

# Capacity and Cutoff Rate for Optical Overlapping Pulse-Position Modulation Channels

Hossam M. H. Shalaby, *Member, IEEE*

**Abstract**—Upper and lower bounds on the capacity and cutoff rate for direct-detection optical overlapping-pulse-position modulation channels are derived. It is shown that these bounds are asymptotically tight, in the sense that the difference between the upper and lower bounds converges to zero as the pulse position multiplicity  $M$  and/or the average photon count per pulse  $Q$  goes to infinity. The tightness of these bounds for finite values of  $M$  and  $Q$  is investigated by providing some numerical examples. Orders of magnitude of the rate of convergence of the difference between the bounds are also estimated in the last section of the paper.

## I. INTRODUCTION

In optical overlapping-pulse-position modulation (OPPM) the input signal modulates the position of a laser pulse within a finite time frame [1–9]. The set of all possible pulse positions within the time frame is assumed to be finite and the adjacent pulse positions are allowed to overlap. An advantage of OPPM over the conventional pulse-position modulation (PPM), where the adjacent pulse positions are disjoint, is the increase of the throughput (bit rate) without decreasing the pulsewidth. This advantage is acquired, however, at the expense of serious degradation in the error-probability performance due to reducing the distance between the symbols in the signal space. Fortunately, we can improve the performance by employing error correcting codes and sacrifice some of the throughput gain [3]. Another way to improve the performance of OPPM is to restrict the overlap between the pulse positions to take values less than half the pulsewidth. In [9] we have shown that we can always find values of the overlapping index (ratio between the overlap and the pulsewidth) in the interval  $[0, 0.5]$  such that OPPM outperforms PPM.

Lower bounds on the capacity and cutoff rate of the self-noise-limited OPPM channel have been derived in [1] when the overlapping index is allowed to take values in the discrete set  $\{0, \frac{1}{2}, \frac{2}{3}, \dots\}$ . In this paper we find tight upper bounds  $C_U$  on the capacity of the above channel when the overlapping index is allowed to take values in the interval  $[0, 0.5]$ . The tightness of these bounds is measured by the difference,  $\Delta C = C_U - C_L$ , between the upper and lower

bounds. We are able to show that this difference is asymptotically zero, in the sense that  $\Delta C \rightarrow 0$  as the pulse-position multiplicity  $M$  and/or the average photon count per pulse  $Q$  goes to infinity. We also derive asymptotically-equal upper and lower bounds for the cutoff rate of the above channel. In the last part of this paper we determine the rates of convergence of  $\Delta C$  and  $\Delta R = R_U - R_L$ .

The paper is thus organized as follows: The channel model and preliminaries are given in Section II. Section III is devoted for the derivation of the capacity upper bound. The cutoff rate bounds are given in Sections IV. Numerical examples are provided in Section V. Finally the conclusion is given in Section VI.

## II. CHANNEL MODEL AND PRELIMINARIES

Our model for OPPM with overlapping index  $r \in [0, 0.5]$  is as follows. A rectangular laser pulse is transmitted in one of  $M$  possible positions  $\{1, 2, \dots, M\}$  within a time frame of duration  $T$ . A pulse of width  $\tau$  is said to be in position  $m$ ,  $m \in \{1, 2, \dots, M\}$  if it extends over the subinterval beginning at time  $(m-1)(1-r)\tau$ . The relation between  $T$ ,  $r$ ,  $M$ , and  $\tau$  is

$$T = (M(1-r) + r)\tau.$$

The above model is called ambiguity and erasure channel [1,9].

Let the random variables  $X$  and  $Y$  denote the position of the transmitted pulse and the demodulator output, respectively. Thus  $Y \in \{m, a(m-1, m), a(m, m+1), e\}$  when  $X = m$ ,  $m \in \{2, 3, \dots, M-1\}$ , where we have denoted the ambiguity between positions  $m-1$  and  $m$  by  $a(m-1, m)$  and the erasure output by  $e$ . On the other hand, if  $X = 1$  or  $M$ , then  $Y \in \{1, a(1, 2), e\}$  or  $Y \in \{M, a(M-1, M), e\}$ , respectively.

