Performance of Uncoded Overlapping PPM Under Communication Constraints

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Abstract—In a direct-detection optical channel, overlapping pulse position modulation (OPPM) offers significant improvement in throughput over ordinary pulse position modulation (PPM) under bandwidth constraint. This improvement is acquired, however, at the expense of performance degradation. In this paper it is shown that under bandwidth and throughput constraints OPPM, with two pulse positions per pulsewidth, outperforms PPM when the overlap is chosen properly and the throughput is greater than 0.2 nats/slot. This enables us to increase the throughput and/or decrease the energy of the OPPM system, while maintaining the same performance as PPM. The energy saving when using OPPM instead of PPM to transmit reliably a given amount of information is determined for different values of overlapping indices. The maximum throughput that can be achieved under average power and bandwidth constraints (with error probability not exceeding 10^{-5}) is also determined.

I. Introduction

Recently interest has been given to overlapping pulse position modulation (OPPM) as an alternative signaling format for the conventional (disjoint or orthogonal) pulse position modulation (PPM) in direct-detection optical channels [1–6]. This type of signaling can be considered as a generalization to PPM, where overlap is allowed between pulse positions. The reason to prefer OPPM over PPM is that the throughput (nats/s) can be improved without increasing the bandwidth [1,2]. Moreover, OPPM retains the advantages of PPM in terms of implementation simplicity. Indeed the transmitter involves only time delaying of the optical pulse, and the receiver does not require knowledge of the signal or noise power. Error probabilities of uncoded OPPM, where the overlap is at least half the pulsewidth, has been derived in [2]. It has been noted that the gain in throughput is accompanied by a serious degradation in error-probability performance. This results from increasing the number of signals in the signal space without increasing its dimension. Georghiades [3], however, showed that by using trellis coded modulation (TCM) one can improve the performance of uncoded OPPM at the expense of losing some of its acquired throughput; he was able to double the throughput and avoid performance loss with the aid of rate 2/3 trellis code and allowing seven positions per pulsewidth.

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Recently we have suggested to restrict the overlap to only two pulse positions per pulsewidth and allow this overlap to take any value between 0 and half the pulsewidth [6]. This restriction decreases the complexity of the system dictated by the requirement to use complex error correcting codes and a highly refined timing, while preserving some of the advantages of OPPM. Lower and upper bounds on the channel capacity of this signal format, with overlapping index (ratio between the overlap and the pulsewidth) less than one half, have been derived in [6]. It has been shown that under fixed pulsewidth, peak power, and throughput capacity, there are values of the overlapping index (r) that allow an increase of more than 100% in OPPM efficiency capacity (in nats/photon) over PPM. Furthermore, under fixed pulsewidth, peak power, and efficiency capacity, one can get an improvement of about 51% in throughput capacity over PPM. The results in [6], however, are expressed in terms of the capacities of the system; which can be reached only with impractical coding. This motivates us to study what one can gain from OPPM if coding is not employed.

In this paper we investigate the performance of uncoded OPPM with different values of overlapping index r in the range $[0, 0.5]$ under communication constraints. Next, we determine the minimum energy required to transmit reliably a fixed amount of information rate, and evaluate the corresponding overlapping index. Finally, we determine the maximum throughput that can be achieved under average power and pulsewidth constraints such that the probability of error be less than 10^{-5}. We are able to show that, for fixed pulsewidth, uncoded OPPM (with suitable $r \in [0, 0.5]$) outperforms PPM under throughput and average power constraints. This enables us to increase the throughput and/or decrease the energy of the OPPM system while maintaining the same performance as PPM.

The paper is organized as follows: Section II is devoted for the system description and the derivation of bit error probabilities for both quantum-limited and background noise cases. Numerical results are given in Section III. In part A of this section we compare between the performance of OPPM system with different values of r under communication constraints. In part B we investigate the variation of OPPM energy/nat with r under throughput, pulsewidth, and bit error rate constraints. In the last part of this section we study the dual problem of the previous part, that is the behavior of OPPM throughput with different values of r under average power, pulsewidth, and bit error rate

constraints. Finally the conclusion is given in Section IV.

