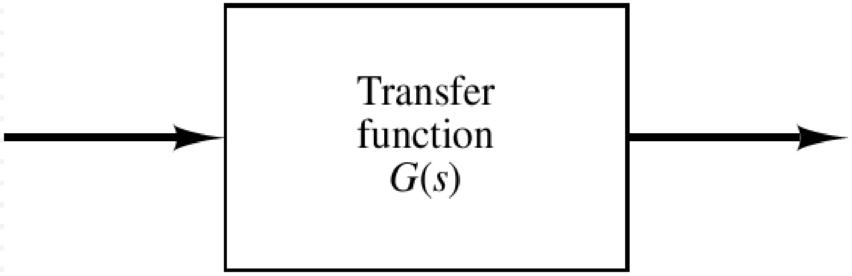
- A control system may consist of a number of components
- To show the functions performed by each component, in control engineering, we commonly use a diagram called the block diagram
- A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals
- A block diagram has the advantage of indicating more realistically the signal flows of the actual system

Block Diagrams

- In a block diagram all system variables are linked to each other through functional blocks
- The transfer functions of the components are usually entered in the corresponding blocks
- Blocks are connected by arrows to indicate the direction of the flow of signals

Note: The dimension of the output signal from the block is the dimension of the input signal multiplied by the dimension of the transfer function in the block

Element of a block diagram



Block Diagrams(2)

- The advantage of the block diagram representation is the simplicity of forming the overall block diagram for the entire system by connecting the blocks of the components according to the signal flow
- A block diagram contains information concerning dynamic behavior, but it does not include any information on the physical construction of the system
- A number of different block diagrams can be drawn for a system

Block Diagram Representation

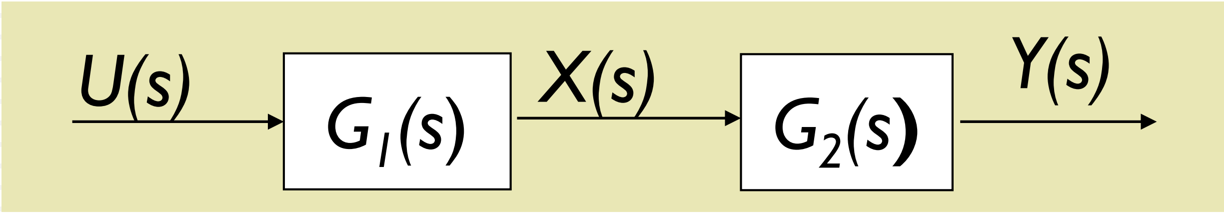
The transfer function relationship

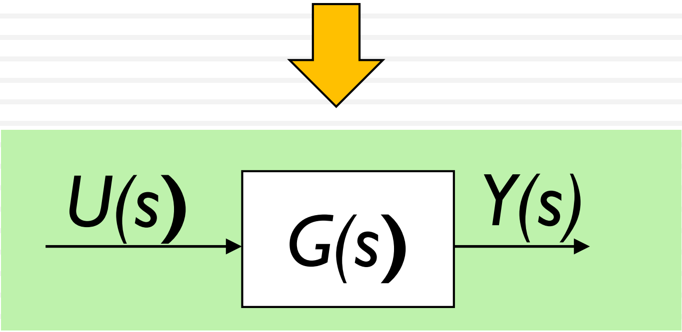
$$Y(s) = G(s)U(s)$$

can be graphically denoted through a block diagram.

Equivalent Transform of Block Diagram

1. Connection in series

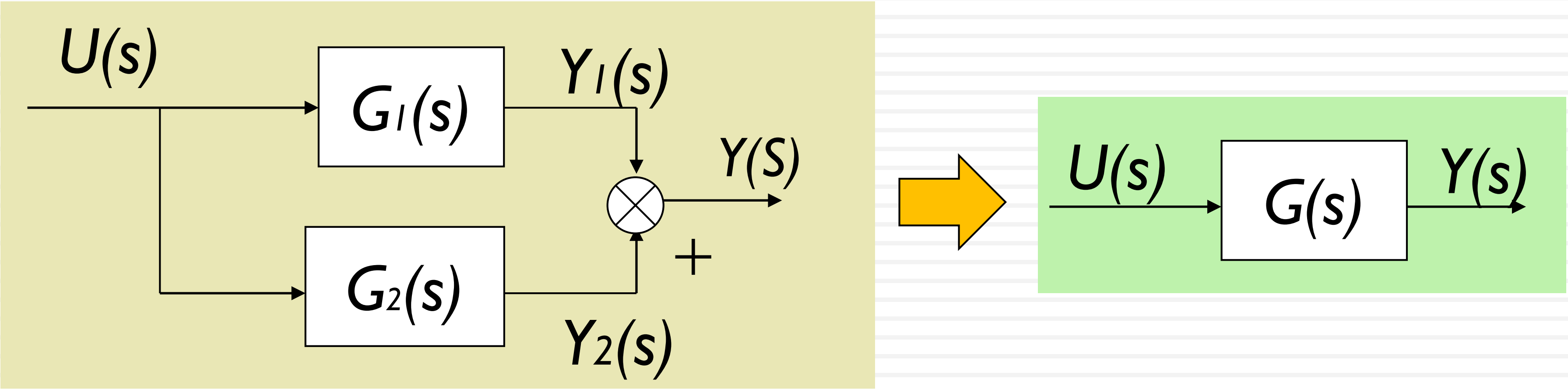




$$G(s) = ?$$

$$G(s) = \frac{Y(s)}{U(s)} = G_1(s) \cdot G_2(s)$$

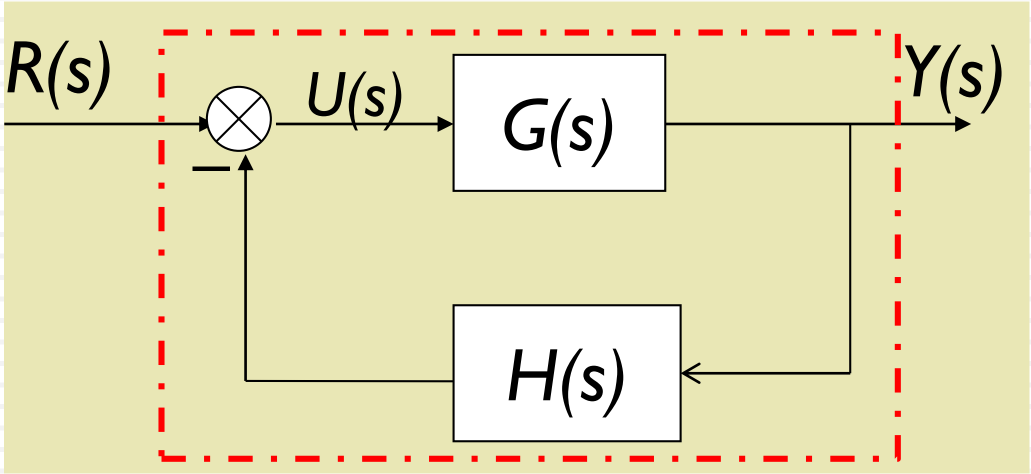
2. Connection in parallel

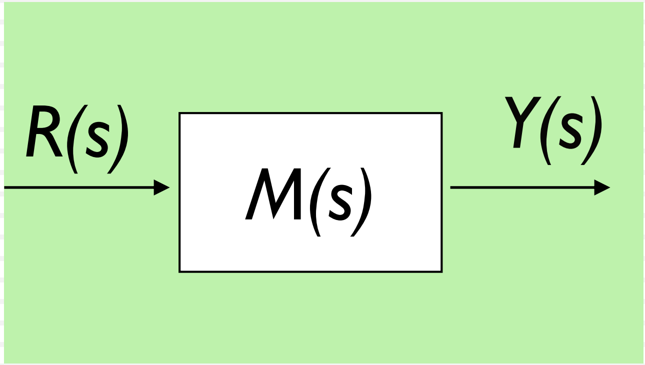


$$G(s) = ?$$

$$G(s) = \frac{Y(s)}{U(s)} = G_1(s) + G_2(s)$$

3. Negative feedback





$$\begin{cases} Y(s) = U(s)G(s) \\ U(s) = R(s) - Y(s)H(s) \end{cases} \Rightarrow Y(s) = \left[R(s) - Y(s)H(s)\right]G(s)$$

Transfer function of a negative feedback system:

$$M(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{\text{gain of the forward path}}{1+\text{gain of the loop}}$$

Matlab Application

- Obtaining Cascaded, Parallel, and Feedback (Closed-Loop) Transfer Functions with MATLAB
  - Suppose that there are two components $G_1(s)$ and $G_2(s)$

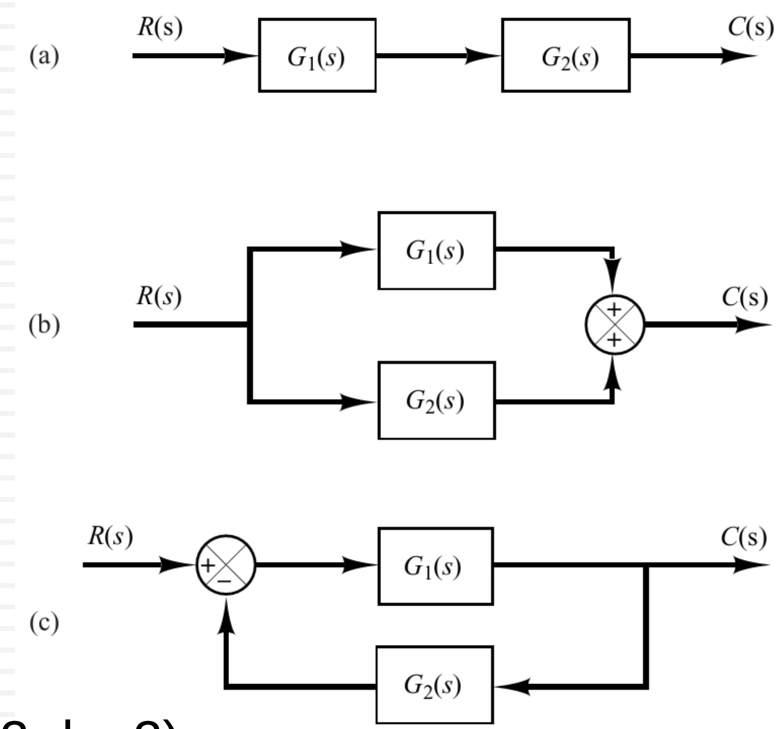
$$G_1(s) = \frac{\text{num1}}{\text{den1}}, \qquad G_2(s) = \frac{\text{num2}}{\text{den2}}$$