It is easy to check that

$$P_{Y|X}(e|m) = \exp[-Q],$$

$$P_{Y|X}(a(m-1, m)|m) = P_{Y|X}(a(m, m+1)|m) \\ = (1 - \exp[-Qr]) \exp[-Q(1-r)],$$

where  $Q$  denotes the average photon count per pulse.

The mutual information for the above channel model is thus given by

$$I(X \wedge Y) \stackrel{\text{def}}{=} \sum_{x,y} P_{XY}(x, y) \log \frac{P_{Y|X}(y|x)}{P_Y(y)}$$

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The author is with the Department of Electrical Engineering, Faculty of Engineering, University of Alexandria, Alexandria 21544, Egypt.

$$\begin{aligned}
 &= -t \sum_{x=1}^{M-1} (P_X(x) + P_X(x+1)) \log(P_X(x) + P_X(x+1)) \\
 &\quad - (s-t) \{P_X(1) \log P_X(1) + P_X(M) \log P_X(M)\} \\
 &\quad - (s-2t) \sum_{x=2}^{M-1} P_X(x) \log P_X(x), \tag{1}
 \end{aligned}$$

where

$$s \stackrel{\text{def}}{=} 1 - \exp[-Q] \quad \text{and} \quad t \stackrel{\text{def}}{=} \exp[-Q(1-r)] - \exp[-Q].$$

In fact  $s$  is the probability that the OPPM pulse is not erased and  $t$  is the probability of occurrence of an ambiguity given that a pulse is transmitted.

We need the following two lemmas in our derivations.

**Lemma 1:** For any  $a, b, Q \geq 0$  and  $\nu \in [0, 1]$ , if  $a\nu \geq b(1-\nu)$ , then

$$a - (a+b) \exp[-Q\nu] + b \exp[-Q] \geq 0.$$

*Proof:* Let  $f(Q) \stackrel{\text{def}}{=} a - (a+b) \exp[-Q\nu] + b \exp[-Q]$ , then  $f(0) = 0$  and  $f(\infty) = a$ . Thus it suffices to show that  $\frac{df}{dQ} \geq 0$  for all  $Q \geq 0$ . Indeed we have

$$f'(Q) = b[e^{-Q\nu} - e^{-Q}] + [a\nu - b(1-\nu)]e^{-Q\nu}$$

which is non-negative under the given hypotheses.  $\square$

**Lemma 2:** For any  $Q \geq 0$  and  $r \in [0, 1]$

$$\frac{\exp[-Q(1-r)] - \exp[-Q]}{1 - \exp[-Q]} \leq r.$$

*Proof:* Let  $f(Q) \stackrel{\text{def}}{=} \frac{\exp[-Q(1-r)] - \exp[-Q]}{1 - \exp[-Q]}$ , then  $f(0) = r$  and  $f(\infty) = 0$ . The derivative  $f'(Q)$  is equal to  $h(Q)/(1 - \exp[-Q])^2$ , where

$$h(Q) = -e^{-Q(1-r)}(1-r - e^{-Qr} + re^{-Q}).$$

Using Lemma 1 with  $a = 1-r$ ,  $b = r$  and  $\nu = r$ , it follows that  $h(Q) \leq 0$  for any  $Q \geq 0$  and  $r \in [0, 1]$ .  $\square$

### III. LOWER AND UPPER BOUNDS ON OPPM CHANNEL CAPACITY

Lower and upper bounds on the ambiguity and erasure channel capacity with  $r \in [0, 0.5]$  are given in the following theorem.

**Theorem 1:** The capacity of the optical OPPM channel with  $M$  pulse positions and overlapping index  $r \in [0, 0.5]$  is lower bounded by

$$\begin{aligned}
 C \geq C_L &\stackrel{\text{def}}{=} (1 - \exp[-Q]) \log M \\
 &\quad - 2 \log 2 \left(1 - \frac{1}{M}\right) \left(\exp[-Q(1-r)] - \exp[-Q]\right)
 \end{aligned}$$

and upper bounded by

$$\begin{aligned}
 C \leq C_U &\stackrel{\text{def}}{=} (1 - \exp[-Q]) \log M \\
 &\quad - 2 \log 2 \left(1 - \frac{1}{M^{1-2r}}\right) \left(\exp[-Q(1-r)] - \exp[-Q]\right),
 \end{aligned}$$

where  $Q$  is the average photon count per pulse.