II. System Description and Theoretical Analysis

A. OPPM Channel Model

The system model for M-ary OPPM with overlapping index $r \in [0, 0.5]$ is as follows. The data symbol is represented by the position of a laser pulse of width τ within time frame of duration T. A pulse is said to be in position $j \in \{1, \ldots, M\}$ if it extends over the subinterval

$$I_j \stackrel{\text{def}}{=} \Big(\big(j-1-(j-1)r\big)\tau,\ \big(j-(j-1)r\big)\tau\Big) .$$

The time frame T is related to r, M, and τ by

$$T = \big(M-(M-1)r\big)\tau .$$

We denote by $N(I)$ the photon count observed in the subinterval I. Each of these counts is a Poisson random variable with mean $\bar{N}(I)$ which equals the average photon count in the interval I. Let K_s and K_b denote the average number of received photons per pulse due to signal and background noise, respectively. It is easy to check that if a laser pulse is transmitted in position k, then $\bar{N}(I_j)$, $j \in \{1, \ldots, M\}$, is given by

$$\bar{N}(I_j) = \begin{cases} K_s + K_b\ ; & \text{if } j = k, \\ K_s r + K_b\ ; & \text{if } j \in \{k-1, k+1\} \cap \{1, \ldots, M\}, \\ K_b\ ; & \text{if } j \in \{1, \ldots, k-2, k+2, \ldots, M\}. \end{cases}$$

B. Bit Error Probability for Quantum-Limited Channel

In the self-noise- (quantum-) limited channel the background noise is negligible ($K_b = 0$) and the channel can be modeled as an ambiguity and erasure one [1,6]. The average bit error probability for equally likely messages is given by

$$P_b = \frac{1}{M}\sum_{j=1}^{M} P_b(j) ,$$

where $P_b(j)$ denotes the bit error probability given that the jth symbol was transmitted. Noticing that $P_b(M) = P_b(1)$ and $P_b(j) = P_b(2)$ for $j \in \{3, \ldots M-1\}$, we can write

$$P_b = \frac{1}{M}\big[2P_b(1) + (M-2)P_b(2)\big] . \qquad (1)$$

It is easy to see that the probabilities of symbol erasure and symbol ambiguity given j was sent are independent of j. Denoting these probabilities by P_e and P_a, respectively, we have

$$P_e = \Pr\{N(I_j) = 0 | j\} = \exp[-K_s]$$

and

$$\begin{aligned} P_a &= \Pr\Big\{N(I_j - I_{j+1}) = 0,\ N(I_j) \neq 0 | j\Big\} \\ &= \Pr\Big\{N(I_j - I_{j+1}) = 0,\ N(I_j \cap I_{j+1}) \neq 0 | j\Big\} \\ &= \Pr\Big\{N(I_j - I_{j+1}) = 0 | j\Big\} \Pr\Big\{N(I_j \cap I_{j+1}) \neq 0 | j\Big\} , \end{aligned}$$

where $N(I_j - I_{j+1})$ is the photon count in the interval I_j and not in I_{j+1}. Since $\bar{N}(I_j - I_{j+1}) = K_s(1-r)$ and $\bar{N}(I_j \cap I_{j+1}) = K_s r$, we obtain

$$P_a = \exp[-K_s(1-r)]\big(1 - \exp[-K_s r]\big) .$$

To determine $P_b(1)$, assume that symbol 1 was sent. If the pulse was erased, then a random choice of symbols produces the wrong one with probability $\frac{M-1}{M}$. It is well known that there is $\frac{M/2}{M-1}$ probability of deciding the wrong bit given symbol error. Thus the bit error probability given an erasure has occurred is equal to 1/2. On the other hand, if an ambiguity has occurred, then there is a chance of 1/2 to choose the wrong symbol between the two candidates 1 and 2. When Grey coding is used, adjacent symbols always differ by only one bit. Thus a bit error occurs with probability $\frac{1}{\log_2 M}$ given symbol error. From this discussion we obtain

$$P_b(1) = \frac{1}{2}P_e + \frac{1}{2\log_2 M}P_a .$$

Similar argument leads to

$$P_b(2) = \frac{1}{2}P_e + \frac{1}{\log_2 M}P_a .$$

Substituting in (1), yields

$$\begin{aligned} P_b &= \frac{1}{2}P_e + \frac{M-1}{M\log_2 M}P_a \\ &= \frac{1}{2}\exp[-K_s] + \\ &\quad + \frac{M-1}{M\log_2 M}\exp[-K_s(1-r)]\big(1-\exp[-K_s r]\big) . \qquad (2) \end{aligned}$$

