Alexandria University
Faculty of Engineering
Division of Communications & Electronics

EE 392: Control Systems & Their Components

Lab 4: Aircraft Pitch: State-Space Methods for Controller Design

System Modeling

Physical setup and system equations

The equations governing the motion of an aircraft are a very complicated set of six nonlinear coupled differential equations. However, under certain assumptions, they can be decoupled and linearized into longitudinal and lateral equations. Aircraft pitch is governed by the longitudinal dynamics. In this example we will design an autopilot that controls the pitch of an aircraft. The basic coordinate axes and forces acting on an aircraft are shown in Figure 1 given below.

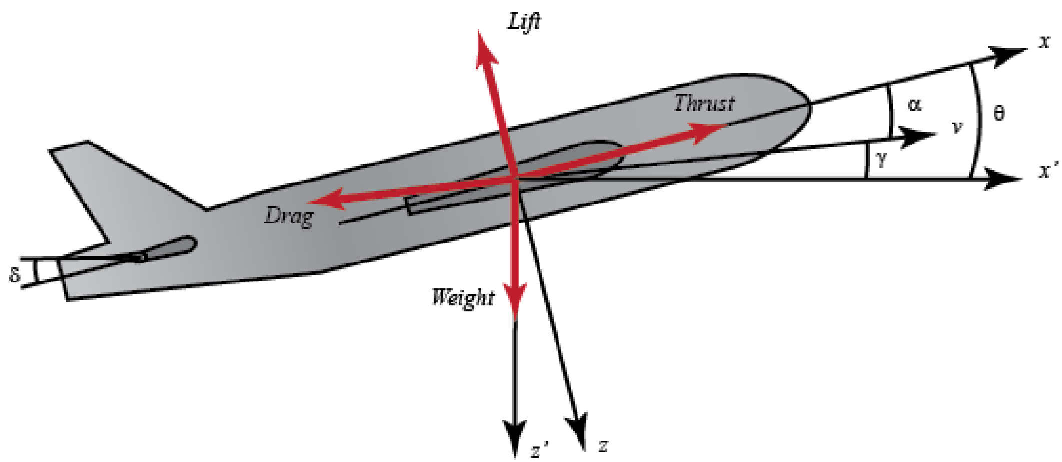

Figure 1: Aircraft Pitch System

We will assume that the aircraft is in steady-cruise at constant altitude and velocity; thus, the thrust, drag, weight and lift forces balance each other in the *x*- and *y*-directions. We will also assume that a change in pitch angle will not change the speed of the aircraft under any circumstance (unrealistic but simplifies the problem a bit). Under these assumptions, the longitudinal equations of motion for the aircraft can be written as follows:

$$\dot{\alpha} = \mu\Omega\sigma[-(C_L + C_D)\alpha + \frac{1}{(\mu - C_L)}q - (C_W \sin\gamma)\theta + C_L] \quad (1)$$

$$\dot{q} = \frac{\mu\Omega}{2i_{yy}}[[C_M - \eta(C_L + C_D)]\alpha + [C_M + \sigma C_M(1 - \mu C_L)]q + (\eta C_W \sin\gamma)\delta]$$

$$\dot{\theta} = \Omega q \quad (2)$$

For this system, the input will be the elevator deflection angle δ and the output will be the pitch angle θ of the aircraft.

Transfer Function and State-Space Model

Before finding the transfer function and state-space models, let's plug in some numerical values to simplify the modeling equations shown above:

$$\dot{\alpha} = -0.313\alpha + 56.7q + 0.232\delta \quad (3)$$

$$\dot{q} = -0.0139\alpha - 0.426q + 0.0203\delta \quad (4)$$

$$\dot{\theta} = 56.7q \quad (5)$$

These values are taken from the data from one of Boeing's commercial aircraft.

Transfer Function

To find the transfer function of the above system, we need to take the Laplace transform of the above modeling equations. Recall that when finding a transfer function, zero initial conditions must be assumed. The Laplace transform of the above equations are shown below.

$$sA(s) = -0.313A(s) + 56.7Q(s) + 0.232\Delta(s) \quad (6)$$

$$sQ(s) = -0.0139A(s) - 0.426Q(s) + 0.0203\Delta(s) \quad (7)$$

$$s\Theta(s) = 56.7Q(s) \quad (8)$$

After few steps of algebra, you should obtain the following transfer function.

$$P(s) = \frac{\Theta(s)}{\Delta(s)} = \frac{1.151s + 0.1774}{s^3 + 0.739s^2 + 0.921s} \quad (9)$$

State-Space Model

Recognizing the fact that the modeling equations above are already in the state-variable form, we can rewrite them as matrices as shown below.