**Remark:** It is obvious that  $\Delta C = C_U - C_L$  converges to zero as  $M \rightarrow \infty$  for any  $r \in [0, 0.5]$ .

*Proof:* The proof of the lower bound is immediate by substituting the uniform distribution in (1). Now we prove the upper bound as follows. Define

$$L(P_X) \stackrel{\text{def}}{=} I(X \wedge Y) - \lambda \left(\sum_{x=1}^M P_X(x) - 1\right),$$

where  $\lambda$  is the Lagrangian multiplier. The distribution  $P_X$  achieving the capacity must satisfy the first order necessary condition  $\frac{\partial L}{\partial P_X} = 0$ , i.e.,

$$(s-t) \log P_X(1) + t \log (P_X(1) + P_X(2)) = -\lambda - s, \tag{2a}$$

$$(s-t) \log P_X(M) + t \log (P_X(M-1) + P_X(M)) = -\lambda - s, \tag{2b}$$

$$\begin{aligned}
 (s-2t) \log P_X(x) + t \log (P_X(x-1) + P_X(x)) \\
 + t \log (P_X(x) + P_X(x+1)) = -\lambda - s, \\
 x = 2, \dots, M-1. \tag{2c}
 \end{aligned}$$

We find an estimate of  $P_X(1)$  as follows. Since  $P_X(x) + P_X(x+1) \leq 1$  and  $s-t \geq s-2t$ , using (2) we obtain for any  $x \in \{1, \dots, M\}$

$$(s-2t) \log P_X(x) \geq -\lambda - s \quad \text{or} \quad P_X(x) \geq \exp\left[-\frac{\lambda + s}{s-2t}\right].$$

Adding over all  $x$ 's yields

$$\lambda + s \geq (s-2t) \log M.$$

Making use of (2a) once more we get that

$$\lambda + s \leq -s \log P_X(1),$$

where we have made use of the fact that  $P_X(2) \geq 0$ . Combining the last two inequalities and making use of Lemma 2, we obtain

$$-\log P_X(1) \geq (1-2t/s) \log M \geq (1-2r) \log M$$

or

$$P_X(1) \leq \frac{1}{M^{1-2r}}.$$

Since  $P_X(1) = P_X(M)$ , we write the mutual information in (1) as

$$I(X \wedge Y) = 2tH(\hat{X}) + (s - 2t)H(X) - 2t \log 2(1 - P_X(1)),$$

where  $H(\cdot)$  is the informational entropy and  $\hat{X}$  is a random variable with probability distribution  $\hat{P}_X$ ,

$$\hat{P}_X(x) \stackrel{\text{def}}{=} \begin{cases} \frac{P_X(x) + P_X(x+1)}{2}; & \text{if } x = 1, \dots, M-1, \\ \frac{P_X(M) + P_X(1)}{2}; & \text{if } x = M. \end{cases}$$

But  $H(X)$  and  $H(\hat{X})$  are not greater than  $\log M$  since  $X, \hat{X} \in \{1, \dots, M\}$ ; hence

$$C \leq (1 - \exp[-Q]) \log M - 2 \log 2(1 - P_X(1))(\exp[-Q(1-r)] - \exp[-Q]).$$

Substituting for  $P_X(1)$  completes the proof.  $\square$

*Remark:* The estimate  $P_X(1) \leq 1/2$  is better than the one given in the proof of the theorem if  $M^{1-2r} < 2$ . This suggests using the tighter upper bound

$$C_U \stackrel{\text{def}}{=} (1 - \exp[-Q]) \log M - 2 \log 2(1 - o(M))(\exp[-Q(1-r)] - \exp[-Q]),$$

where  $o(M) \stackrel{\text{def}}{=} \min\{\frac{1}{2}, \frac{1}{M^{1-2r}}\}$ .