  - MATLAB has convenient commands to obtain the cascaded, parallel, and feedback (closed-loop) transfer functions.
  - Example:

$$G_1(s) = \frac{10}{s^2 + 2s + 10} = \frac{\text{num1}}{\text{den1}}, \qquad G_2(s) = \frac{5}{s + 5} = \frac{\text{num2}}{\text{den2}}$$

Matlab Application (2)

- To obtain the transfer functions of the cascaded system, parallel system, or feedback (closed-loop) system
- The following commands may be used

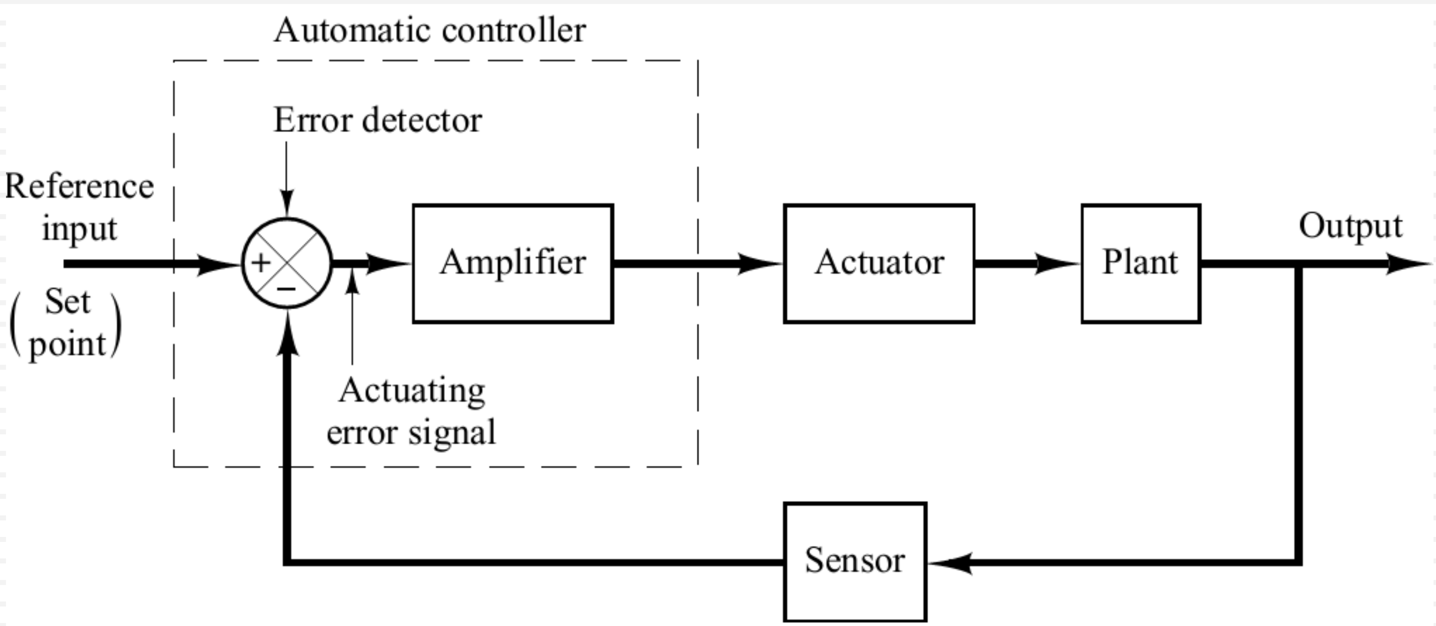



```
[num, den] = series(num1,den1,num2,den2)
[num, den] = parallel(num1,den1,num2,den2)
[num, den] = feedback(num1,den1,num2,den2)
```

- Check Matlab Program 2-1 in the textbook

Automatic Controllers

- An automatic controller compares the actual value of the plant output with the reference input (desired value), determines the deviation, and produces a control signal that will reduce the deviation to zero or to a small value.



Classifications of Industrial Controllers

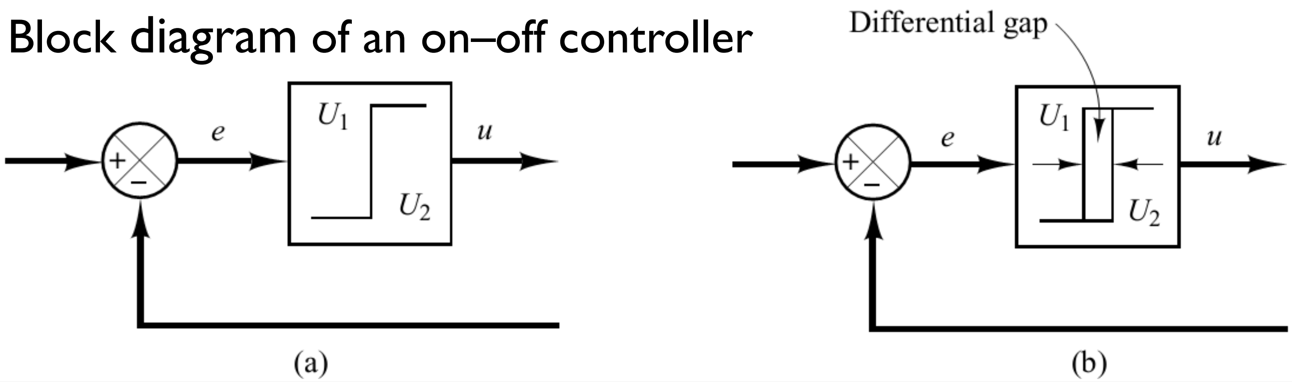
1. Two-position or on–off controllers
2. Proportional controllers
3. Integral controllers
4. Proportional-plus-integral controllers
5. Proportional-plus-derivative controllers
6. Proportional-plus-integral-plus-derivative controllers

Two-Position or On–Off Control Action

- In a two-position control system, the actuating element has only two fixed positions, which are, in many cases, simply on and off, e.g. Liquid-level control system
- Let the output signal from the controller be u(t)and the actuating error signal be e(t)

$$
\begin{aligned}
u(t) &= U_1, \qquad \text{for } e(t) > 0 \\
&= U_2, \qquad \text{for } e(t) < 0
\end{aligned}
$$

Block diagram of an on–off controller



Proportional Control Action

- For a controller with proportional control action, the relationship between the output of the controller u(t)and the actuating error signal e(t)is

$$u(t) = K_p e(t) \Longrightarrow \frac{U(s)}{E(s)} = K_p$$

Where K_p is termed the proportional gain

- Whatever the actual mechanism may be and whatever the form of the operating power, the proportional controller is essentially an amplifier with an adjustable gain

Integral Control Action

- In a controller with integral control action, the value of the controller output u(t)is changed at a rate proportional to the actuating error signal e(t)

$$\frac{du(t)}{dt} = K_i e(t) \Longrightarrow u(t) = K_i \int_0^t e(t)\,dt$$

$$\frac{U(s)}{E(s)} = \frac{K_i}{s}$$

where K_i is an adjustable constant

Other Control Actions

- Proportional-Plus-Integral Control Action

$$u(t) = K_p e(t) + \frac{K_p}{T_i}\int_0^t e(t)\,dt \Longrightarrow \frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s}\right)$$

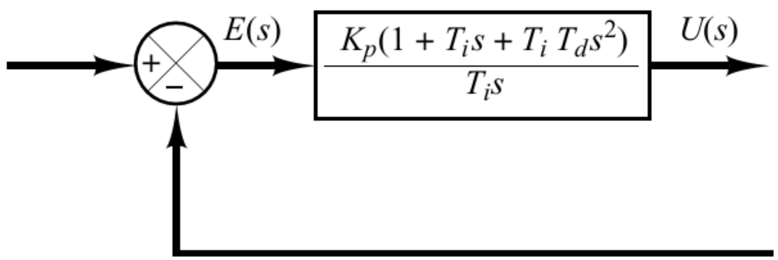
- Proportional-Plus-Derivative Control Action

$$u(t) = K_p e(t) + K_p T_d \frac{de(t)}{dt} \Longrightarrow \frac{U(s)}{E(s)} = K_p(1 + T_d s)$$

- Proportional-Plus-Integral-Plus-Derivative Control Action

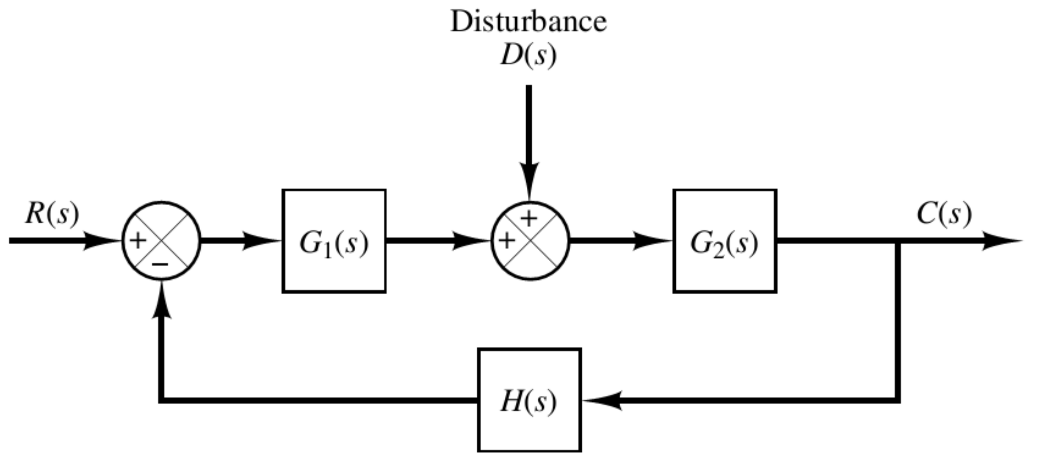
$$u(t) = K_p e(t) + \frac{K_p}{T_i}\int_0^t e(t)\,dt + K_p T_d \frac{de(t)}{dt} \Longrightarrow \frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