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} [\delta] \quad (10)$$

Since our output is pitch angle, the output equation is the following.

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + [0][\delta] \qquad (11)$$

where the input is elevator deflection angle δ and the output is the aircraft pitch angle θ. The above equations match the general, linear state-space form.

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + Bu \qquad (12)$$

$$y = C\mathbf{x} + Du \qquad (13)$$

In this part we will apply a state-space controller design technique. In particular, we will attempt to place the closed-loop poles of the system by designing a controller that calculates its control based on the state of the system.

Controllability and Observability

A system is controllable if there exists a control input, u(t), that transfers any state of the system to zero in finite time. It can be shown that an LTI system is controllable if and only if its controllabilty matrix, CO, has full rank (i.e. if rank(CO) = n where n is the number of states).

$$CO = [B|AB|A^2B|...|A^{n-1}B]; \qquad (14)$$

All the state variables of a system may not be directly measurable, for instance if the component is in an inaccessible location. In these cases it is necessary to estimate the values of the unknown internal state variables using only the available system outputs. A system is observable if the initial state, $x(t_0)$, can be determined from the system output, $y(t)$, over some finite time $t_0<t<t_f$. For LTI systems, the system is observable if and only if the observability matrix, OB, has full rank (i.e. if rank(OB) = n where n is the number of states).

$$OB = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad (14)$$

State Feedback Control Design

The schematic of a full-state feedback control system is shown in Figure 2 (with $D = 0$).

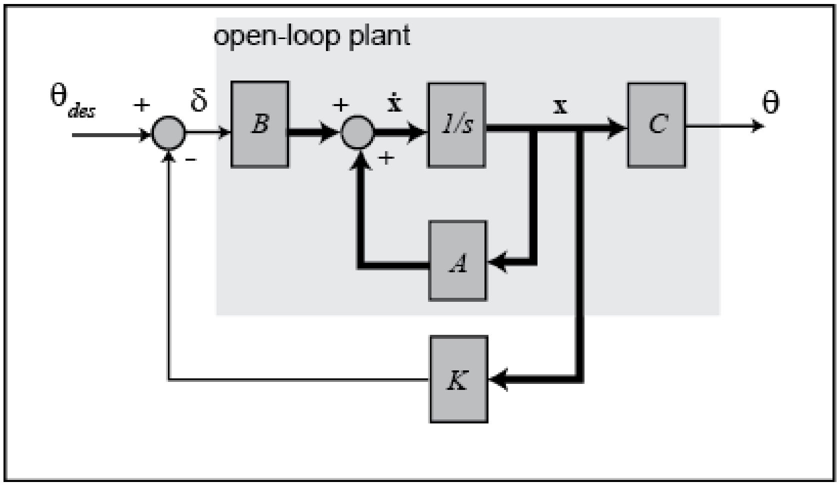

Figure 2: State feedback control system

where

K = control gain matrix

$\mathbf{x} = [\ \alpha, q, \theta\ ]'$ = state vector

θ_{des} = reference (r)

$\delta = \theta_{des} - K\ \mathbf{x}$ = control input (u)

θ = output (y)

Referring back to the state-space equations at the top of the page, we see that substituting the state-feedback law $\delta = \theta_{des} - K\ \mathbf{x}$ for δ leads to the following.

$$\dot{\mathbf{x}} = (A - BK)\mathbf{x} + B\theta_{des} \qquad (16)$$

$$\theta = C\mathbf{x} \qquad (17)$$

Based on the above, matrix $A - BK$ determines the closed-loop dynamics of our system. Specifically, the roots of the determinant of the matrix $[\ sI - (\ A - BK\ )\ ]$ are the closed-loop poles of the system. Since the determinant of $[\ sI - (\ A - BK\ )\ ]$ is a third-order polynomial, there are three poles we can place and since our system is completely state controllable, we can place the poles anywhere we like. Recall that a "pole-placement" technique can be used to find the control gain matrix K to place the closed-loop poles in the desired locations. Note that this feedback law presumes that all of the state variables in the vector **x** are measured, even though θ is our only output. If this is not the case, then an observer needs to be designed to estimate the other state variables.