#### IV. BOUNDS ON THE CUTOFF RATE OF THE OPPM CHANNEL

In this section we derive lower and upper bounds on the cutoff rate of the OPPM channel with overlapping index  $r \in [0, 0.5]$ . The cutoff rate for discrete memoryless channels is defined as  $R_0 = -\log \Phi^*$ , where

$$\Phi^* \stackrel{\text{def}}{=} \min_{P_X} \Phi(P_X)$$

and

$$\Phi(P_X) \stackrel{\text{def}}{=} \sum_{y \in \mathcal{Y}} \left[ \sum_{x \in \mathcal{X}} P_X(x) \sqrt{P_{Y|X}(y|x)} \right]^2.$$

Here  $\mathcal{X}$  and  $\mathcal{Y}$  are the cardinalities of the random variables  $X$  and  $Y$ , respectively. It is easy to see that for our channel:

$$\begin{aligned} \Phi(P_X) &= 1 - s + (s - t) \left( P_X^2(1) + P_X^2(M) \right) \\ &\quad + (s - 2t) \sum_{x=2}^{M-1} P_X^2(x) \\ &\quad + t \sum_{x=1}^{M-1} \left( P_X(x) + P_X(x+1) \right)^2. \end{aligned} \quad (3)$$

*Theorem 2:* The cutoff rate of the optical OPPM channel with  $M$  pulse positions and overlapping index  $r \in [0, 0.5]$  is lower bounded by

$$R_0 \geq -\log \left[ \exp[-Q] + \frac{1}{M}(1 - \exp[-Q]) + \frac{2(M-1)}{M^2} \left( \exp[-Q(1-r)] - \exp[-Q] \right) \right]$$

and upper bounded by

$$R_0 \leq -\log \left[ \exp[-Q] + \frac{1}{M}(1 - \exp[-Q]) + \frac{2(M - (1+2r)^2)}{M^2} \left( \exp[-Q(1-r)] - \exp[-Q] \right) \right].$$

*Proof:* The lower bound is immediate by substituting the uniform distribution in (3). Now let

$$L(P_X) \stackrel{\text{def}}{=} \Phi(P_X) - \lambda \left( \sum_{x=1}^M P_X(x) - 1 \right),$$

where  $\lambda$  is the Lagrangian multiplier. The first order necessary conditions are thus

$$\begin{aligned} 2sP_X(1) + 2tP_X(2) &= \lambda, \\ 2sP_X(M) + 2tP_X(M-1) &= \lambda, \\ 2sP_X(x) + 2t(P_X(x-1) + P_X(x+1)) &= \lambda, \\ & \quad x = 2, \dots, M-1. \end{aligned} \quad (4)$$

Adding the above equations and using the fact that  $P_X(M) = P_X(1)$  (because of symmetry) we obtain

$$M\lambda = 2(s + 2t) - 4tP_X(1).$$

From (4) we can see that  $\lambda \geq 2sP_X(1)$ . Substituting in the last equation, we get

$$P_X(1) \leq \frac{s + 2t}{Ms + 2t} \leq \frac{s + 2t}{Ms} \leq \frac{1 + 2r}{M},$$

where the last two inequalities are because  $t \geq 0$  and  $t/s \leq r$ , respectively. Now, we can rewrite  $\Phi(P_X)$  as

$$\begin{aligned} \Phi(P_X) &= 1 - s + (s - 2t) \sum_{x=1}^M P_X^2(x) \\ &\quad + 4t \sum_{x=1}^M \hat{P}_X^2(x) - 2tP_X(1)P_X(M), \end{aligned}$$

where  $\hat{P}_X$  is the distribution on  $\mathcal{X}$  defined previously. Noticing that  $P_X(1) = P_X(M)$  and  $\min\{u + v\} \geq \min u + \min v$  we obtain

$$\begin{aligned} \Phi^* &= \min_{P_X} \Phi(P_X) \geq 1 - s + (s - 2t) \min_{P_X} \sum_{x=1}^M P_X^2(x) \\ &\quad + 4t \min_{\hat{P}_X} \sum_{x=1}^M \hat{P}_X^2(x) - 2tP_X^2(1). \end{aligned}$$