C. Bit Error Probability with Background Noise

When background noise is significant the ambiguity and erasure channel model no longer holds. We derive here an upper bound on the bit error rate. The symbol error probability is given by

$$P_S = \frac{1}{M}\sum_{j=1}^{M} P_S(j) = \frac{1}{M}\big[2P_S(1) + (M-2)P_S(2)\big] ,$$

where $P_S(j)$ is the symbol error probability given j was sent. This can be bounded as

$$\begin{aligned} P_S(j) &= \Pr\Big\{\bigcup_{i \neq j}[N(I_j) \leq N(I_i) | j]\Big\} \\ &\leq \sum_{i \neq j} \Pr\{N(I_j) \leq N(I_i) | j\} . \end{aligned}$$

When $j = 1$ we have

$$\begin{aligned} P_S(1) &\leq \Pr\Big\{N(I_1) \leq N(I_2) | 1\Big\} + \sum_{i=3}^{M} \Pr\Big\{N(I_1) \leq N(I_i) | 1\Big\} \\ &= \Pr\Big\{N(I_1 - I_2) \leq N(I_2 - I_1) | 1\Big\} + \\ &\quad + (M-2)\Pr\Big\{N(I_1) \leq N(I_3) | 1\Big\} . \end{aligned}$$

Define the following two quantities:

$$p \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \exp[-K_b(1-r)] \frac{(K_b(1-r))^k}{k!} \times \times \sum_{i=0}^{k} \exp[-(K_s+K_b)(1-r)] \frac{((K_s+K_b)(1-r))^i}{i!}$$

and

$$q \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \exp[-K_b] \frac{K_b^k}{k!} \sum_{i=0}^{k} \exp[-(K_s+K_b)] \frac{(K_s+K_b)^i}{i!} .$$

Hence the bound on $P_S(1)$ can be rewritten as

$$P_S(1) \leq p + (M-2)q .$$

In a similar way we can show that $P_S(2) \leq 2p + (M-3)q$ which leads to the upper bound on the symbol error probability $P_S \leq \frac{M-1}{M}[2p + (M-2)q]$. The bit error rate is thus

$$P_b \leq p + \frac{M-2}{2} q . \tag{3}$$

III. Numerical Results

A. Bit Error Rate

In order to compare between the performance of PPM and OPPM for different values of r, eqs. (2) and (3) were evaluated numerically as functions of the average signal photons per nat (μ) and the overlapping index (r) under pulsewidth (τ) and throughput (R_0 nats/slot) constraints. The alphabet size M is chosen so as to give the best performance satisfying the throughput constraint, that is we choose M that solves the following optimization problem:

$$P_b^*(r) = \min_{\substack{M: \\ R(r) \geq R_0}} P_b(r) , \tag{4}$$

where $P_b(r)$ is given by (2) or (3) with $K_s = \mu \log M$; and

$$R(r) \stackrel{\text{def}}{=} \frac{\log M}{M - (M-1)r} \quad \text{nats/slot}$$

is the throughput of OPPM with overlapping index r. Figs. 1 and 2 show the quantum-limited bit error probability, $P_b^*(r)$, with $R_0 =$ 0.3 and 0.35 nats/slot, respectively. When $R_0 = 0.35$, the best performance is obtained at $r = 0.3$ whereas the worst occurs at $r = 0$ (PPM). When $R_0 = 0.25$ or 0.3 the best performance occurs at $r = 0.1$ and the worst at $r = 0.5$. The performance of PPM is close to the worst when $R_0 = 0.3$ and close to the best when $R_0 = 0.25$. On the other hand PPM starts to perform better than OPPM when R_0 decreases under 0.2; moreover, the performance gets worse by increasing r.

