Lecture 9

9.1 Wave Aspects of Material Particles:

De Broglie Hypthesis:

He assumed that the dual aspects are cot confined to EM waves only, but apply as well to material particles.

De Broglie Wave

If a particle ha an energy $E = mc^2$ and a momentum $p = mv$, then one can assume a wave moving in place of the particle and carrying the same amount of energy mc^2.

$$h\upsilon = mc^2$$

$$\therefore \upsilon_D = \frac{mc^2}{h}$$

The above frequency is known as De Broglie frequency υ_D. The same thing can be said about the momentum.

$$P_D = p_{particle}$$

$$\therefore \frac{h}{\lambda} = mv$$

$$hence \quad \lambda_D = \frac{h}{mv}$$

The velocity of propagation of De Broglie wave can be simply obtained as:

$$v_D = \lambda_D \upsilon_D$$

$$v_D = \frac{h}{mv}\frac{mc^2}{h} = \frac{c^2}{v}$$

Two main issues can be pinpointed from this results:

1. $v_D > c$, but no energy or signal can be transmitted by this velocity. Energy is being transmitted by the particle's velocity.
2. $v_D > v$, which means that the De Broglie wave gets away from the particle. This is not true, the particle is effectively inside a wave packet or a group of waves whose velocity is equal to the particle's velocity.

So, we conclude that the De Broglie velocity v_D does not represent a velocity with which a matter can be transferred, and from now on will be called the phase velocity v_{ph}. One main unique characteristic of the phase velocity of the waves associated with the particle is that id depends on the wavelength.

Proof:

$$E^2 = p^2c^2 + m_o^2c^4 \qquad (1)$$

According to De Broglie $v_{ph} = \frac{c^2}{v} = \frac{mc^2}{mv} = \frac{E}{p}$

$\therefore E = v_{ph} p$

into equation (1)