Block diagram of a proportional-plus-integral-plus-derivative controller



Closed-Loop System Subjected to a Disturbance

- Figure shows a closed-loop system subjected to a disturbance



- We can use superposition of LTI systems to obtain the output

$$\frac{C_D(s)}{D(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$ For $R(s)=0$

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$ For $D(s)=0$

Closed-Loop System Subjected to a Disturbance

- The response to both inputs is

$$C(s) = C_R(s) + C_D(s) = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}\left[G_1(s)R(s) + D(s)\right]$$

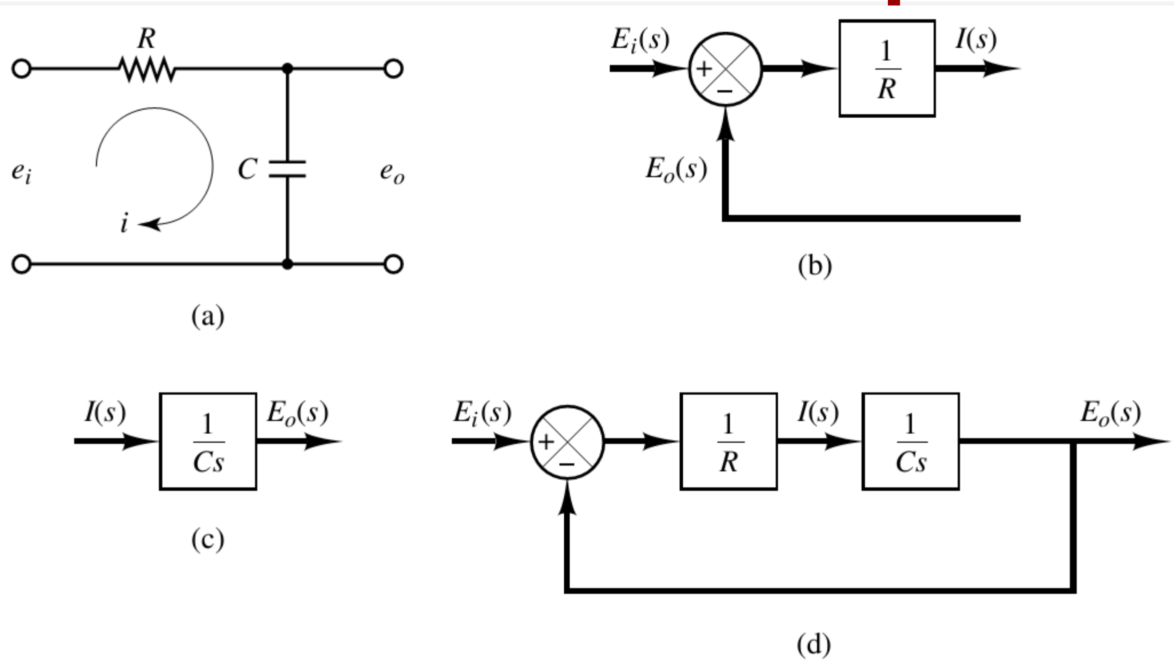
- Consider now the case where $|G_1(s)H(s)|>>1$ and $|G_1(s)G_2(s)H(s)|>>1$.
  - In this case, the closed-loop transfer function $C_D(s)/D(s)$ becomes almost zero, and the effect of the disturbance is suppressed. This is an advantage of the closed-loop system
  - The closed-loop transfer function $C_R(s)/R(s)$ approaches $1/H(s)$ as the gain of $G_1(s)G_2(s)H(s)$ increases
  - This means that if $|G_1(s)G_2(s)H(s)|>>1$, then the closed-loop transfer function $C_R(s)/R(s)$ becomes independent of $G_1(s)$ and $G_2(s)$ and inversely proportional to $H(s)$

Procedures for Drawing a Block Diagram

1. Write the equations that describe the dynamic behavior of each component
2. Take the Laplace transforms of these equations, assuming zero initial conditions
3. Represent each Laplace-transformed equation individually in block form
4. Assemble the elements into a complete block diagram.

Example

- Consider the RC circuit shown in Figure



- The equations for this circuit are

$$i = \frac{e_i - e_o}{R}$$

$$e_o = \frac{\int i\,dt}{C}$$

$$I(s) = \frac{E_i(s) - E_o(s)}{R}$$

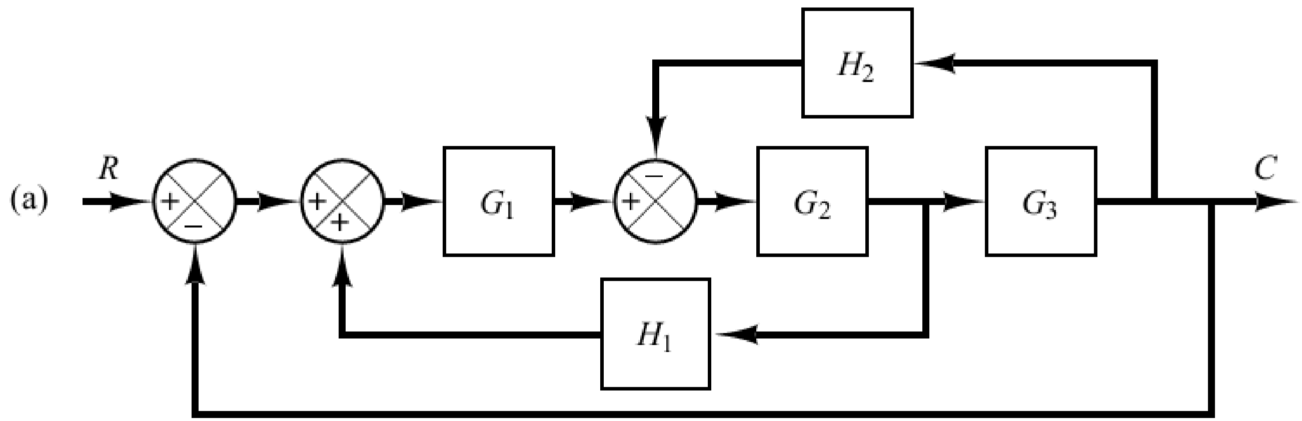
$$E_o(s) = \frac{I(s)}{Cs}$$

Block Diagram Reduction

- Any number of cascaded blocks can be replaced by a single block, the transfer function of which is simply the product of the individual transfer functions
- Blocks can be connected in series only if the output of one block is not affected by the next following block (no feedback)
- A complicated block diagram involving many feedback loops can be simplified by a step-by-step rearrangement
- Simplification of the block diagram by rearrangements considerably reduces the labor needed for subsequent mathematical analysis

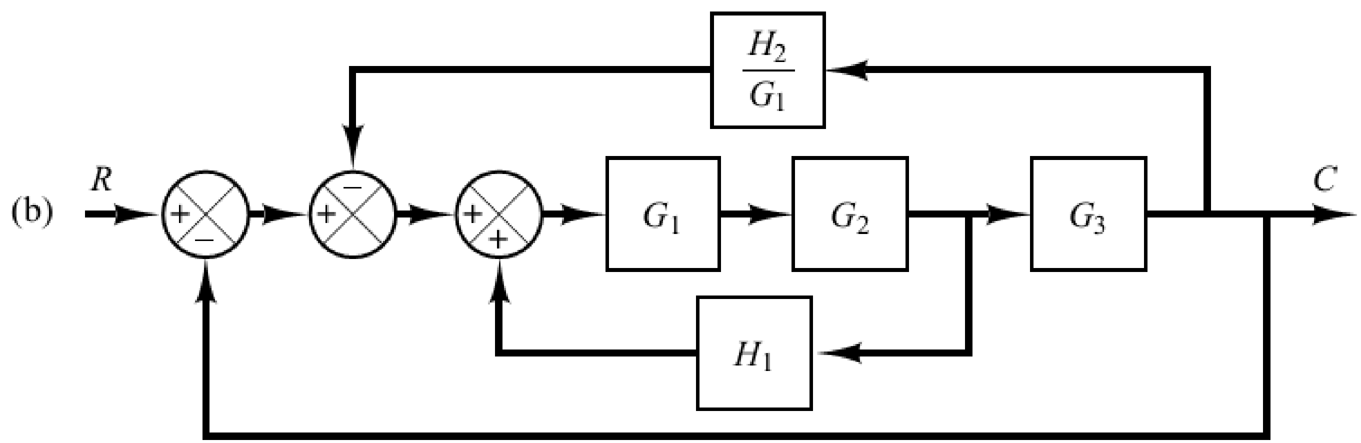
Example

- Simplify this diagram



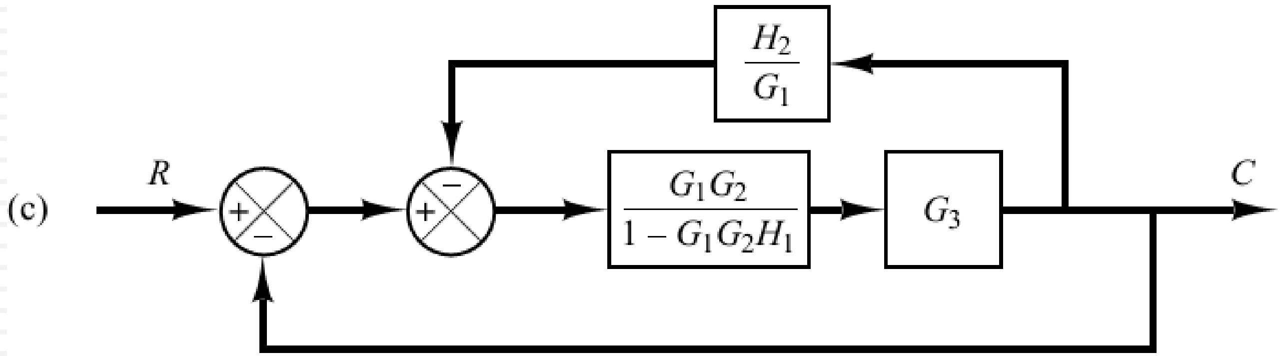
Solution:

- By moving the summing point of the negative feedback loop containing H_2 outside the positive feedback loop containing H_1, we obtain

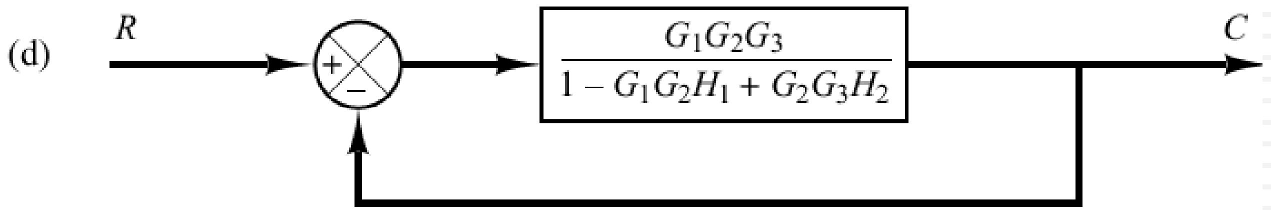


Example (2)

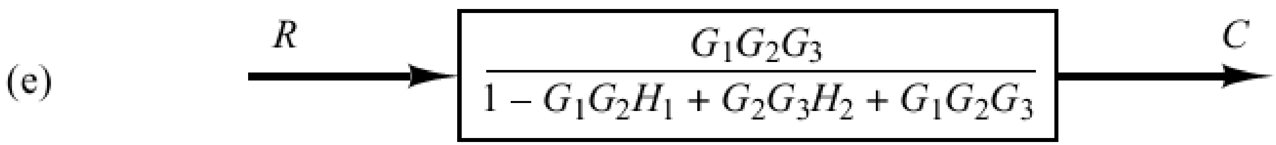
- Eliminating the positive feedback loop



- The elimination of the loop containing H_2/G_1 gives



- Finally, eliminating the feedback loop results in



Example (3)

- Notice that the numerator of the closed-loop transfer function *C(s)/R(s)* is the product of the transfer functions of the feed-forward path.
- The denominator of *C(s)/R(s)* is equal to

$$1 + \sum (\text{product of the transfer functions around each loop})$$

$$= 1 + (-G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3)$$

$$= 1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3$$

The positive feedback loop yields a negative term in the denominator

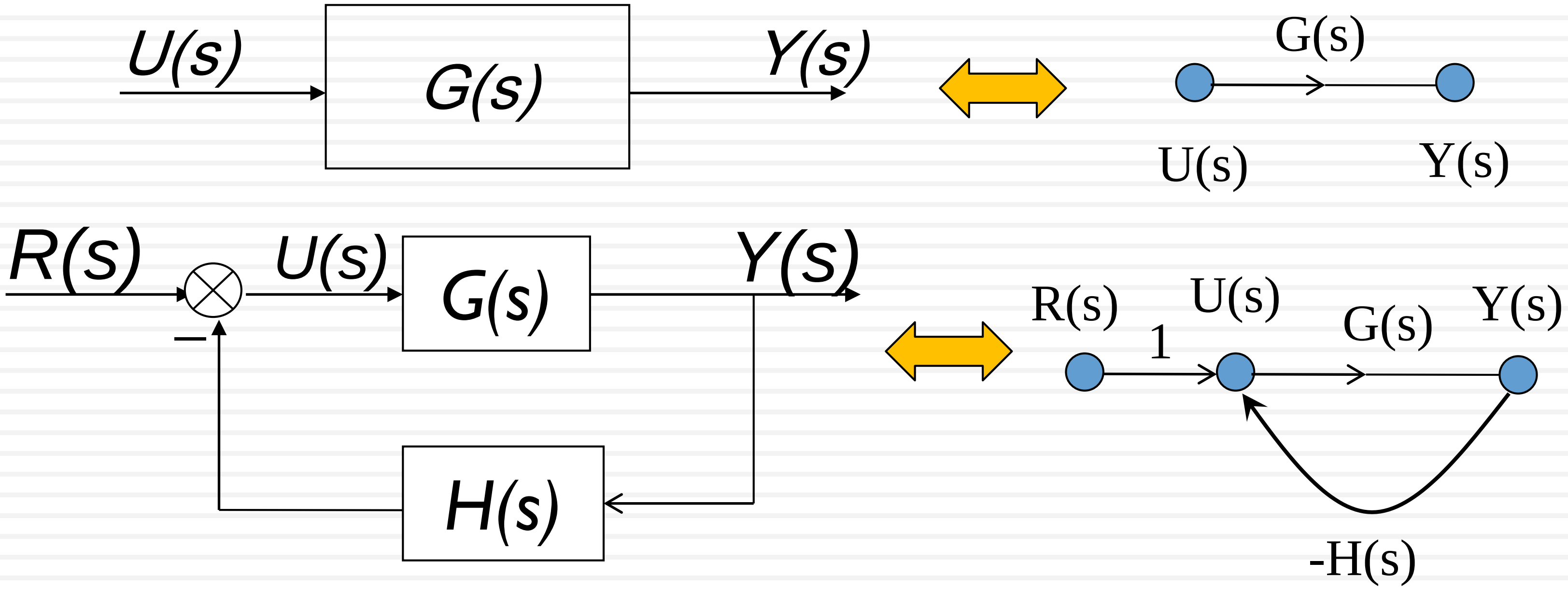
Signal Flow Graph

Note: This part is not included in the textbook (Slides 69 - 76)

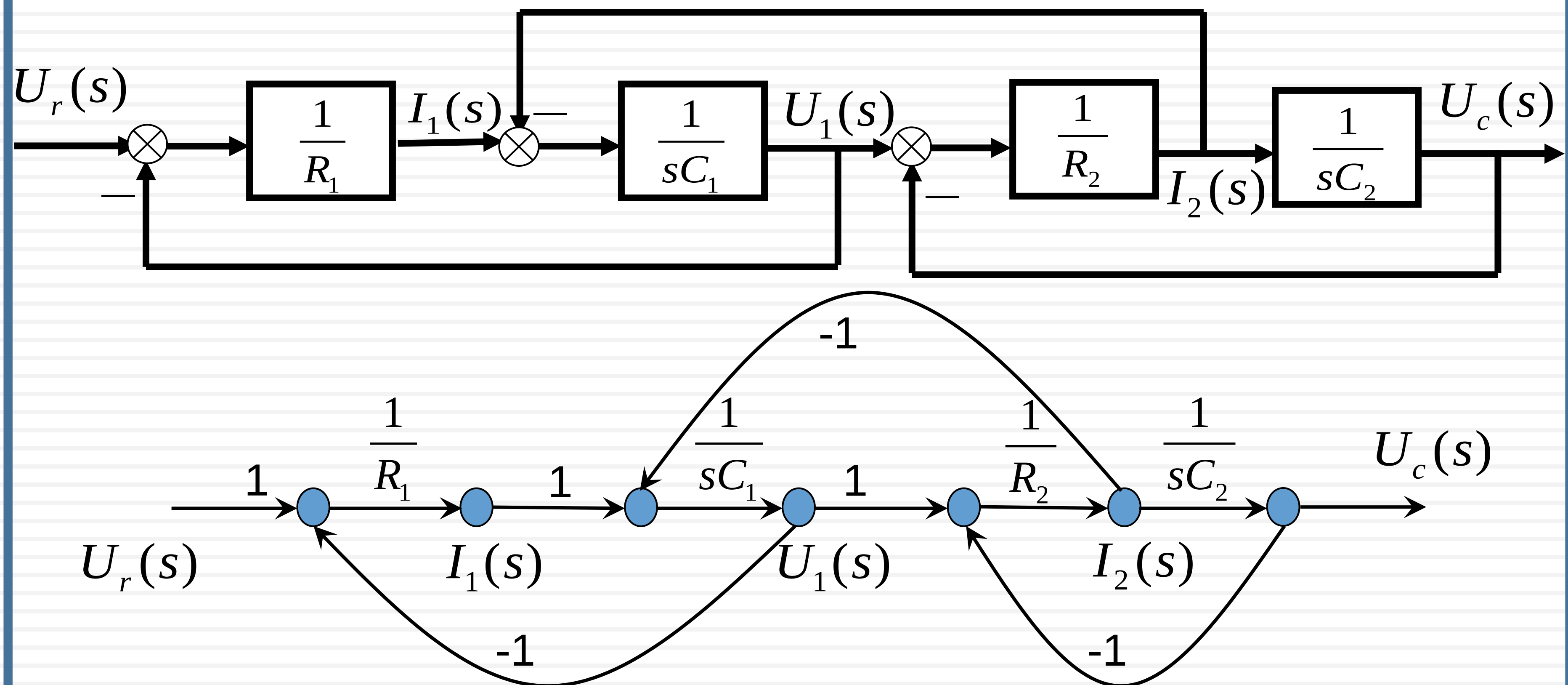
Signal Flow Graph (SFG)

SFG was introduced by S.J. Mason for the cause-and-effect representation of linear systems.