Our explanation of this behavior is as follows. For large values of throughput ($R_0 > 0.2$) the alphabet size M, satisfying the throughput constraint and achieving minimum error rate as described by (4), increases with r. This leads to a decrease in the bit error probability. As r increases above some value, however, the increase in M is insufficient to continue improving the error probability and $P_b^*(r)$ begins to increase with r. When r reaches 0.5, the increase in the error probability becomes maximum; at this point $P_b^*(0.5)$ might still be better than $P_b^*(0)$ (as in $R_0 = 0.35$) or be worse (as in $R_0 = 0.3$or 0.25). On the other hand if the throughput constraint is low ($\leq$ 0.2 nats/slot), M values achieving the best performance are large; and hence the increment in M with r becomes insensitive. This leads to insufficient improvement in the bit error rate and the performance gets worse as the overlap increases.

Under pulsewidth constraint it is not possible to increase PPM throughput above $\frac{\log 3}{3} = 0.3662$ nats/slot; hence OPPM with suitable r is a practical solution to increase the throughput. Bit error rates for OPPM with $R_0 =$ 0.4 nats/slot are evaluated for different values of μ and $r \in [0.1, 0.5]$. We have found that the differences between the curves are insignificant and the best performance is obtained for $r = 0.4$ or 0.5.

Similar results can be obtained when the background noise is significant. Upper bounds on $P_b^*(r)$ with one noise photon per slot ($K_b = 1$) are calculated for $R_0 = 0.35$; the best performance occurs at $r = 0.3$.

In order to figure out the implementation simplicity of the OPPM detection system compared to PPM, denote by $M_{opt}(r)$ the optimizing alphabet size corresponding to an overlapping index r. We found that if $R_0 = 0.35$, then $M_{opt}(0.3) = 8$ while $M_{opt}(0) = 3$. For $R_0 = 0.3$, $M_{opt}(0.1) =$ 7 while $M_{opt}(0) = 5$. For $R_0 = 0.25$, $M_{opt}(0.1) = 10$ while $M_{opt}(0) = 8$.

B. Minimum Energy per Information Nat

In this section we search for the minimum energy required to transmit, *reliably*, a given amount of information per second. Reliability is accounted for by requiring the final bit error rate to be less than 10^{-5}. We can formulate the problem as follows:

$$\mu^*(r) \stackrel{\text{def}}{=} \min_{\substack{M, K_s: \\ P_b(r) \leq 10^{-5},\ R(r) \geq R_0}} \frac{K_s}{\log M} .$$

Here $\mu^*(r)$ represents the optimum energy in photons/nat and $P_b(r)$ is given by eq. (2). This energy is plotted (Fig. 3) versus throughput constraint for different values of r. For throughput constraints less than 0.2 nats/s the gain in using OPPM is insignificant; but after this value OPPM starts to become superior to PPM. For example if $R_0 = 0.35$, 30% of energy per nat can be saved when using OPPM with $r = 0.3$ instead of PPM. In other words we can transmit about 40% more nats per photon when using OPPM with r=0.3 . This should be compared with our results in [6] where we have shown that over 100% in efficiency capacity (in nats/photon) can be gained when using OPPM with throughput capacity of about 0.35 . This much information (capacity) cannot be achieved, however, without involving complex methods of

encoding techniques but the 40% increase can be obtained with uncoded OPPM. Throughput more than 0.3662 are not shown in the figure because (as mentioned previously) it is not possible to use PPM with such values of throughput. The abrupt jumps in the curves correspond to the instants at which M should be decreased to satisfy the throughput constraint. At these instants the energy must be increased by a suitable amount to maintain the bit error rate below 10^{-5}. It is clear also from the figure that the minimum energy is always achieved for $r \leq 0.3$, which justifies our restriction of the overlap to only two positions per pulsewidth. When r is increased over 0.5, error correcting codes will be mandatory [2,3]. In Fig. 4 we plot $\mu^*(r)$ versus r for different values of throughput. This figure is useful in determining the values of r that will achieve the minimum energy required to transmit prescribed values of R_0. For example these optimum values are approximately 0.3, 0.155, and 0.09 when $R_0 = 0.35$, 0.3, and 0.25, respectively. The jumps in these curves identify the values of r where it is possible to increase M without disturbing the throughput constraint. At these values energy can be decreased and still have the error probability under the threshold. $\mu^*(r)$ begins to increase again next to each jump because M cannot be increased continuously with r but it should be held fixed for some interval immediately after a jump. In this interval energy must be increased to equalize the degradation in performance due to increasing the overlap.