$v_{ph}^2 p^2 = p^2 c^2 + m_o^2 c^4$

$\therefore v_{ph}^2 = c^2 + c^2 \frac{m_o^2 c^2}{p^2}$

$v_{ph} = c\sqrt{1 + \frac{m_o^2 c^2}{p^2}}$

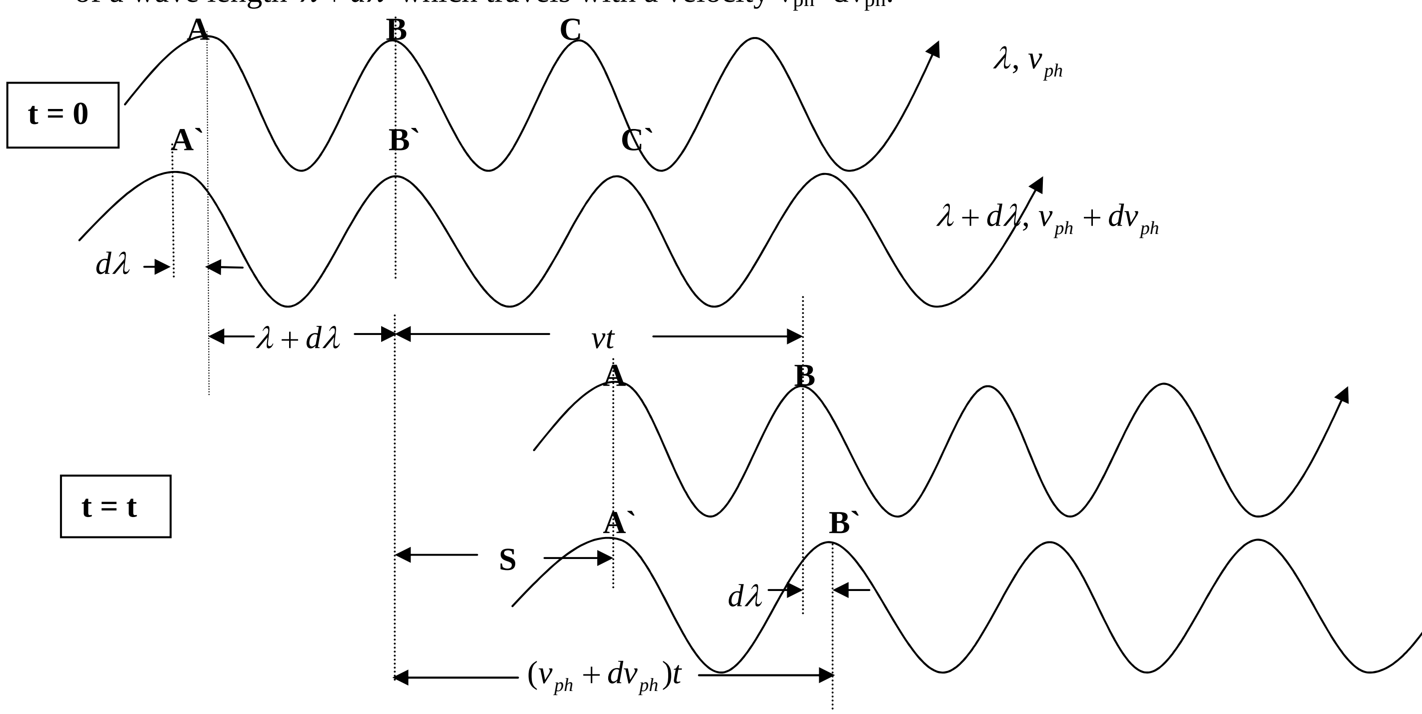
but $p = \frac{h}{\lambda}$

$\therefore v_{ph} = c\sqrt{1 + \frac{m_o^2 c^2}{h^2}\lambda^2}$ This is known as a dispersion relation.

It is clear from this relation that the phase velocity is a function of the wavelength , for a photon m_o=0, which gives v_{ph} = c.

9.2 Phase and Group Velocities:

Now, we have known that the particle is actually inside a wave packet composed of many wavelets each of which travels with a phase velocity v_{ph}, but the whole wave packet travels with a velocity equal to that of the particle. Consider now a dispersive medium where the phase velocity depends on the wavelength. Consider two wavelets, one of a wave length λ and travels with a velocity v_{ph}, and the other one of a wave length $\lambda + d\lambda$ which travels with a velocity v_{ph}+dv_{ph}.



$v_g = \frac{S}{t}$

From the figure, it is clear that:

$$S = v_{ph}t - \lambda$$

$$\therefore v_g = \frac{S}{t} = v_{ph} - \frac{\lambda}{t}$$

It is also clear from the above figure that:

$$d\lambda = (v_{ph} + dv_{ph})t - v_{ph}t$$

$$d\lambda = t\,dv_{ph}$$

$$t = \frac{d\lambda}{dv_{ph}}$$

Giving $v_g = v_{ph} - \lambda \frac{dv_{ph}}{d\lambda}$

Which is the relation between the group velocity and the phase velocity. For nondipersive medium, vacuum for example, the phase velocity does not depend on the wavelength λ.

$$\frac{dv_{ph}}{d\lambda} = 0$$

$$\therefore v_g = v_{ph}$$

We aim now at proving that the group velocity is equal to the particle velocity. We previously prove that:

$$v_{ph} = \sqrt{1 + \frac{m_o^2 c^2}{h^2}\lambda^2}$$

From which the group velocity can be obtained. First we differentiate v_{ph} with respect to λ, then we substitute in the above relation.

$$\frac{dv_{ph}}{d\lambda} = \frac{c\dfrac{m_o^2 c^2}{h^2}\lambda}{\sqrt{1 + \dfrac{m_o^2 c^2}{h^2}\lambda^2}}$$

$$v_g = v_{ph} - \lambda\frac{dv_{ph}}{d\lambda} = c\sqrt{1 + \frac{m_o^2 c^2}{h^2}\lambda^2} - \lambda\,\frac{c\dfrac{m_o^2 c^2}{h^2}\lambda}{\sqrt{1 + \dfrac{m_o^2 c^2}{h^2}\lambda^2}}$$

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$$\therefore v_g = \frac{c^2}{v_{ph}} = \frac{c^2}{c^2/v} = v$$