1. Each signal is represented by a node.
2. Each transfer function is represented by a branch.



Block Diagram and Signal-Flow Graph



- Note: A signal flow graph and a block diagram contain exactly the same information (there is no advantage to one over the other; there is only personal preference)

Mason's Rule

$$M(s) = \frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_{k=1}^{N} M_k \Delta_k$$

N = Total number of forward paths between output $Y(s)$ and input $U(s)$

M_k = Path gain of the k^{th} forward path

$\Delta = 1 - \sum($ all individual loop gains)

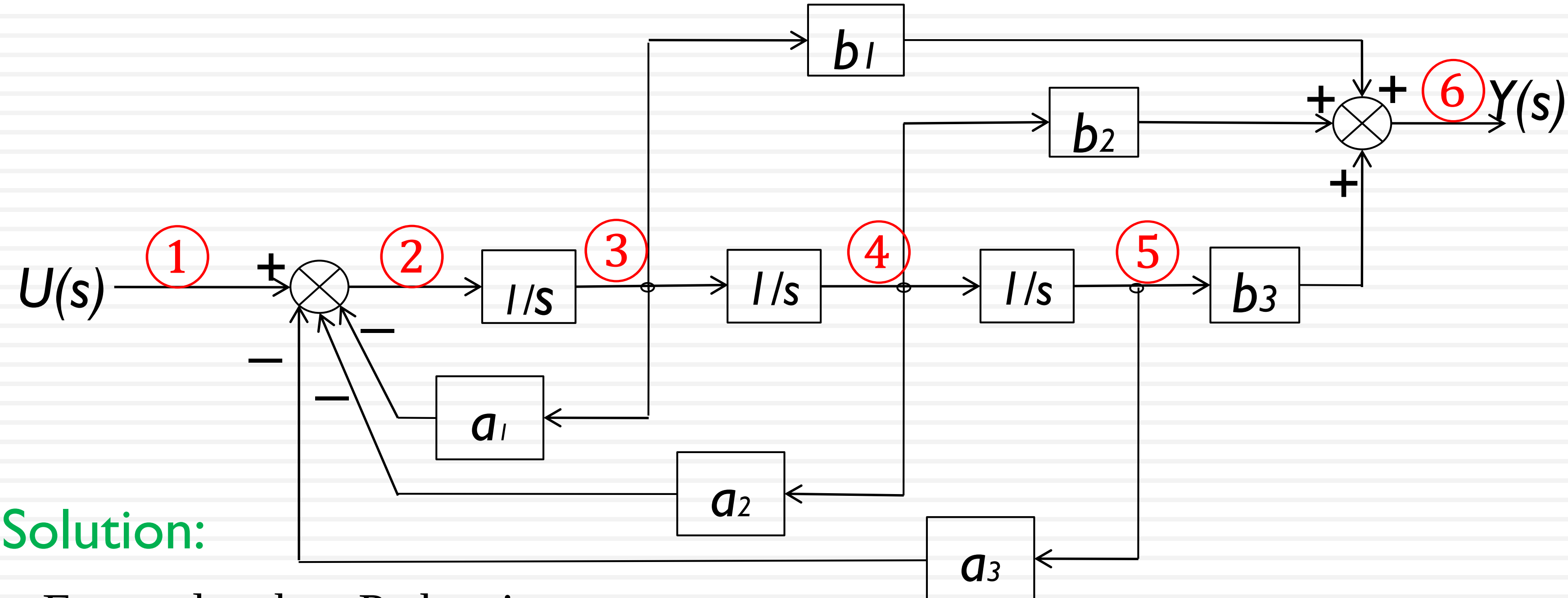
$+\sum($ gain products of all possible two loops that do not touch)

$-\sum($ gain products of all possible three loops that do not touch)

$+\cdots$

Δ_k = Value of Δ for that part of the block diagram that does not touch the k^{th} forward path

Example 1 Find the transfer function for the following block diagram



Solution:

Forward path	Path gain
1236	$M_1 = 1\left(\frac{1}{s}\right)(b_1)(1)$
12346	$M_2 = 1\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)(b_2)(1)$
123456	$M_3 = 1\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)(b_3)(1)$

and the determinates are

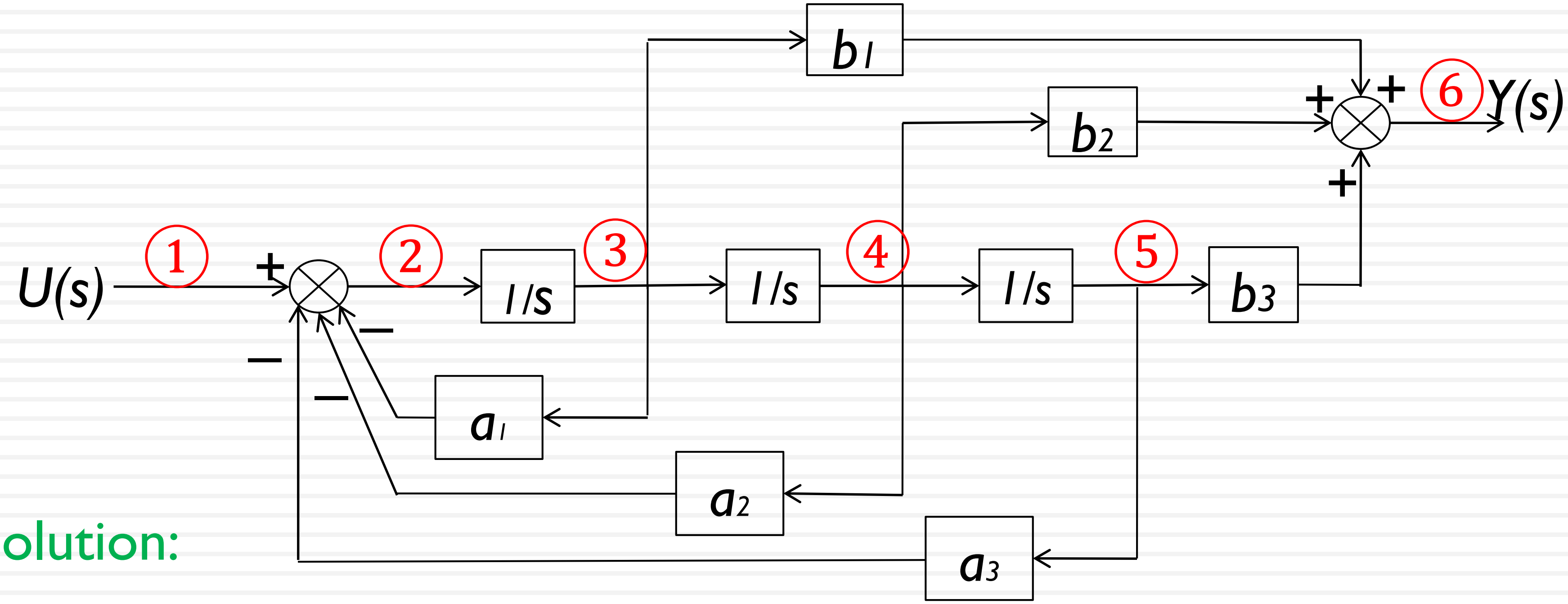
$$\Delta = 1 - \left(-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3} \right) + 0$$

$$\Delta_1 = 1 - 0$$

$$\Delta_2 = 1 - 0$$

$$\Delta_3 = 1 - 0$$

Example 1 **Find the transfer function for the following block diagram**



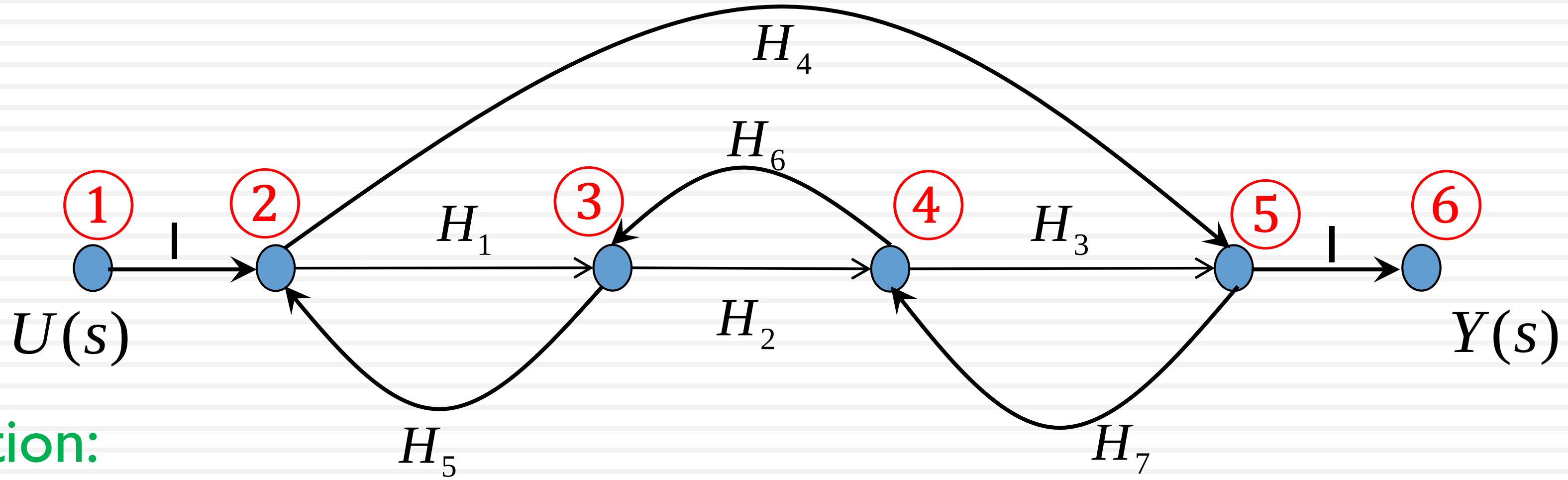
Solution:

Applying Mason's rule, we find the transfer function to be

$$M(s)=\frac{Y(s)}{U(s)}=\sum_{k=1}^{N}\frac{M_k\Delta_k}{\Delta}$$

$$=\frac{b_1 s^2+b_2 s+b_3}{s^3+a_1 s^2+a_2 s+a_3}$$

Example 2 **Find the transfer function for the following SFG**



Solution:

Forward path	Path gain
123456	$M_1 = H_1 H_2 H_3$
1256	$M_2 = H_4$

Loop path	Path gain
232	$l_1 = H_1 H_5$
343	$l_2 = H_2 H_6$
454	$l_3 = H_3 H_7$
25432	$l_4 = H_4 H_7 H_6 H_5$

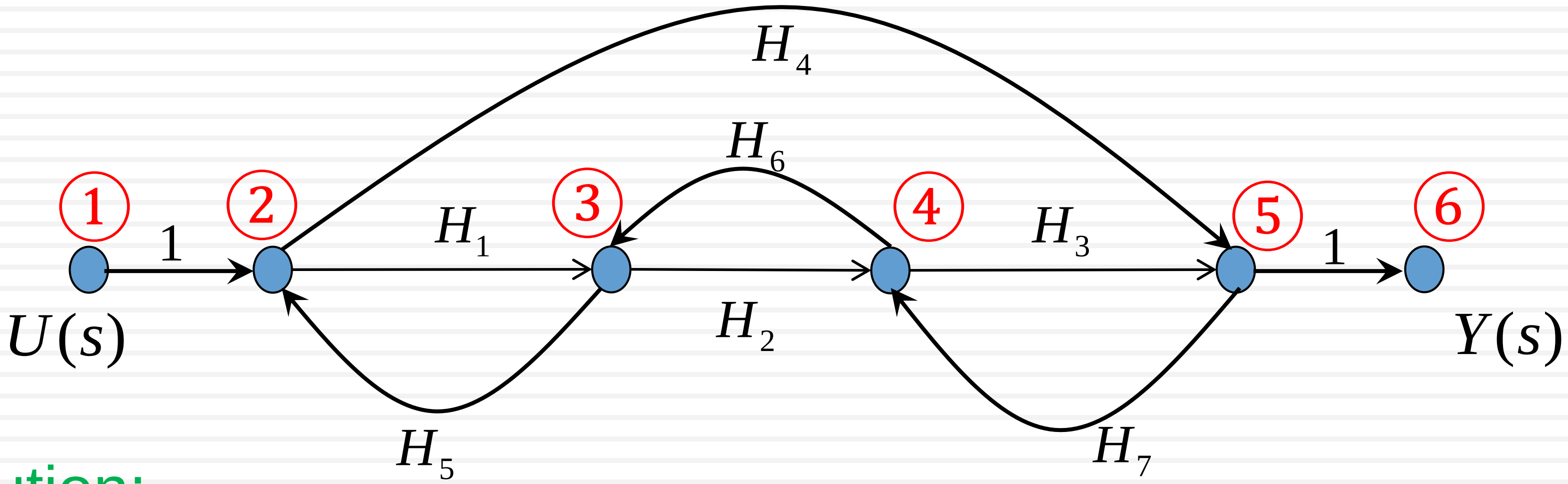
and the determinates are

$$\Delta = 1 - \left(l_1 + l_2 + l_3 + l_4\right) + (l_1 l_3)$$

$$\Delta_1 = 1 - 0$$

$$\Delta_2 = 1 - H_2 H_6$$

Example 2 Find the transfer function for the following SFG



Solution:

Applying Mason's rule, we find the transfer function to be

$$M(s) = \frac{Y(s)}{U(s)} = \sum_{k=1}^{N} \frac{M_k \Delta_k}{\Delta}$$

$$= \frac{H_1H_2H_3 + H_4 - H_4H_2H_6}{1 - H_1H_5 - H_2H_6 - H_3H_7 - H_4H_7H_6H_5 + H_1H_5H_3H}$$

MODELING IN STATE SPACE

- The modern trend in engineering systems is toward greater complexity
- Complex systems may have multiple inputs and multiple outputs (MIMO) and may be nonlinear and/or time varying
- Because of the increase in system complexity, and easy access to large scale computers, modern control theory has been developed since around 1960
- This new approach is based on the concept of state
- Modern control theory is essentially time-domain approach
- Modern control theory addresses MIMO, nonlinear, time varying control problems with non-zero initial conditions in addition to classical problems

State-Space Terminology

- A **linear combination** of n variables, x_i, for $i=1$ to n, is given by the following sum, S:

$$S = K_n x_n + K_{n-1} x_{n-1} + \cdots + K_1 x_1$$

- A set of variables is said to be **linearly independent** if none of the variables can be written as a linear combination of the others
- Formally, variables x_i, for $i=1$ to n, are said to be **linearly independent** if their linear combination, S, equals zero only if every $K_i=0$ and no $x_i=0$ for all $t \geq 0$
- A system variable is any variable that responds to an input or initial conditions in a system

State-Space Terminology (2)

- State variables are the smallest set of linearly independent system variables such that the values of the members of the set at time t_0 along with known forcing functions completely determine the value of all system variables for all $t \geq t_0$
- State vector is the vector whose elements are the state variables
- State-space is the *n*-dimensional space whose axes are the state variables

State-Space Terminology (3)

- State equations are a set of n simultaneous, first-order differential equations with n variables, where the n variables to be solved are the state variables
- Output equation is the algebraic equation that expresses the output variables of a system as linear combinations of the state variables and the inputs.

State-Space Representation

- An LTI system is represented in state-space by the following equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\dot{\mathbf{y}}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$x(t)$ = state vector

$\dot{x}(t)$ = derivative of the state vector with respect to time

$y(t)$ = output vector

$u(t)$ = input or control vector

A = system matrix

B = input matrix

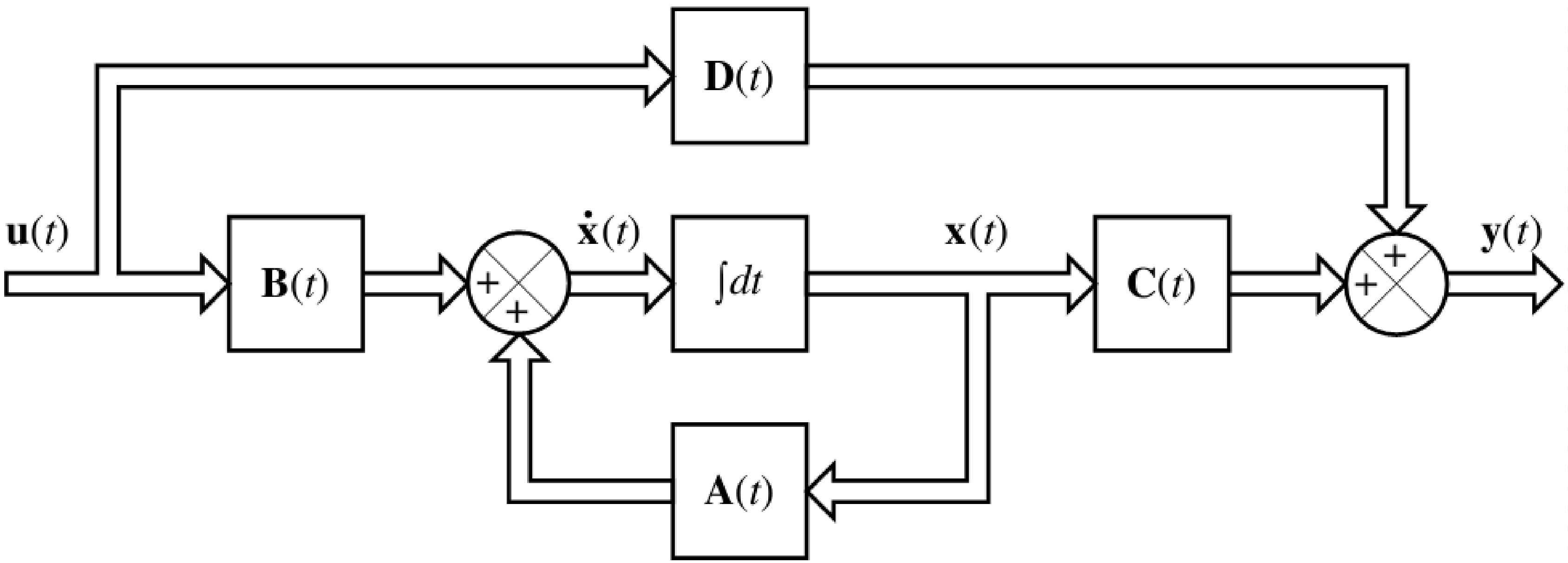
C = output matrix

D = feedforward or direct transmission matrix

Block diagram of the system in state-space.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\dot{\mathbf{y}}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$



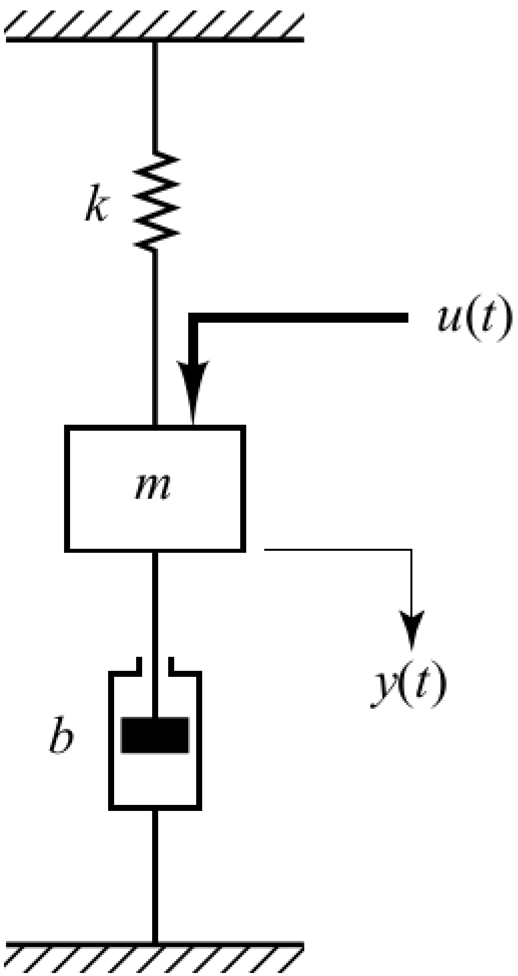
Example

- The external force u(t)is the input to the system, and the displacement y(t) of the mass is the output
- The system equation is

$$m\ddot{y} + b\dot{y} + ky = u$$

- This system is of second order. This means that the system involves two integrators.
- Let us define state variables $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$