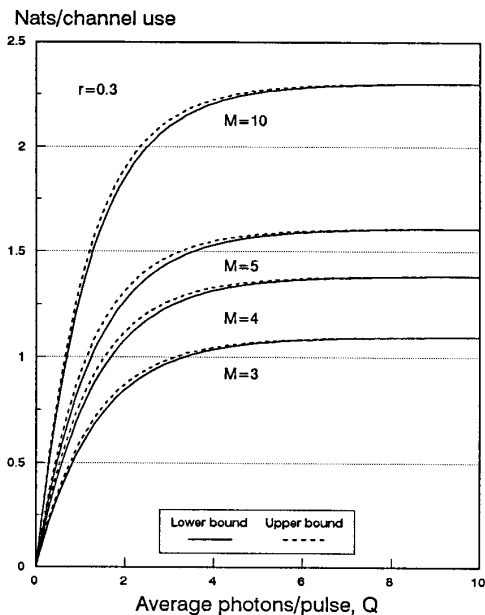


Fig. 1. Capacity bounds versus  $Q$  for OPPM with  $r = 0.3$ .

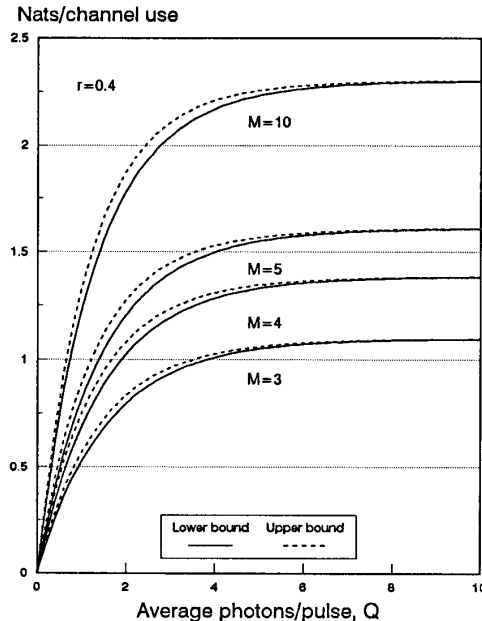


Fig. 2. Capacity bounds versus  $Q$  for OPPM with  $r = 0.4$ .

It is easy to see that

$$\min_{P_X} \sum_{x=1}^M P_X^2(x) = \frac{1}{M} .$$

Hence

$$\Phi^* \geq 1 - s + \frac{s + 2t}{M} - 2t \frac{(1 + 2r)^2}{M^2} . \quad \square$$

*Remark:* If  $M < 2(1 + 2r)$ , then the estimate  $P_X(1) \leq 1/2$  is better than the one given in the proof of the theorem. Hence the following upper bound is tighter than the above as long as  $M < 2(1 + 2r)$

$$R_0 \leq -\log \left[ \exp[-Q] + \frac{1}{M}(1 - \exp[-Q]) + \frac{4 - M}{2M} (\exp[-Q(1 - r)] - \exp[-Q]) \right] .$$

### V. NUMERICAL RESULTS

As mentioned early the above bounds are asymptotically tight in the sense that the difference between the upper and lower bounds converges to zero as  $M$  and/or  $Q$  goes to infinity. In this section we examine the tightness of these bounds for small values of  $M$  and  $Q$ . We have evaluated numerically the bounds given in the above theorems for different values of  $M$ ,  $Q$ , and  $r$ ; some of our results are plotted in Figs. 1-4. A general observation on the capacity bounds (Figs. 1 and 2) is the closeness of the bounds even for small values of  $M$  and/or  $Q$ . It can be seen from Fig.

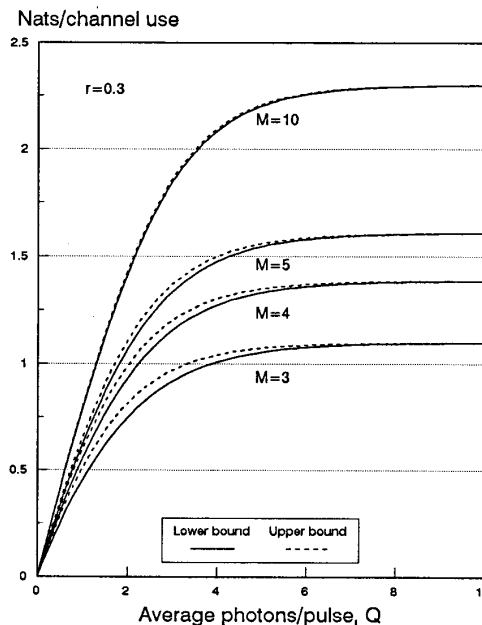


Fig. 3. Cutoff rate bounds versus  $Q$  for OPPM with  $r = 0.3$ .