C. The Maximum Throughput

The objective here is to obtain maximum throughput (in nats/slot), given average power, pulsewidth, and bit error rate constraints. In other words, we consider:

$$R^*(r) = \max_{\substack{M: \\ P_b(r) \leq 10^{-5}, \\ K_s \leq n_a(M-(M-1)r)}} \frac{\log M}{M-(M-1)r},$$

where $P_b(r)$ is given by (2) and n_a denotes the average power in photons/slot. This maximization was performed numerically for different values of n_a and r. The results are shown in Fig. 5. It is seen that for $n_a \leq 3$ the maximum throughput is almost independent of r. Above this value the the maximum achievable throughput increases with r. For example if $r = 0.5$ and $n_a \geq 8.5$, one can have about 51% more throughput than PPM. Furthermore energy can be decreased when increasing the throughput as can be seen from

$$\mu(r) = \frac{K_s}{\log M} = \frac{n_a}{R^*(r)}.$$

Thus increasing $R^*(r)$ for fixed n_a will cause μ to decrease (i.e., less photons per nat will be needed with OPPM).

IV. Conclusions

The performance of uncoded overlapping PPM with at most two pulse positions per pulsewidth has been investigated for an optical direct-detection channel under communication constraints. It was shown that under pulsewidth and throughput constraints, uncoded OPPM (with overlapping index less than 0.3) is superior to ordinary PPM in terms of bit error rate. This means that we can decrease the energy required to transmit a given amount of information without sacrificing its performance or throughput. The overlapping index r that offers the minimum energy varies significantly with the throughput constraint and should be identified from characteristic curves as those given in Fig. 4. OPPM loses its advantages when the throughput constraint decreases below some threshold ≈ 0.2, i.e., $r = 0$ is the optimum index in this case.

Under pulsewidth, average power, and bit error rate constraints, OPPM offers better throughput and efficiency than PPM. The optimum value of r in this case is close to 0.5 when the average power exceeds 3 photons/slot.

Our results in parts B and C of the previous section were obtained under negligible background noise (quantum-limited case). With the aid of eq. (3), similar results can be obtained when the background noise is significant.

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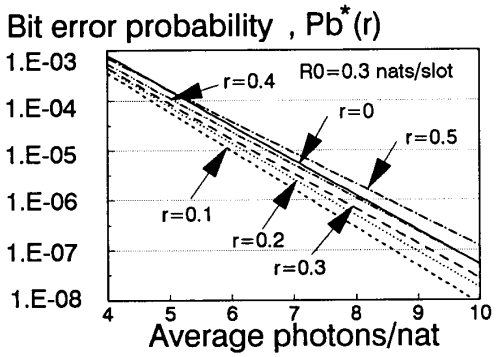

Fig. 1. Bit error prob. for OPPM with R0=0.3, Kb=0.

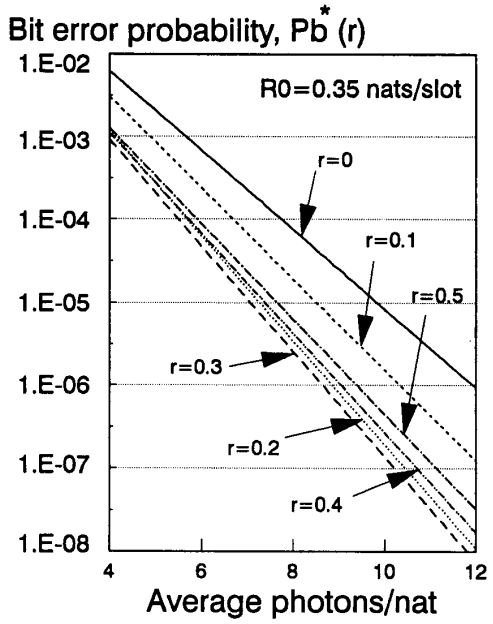

Fig. 2. Bit error probability for OPPM with R0 =0.35 and Kb = 0.