Example (2)

- Then we obtain

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

$$\Longrightarrow \quad \dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

- The output equation is

$$y = x_1$$

Example (3)

- In a vector-matrix form, the state and output equations can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- They are in the standard form

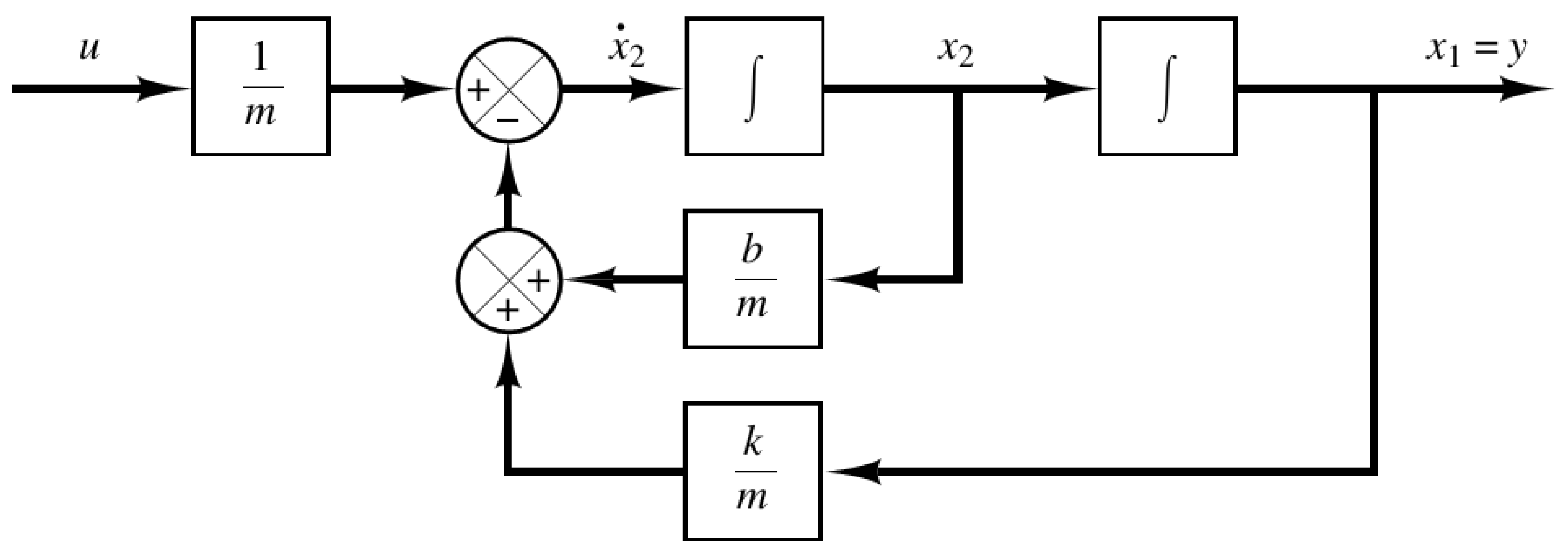
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Example (4)

- We can draw the block diagram of the mechanical system as shown in Figure



Transfer Functions and State-Space

- Let us consider the system whose transfer function is given by

$$\frac{Y(s)}{U(s)} = G(s)$$

- This system may be represented in state space by the following equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

- The Laplace transforms of the state-space equations is

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$

$$Y(s) = \mathbf{C}\mathbf{X}(s) + DU(s)$$

Transfer Functions and State-Space (2)

- To get the transfer function from the state-space equations, set initial conditions to zero (*x(0)=0*)

$$s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) = \mathbf{B}U(s)$$

or

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$

$$\longrightarrow \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$

Substitute in the output equation

$$Y(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\right]U(s)$$

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

The eigenvalues of A are identical to the poles of G(s)

Example

- Obtain the transfer function for the mechanical system in the previous example from the state-space equations.

Solution:

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \right\}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

Example (2)

Note:

$$\begin{bmatrix} s & -1 \\ \frac{k}{m} & s+\frac{b}{m} \end{bmatrix}^{-1} = \frac{1}{s^2+\frac{b}{m}s+\frac{k}{m}} \begin{bmatrix} s+\frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$$

Thus

$$G(s) = [1 \quad 0] \frac{1}{s^2+\frac{b}{m}s+\frac{k}{m}} \begin{bmatrix} s+\frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$= \frac{1}{ms^2 + bs + k}$$

STATE-SPACE REPRESENTATION

- An n^{th}-order differential equation may be expressed by a first-order vector-matrix differential equation
- If n elements of the vector are a set of state variables, then the vector-matrix differential equation is a state equation
- In this section we will present methods for obtaining state-space representations of continuous-time systems

State-Space Representation of n^th-Order Systems

- Linear Differential Equations in which the Forcing Function Does Not Involve Derivative Terms:

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u$$

Let us define

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ &\cdot \\ &\cdot \\ &\cdot \\ x_n &= \overset{(n-1)}{y} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\cdot \\ &\cdot \\ &\cdot \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_n x_1 - \cdots - a_1 x_n + u \end{aligned}$$

State-Space Representation of n^th-Order Systems (2)

This equation is on the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

State-Space Representation of n^th-Order Systems (3)

The output can be given by

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

Or $y = \mathbf{C}\mathbf{x}$ where $\mathbf{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$

Note that the transfer function of the system is

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

State-Space Representation of n^th^-Order Systems (4)

- Linear Differential Equations in which the Forcing Function Involves Derivative Terms:

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

- The main problem in defining the state variables for this case lies in the derivative terms of the input u

- The state variables must be expressed such that they will eliminate the derivatives of *u* in the state equation

State-Space Representation of n[th]-Order Systems (5)

- Let us define the following *n* variables as a set of *n* state variables:

$$
\begin{aligned}
x_1 &= y - \beta_0 u \\
x_2 &= \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \\
x_3 &= \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u \\
&\cdot \\
&\cdot \\
&\cdot \\
x_n &= \overset{(n-1)}{y} - \beta_0 \overset{(n-1)}{u} - \beta_1 \overset{(n-2)}{u} - \cdots - \beta_{n-2} \dot{u} - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u
\end{aligned}
$$

State-Space Representation of n^th-Order Systems (6)

where $\beta_0, \beta_1, \beta_2, \ldots, \beta_{n-1}$ are determined from

$$
\begin{aligned}
\beta_0 &= b_0 \\
\beta_1 &= b_1 - a_1\beta_0 \\
\beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\
\beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 \\
&\;\cdot \\
&\;\cdot \\
&\;\cdot \\
\beta_{n-1} &= b_{n-1} - a_1\beta_{n-2} - \cdots - a_{n-2}\beta_1 - a_{n-1}\beta_0
\end{aligned}
$$

State-Space Representation of n^th-Order Systems (7)

- With this choice of state variables the existence and uniqueness of the solution of the state equation is guaranteed
- With the present choice of state variables, we obtain

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \beta_1 u \\
\dot{x}_2 &= x_3 + \beta_2 u \\
&\cdot \\
&\cdot \\
&\cdot \\
\dot{x}_{n-1} &= x_n + \beta_{n-1} u \\
\dot{x}_n &= -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + \beta_n u
\end{aligned}
$$

where β_n is given by

$$\beta_n = b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_{n-1}\beta_0$$

State-Space Representation of n^th-Order Systems (8)

- The state and output equations can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \beta_0 u$$

State-Space Representation of n^th-Order Systems (9)

Or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

Where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0 \quad \cdots \quad 0], \qquad D = \beta_0 = b_0$$

Note that the transfer function of the system is

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

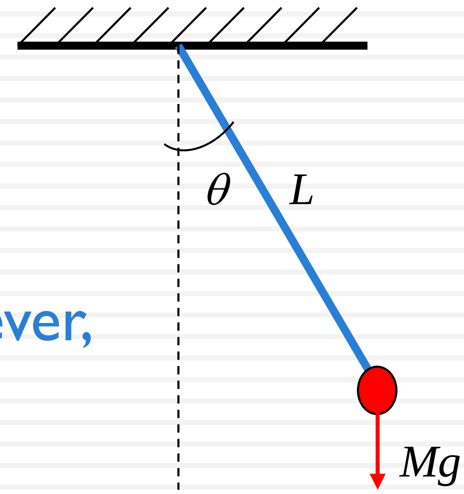
LINEARIZATION OF NONLINEAR SYSTEMS

- A system is nonlinear if the principle of superposition does not apply
- Most linear systems are really linear only in limited operating ranges
- In practice, many systems involve nonlinear relationships among the variables
- For example, the output of a component may saturate for large input signals
- There may be a dead space that affects small signals
- Square-law nonlinearity may occur in some components

Nonlinear Systems

- In reality, most systems are indeed nonlinear, e.g. the pendulum system, which is described by nonlinear differential equations.

- Example



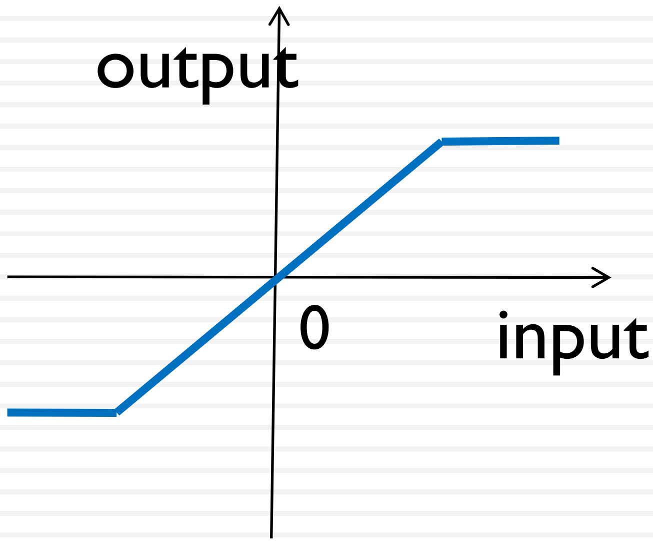
$$ML\frac{d^2\theta}{dt^2} + Mg\sin\theta(t) = 0$$

- It is difficult to analyze nonlinear systems, however, we can linearize the nonlinear system near its equilibrium point under certain conditions.