We know from the above that we can place the closed-loop poles of the system anywhere we would like. The question then that is left is, where should we place them? If we have a standard first- or second-order system, we then have relationships that directly relate pole locations to characteristics of the step response and can use these relations to place the poles in order to meet our given requirements. This process becomes more difficult if we have a higher-order system or zeros. With a higher-order system, one approach is to place the higher-order poles 5-10 times farther to the left in the complex plane than the dominant poles, thereby leading them to have negligible contribution to the transient response. The effect of zeros is more difficult to address using a pole-placement approach to control. Another limitation of this pole-placement approach is that it doesn't explicitly take into account other factors like the amount of required control effort.

Linear Quadratic Regulation

We will use a technique called the **Linear Quadratic Regulator (LQR)** method to generate the "best" gain matrix K, without explicitly choosing to place the closed-loop poles in particular locations. This type of control technique optimally balances the system error and the control effort based on a cost that the designer specifies that defines the relative importance of minimizing errors and minimizing control effort. In the case of the regulator problem, it is assumed that the reference is zero. Therefore, in this case the magnitude of the error is equal to the magnitude of the state. To use this LQR method, we need to define two parameters: the state-cost weighted matrix (Q) and the control weighted matrix (R). For simplicity, we will choose the control weighted matrix equal to 1 (R=1), and the state-cost matrix (Q) equal to $pC'C$. Employing the vector C from the output equation means that we will only consider those states in the output in defining our cost. In this case, θ is the only state variable in the output. The weighting factor (p) will be varied in order to modify the step response. In this case, R is a scalar since we have a single input system.

Referring to the closed-loop state equations given above assuming a control law with non-zero reference, $\delta = \theta_{des} - K\,\mathbf{x}$, we can then generate the closed-loop step response. Note that the response is scaled to model the fact that the pitch angle reference is a 0.2 radian (11 degree) step.

Examination of the above demonstrates that the response is too slow. We can tune the performance of our system to be faster by weighting the importance of the error more heavily than the importance of the control effort. More specifically, this can be done by increasing the weighting factor *p*. After some trial and error, we settle on a value of *p* = 50.

Adding Precompensation

Unlike other design methods, the full-state feedback system does not compare the output to the reference; instead, it compares all states multiplied by the control matrix (*K* **x**) to the reference (see the schematic shown above). Thus, we should not expect the output to equal the commanded reference. To obtain the desired output, we can scale the reference input so that the output does equal the reference in steady state. This can be done by introducing a precompensator scaling factor called *N_bar*. The basic schematic of our state-feedback system with scaling factor (*N_bar*) is shown below.

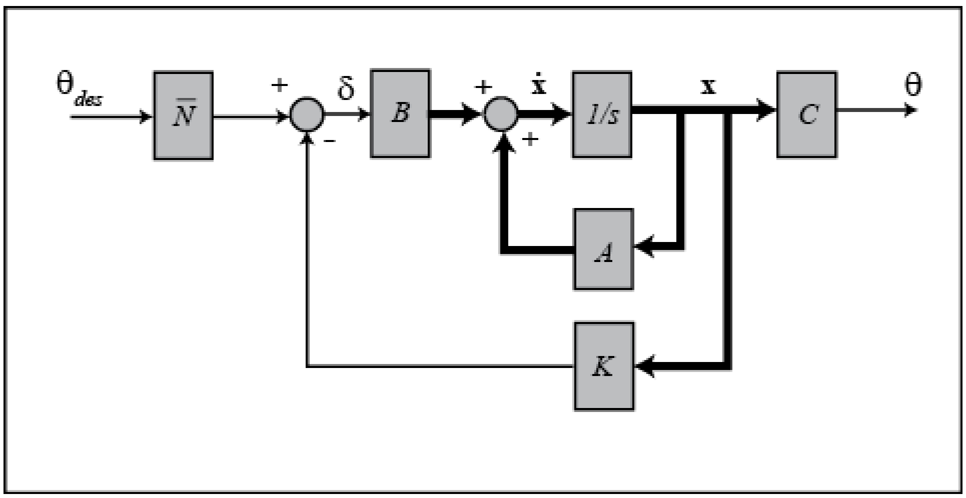

Figure 3: State feedback with input precompensation

Observer Design

When we can't measure all the states x (often the case in practice), we can build an **observer** to estimate them, while measuring only the output *y* = *C x*. For the magnetic ball example, we will add three new, estimated states to the system. The schematic is as follows:

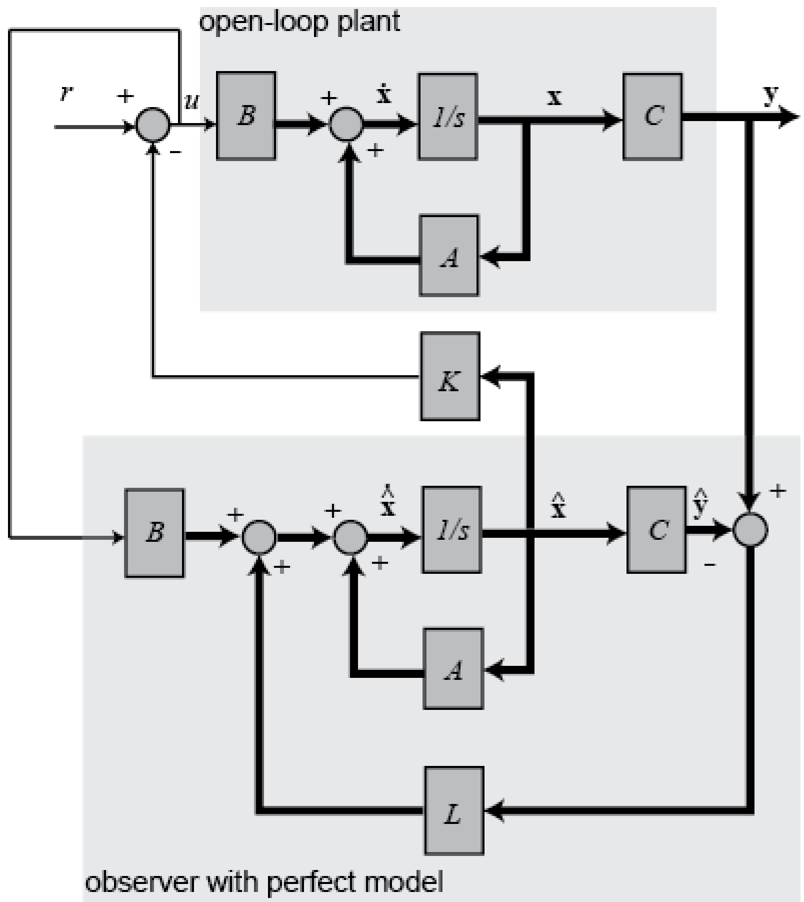

Figure 4: State estimator (observer)

The observer is basically a copy of the plant; it has the same input and almost the same differential equation. An extra term compares the actual measured output y to the estimated output $\hat{y}$; this will cause the estimated states $\hat{x}$ to approach the values of the actual states x. The error dynamics of the observer are given by the poles of (A-LC).

First, we need to choose the observer gain L. Since we want the dynamics of the observer to be much faster than the system itself, we need to place the poles at least five times farther to the left than the dominant poles of the system. If we want to use place, we need to put the three observer poles at different locations. Because of the duality between controllability and observability, we can use the same technique used to find the control matrix, but replacing the matrix B by the matrix C and taking the transposes of each matrix.

Lab Requirements

During the Lab period, you are required to:

1. Model the aircraft pitch system shown in Figure 1 and Equation (10) in state space using MATLAB & SimuLink and draw the open loop step response. The parameters for the system are given in equation (10).
2. Check the system stability, controllability, and observability using the appropriate Matlab commands.
3. Find the value of the state feedback gain K using the LQR method and draw the system closed loop response. For a step reference of 0.2 radians, the design criteria are the following.
   - Overshoot less than 10%
   - Rise time less than 2 seconds
   - Settling time less than 10 seconds
   - Steady-state error less than 2%
4. Find the value of the precompensator gain and plot the closed loop step response.
5. Find the gain L of the state estimator and remodel the whole system and plot its unit step response for the reference input given in step 1.

Useful MATLAB Commands

`ss , ctrb , rank , lqr , step`