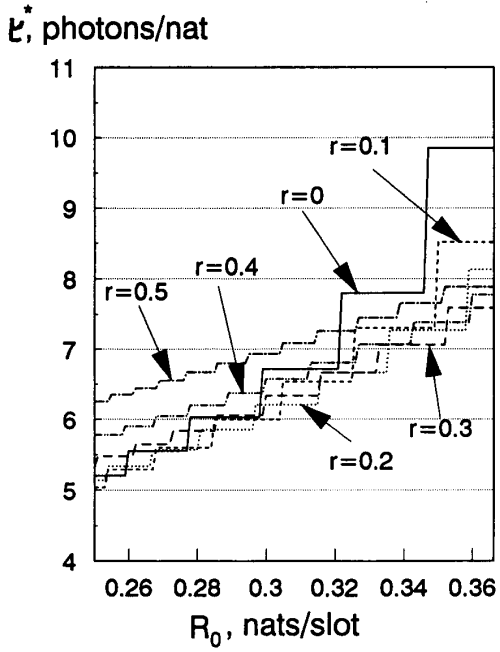

Fig. 3. Minimum energy vs. throughput.

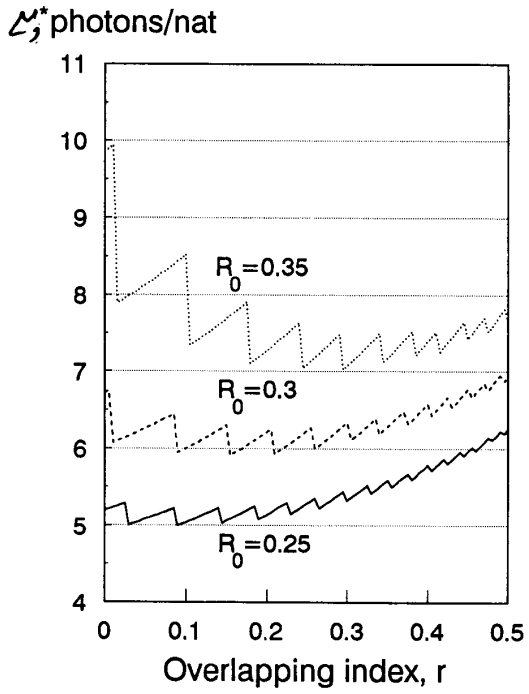

Fig. 4. Minimum energy vs. the overlap.

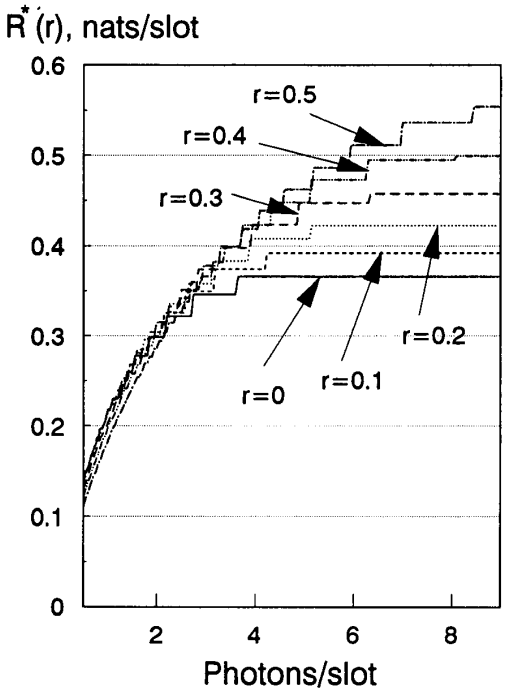

Fig. 5. Maximum throughput vs. av. power.