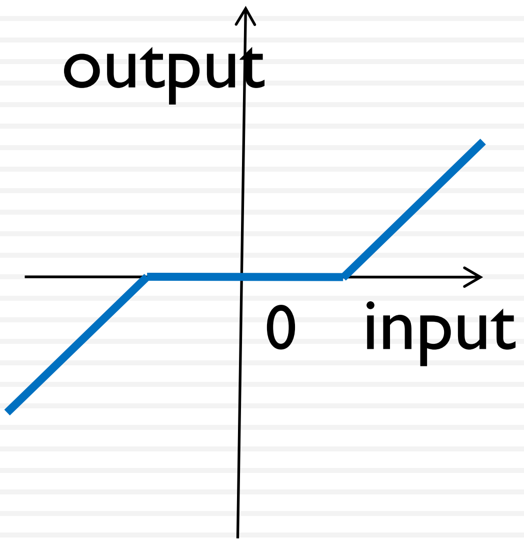
$$ML\frac{d^2\theta}{dt^2} + Mg\theta(t) = 0 \quad (\text{when } \theta \text{ is small})$$

Linearization of Nonlinear Systems

Several typical nonlinear characteristics in control system.



Saturation (Amplifier)



Dead-zone (Motor)

Linearization Methods

（1） Weak nonlinearity neglected

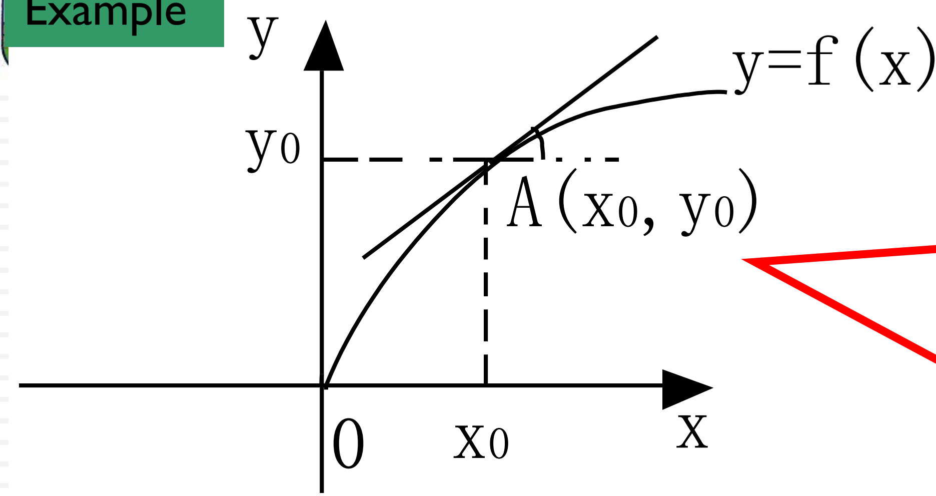
If the nonlinearity of the component is not within its linear working region, its effect on the system is weak and can be neglected.

（2） Small perturbation/error method

Assumption: In the system control process, there are just small changes around the equilibrium point in the input and output of each component.

This assumption is reasonable in many practical control system: in closed-loop control system, once the deviation occurs, the control mechanism will reduce or eliminate it. Consequently, all the components can work around the equilibrium point.

Example



A(x0,y0) is equilibrium point. Expanding the nonlinear function y=f(x) into a Taylor series about A(x0,y0) yields

$$y = f(x) = y_0 + \left.\frac{dy}{dx}\right|_{x_0}(x - x_0) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_0}(x - x_0)^2 + \cdots\cdots$$

The input and output only have small variance around the equilibrium point.

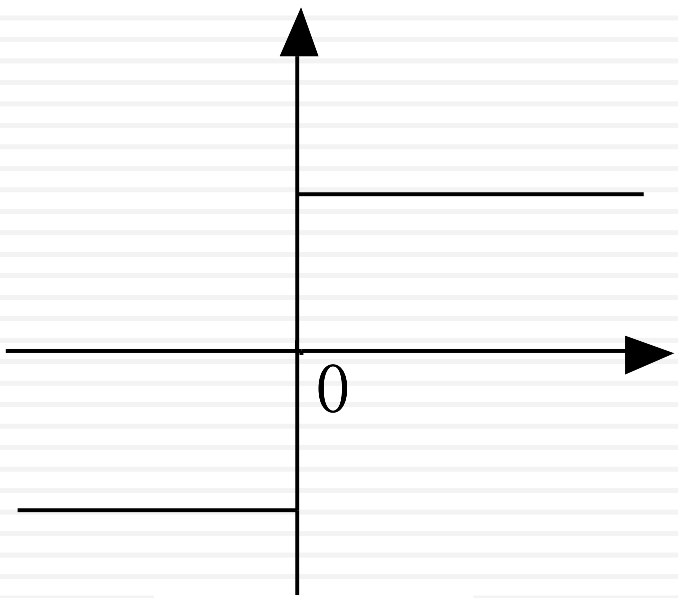
$$\Delta x = (x - x_0), (\Delta x)^n \to 0$$

$$y = y_0 + \left.\frac{dy}{dx}\right|_{x_0}(x - x_0)$$

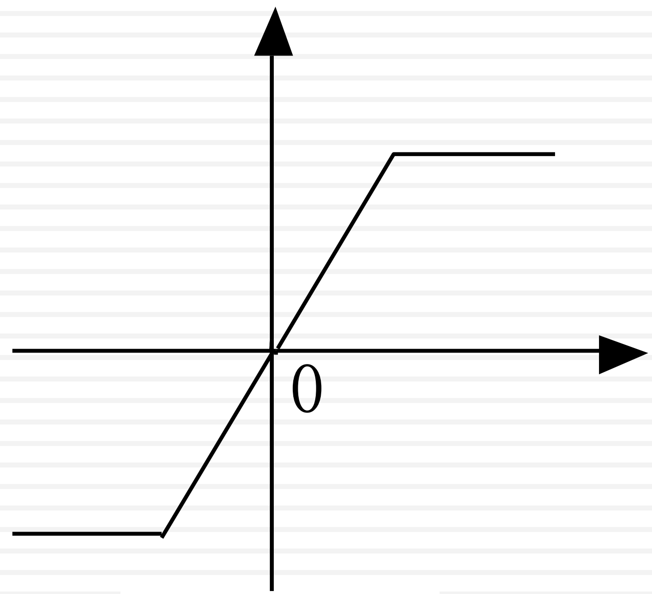
$$\Delta y = k\Delta x$$

This is linearized model of the nonlinear component.

Note: this method is only suitable for systems with weak nonlinearity.



Relay



Saturation

For systems with strong nonlinearity, we cannot use such linearization method.

Linear Approximation of Nonlinear Mathematical Models

- To obtain a linear mathematical model for a nonlinear system, we assume that the variables deviate only slightly from some operating condition
- Consider a system whose input is *x(t)* and output is *y(t)*. The relationship between *y(t)* and *x(t)* is given by

$$y = f(x)$$

The output equation may be expanded into a Taylor series about this equilibrium point as follows

$$y = f(\bar{x}) + \frac{df}{dx}(x - \bar{x}) + \frac{1}{2!}\frac{d^2 f}{dx^2}(x - \bar{x})^2 + \cdots$$

where the derivatives $df/dx, d^2f/dx^2, \ldots$ are evaluated at $x = \bar{x}$.

Linear Approximation of Nonlinear Mathematical Models (2)

If the variation $x - \bar{x}$ is small, we may neglect the higher-order terms in $x - \bar{x}$

The output equation can be rewritten as

$$y = \bar{y} + K(x - \bar{x})$$

Or

$$y - \bar{y} = K(x - \bar{x})$$

where $\bar{y} = f(\bar{x})$, $K = \left.\frac{df}{dx}\right|_{x=\bar{x}}$

This indicates that $y - \bar{y}$ is proportional to $x - \bar{x}$ which gives a linear mathematical model for the nonlinear system

Linear Approximation of Nonlinear Mathematical Models (3)

- The linearization technique presented here is valid in the vicinity of the operating condition
- If the operating conditions vary widely, however, such linearized equations are not adequate, and nonlinear equations must be dealt with

Example

- Linearize the nonlinear equation: $z = xy$

in the region $5 \le x \le 7$, $10 \le y \le 12$. Find the error if the linearized equation is used to calculate the value of z when $x=5$, $y=10$

<u>Solution:</u>

Choose $\bar{x}$, $\bar{y}$ as the average values of the given ranges

$$\bar{x} = 6, \bar{y} = 11$$

Then

$$\bar{z} = \bar{x}\bar{y} = 66.$$

Example (2)

Expanding the nonlinear equation into a Taylor series about points $x = \bar{x}$, $y = \bar{y}$ and neglecting the higher-order terms

$$z - \bar{z} = a(x - \bar{x}) + b(y - \bar{y})$$

Where

$$a = \left.\frac{\partial(xy)}{\partial x}\right|_{x=\bar{x},\, y=\bar{y}} = \bar{y} = 11$$

$$b = \left.\frac{\partial(xy)}{\partial y}\right|_{x=\bar{x},\, y=\bar{y}} = \bar{x} = 6$$

Example (3)

Hence the linearized equation is

$$z - 66 = 11(x - 6) + 6(y - 11)$$

or

$$z = 11x + 6y - 66$$

When $x=5$, $y=10$,the value of z given by the linearized equation is

$$z = 11x + 6y - 66 = 55 + 60 - 66 = 49$$

The exact value of z is $z = xy = 50$

The error is thus $50-49=1$ or 2%

Example Problems and Solutions

- Check the textbook examples and solutions (Pages 46-60)
- Sheet #1 includes Chapter 2 problems.