1 that, for overlapping index equals 0.3, the difference  $\Delta C$  is not greater than 0.097 of the true capacity. This ratio is at most 0.143 when  $r = 0.4$ , Fig. 2. On the other hand, the difference ( $\Delta R$ ) between the bounds of the cutoff rate may be significant for small values of  $M$  and  $Q$ . As can be seen from Figs. 3 and 4,  $\Delta R$  is less than 0.156 of the

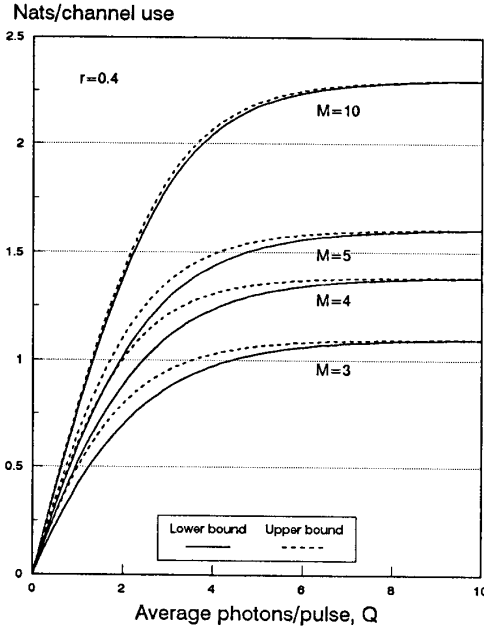


Fig. 4. Cutoff rate bounds versus  $Q$  for OPPM with  $r = 0.4$ .

true cutoff rate when  $r = 0.3$  and less than  $0.227 R_0$  when  $r = 0.4$ . For moderate values of  $M$  and/or  $Q$ , however, these bounds are becoming tight and the difference  $\Delta R$  converges so fast to zero.

## VI. EXTENSIONS AND CONCLUDING REMARKS

We have derived asymptotically tight lower and upper bounds on the channel capacity and cutoff rate of the direct detection self-noise-limited OPPM channel. We have shown by numerical examples that these bounds are also tight for moderate values of  $M$  and/or  $Q$ . The main idea in our derivations is to find an over-estimate of the optimizing probability of the input symbol  $P_X(1)$ . In the case of the capacity we have shown that  $P_X(1) \leq \frac{1}{M^{1-2r}}$ , whereas in the case of the cutoff rate we got that  $P_X(1) \leq \frac{1+2r}{M}$ .

To provide an estimate of the rate of convergence of the difference in capacity bounds, we have

$$\begin{aligned} \frac{\Delta C}{C} &\leq \frac{\Delta C}{C_L} = \frac{2 \log 2 (o(M) - \frac{1}{M}) t}{s \log M - 2 \log 2 (1 - \frac{1}{M}) t} \\ &= \frac{2 \log 2 (o(M) - \frac{1}{M}) t / s}{\log M - 2 \log 2 (1 - \frac{1}{M}) t / s} \\ &\leq \frac{2 \log 2 (o(M) - \frac{1}{M}) r}{\log M - 2 \log 2 (1 - \frac{1}{M}) r}. \end{aligned}$$

The equality in the last expression holds when  $Q = 0$ . We can further increase the upper bound as follows

$$\frac{\Delta C}{C} \leq \frac{2r \log 2}{M^{1-2r} (\log M - 2r \log 2)} \leq \frac{\log 2}{M^{1-2r} \log(M/2)}.$$

Similarly, to give an order of magnitude of the rate of convergence of the difference between cutoff rate bounds, we notice that

$$\begin{aligned} \frac{\Delta R}{R} &\leq \frac{\Delta R}{R_L} = \frac{\Delta R}{-\log \left[ 1 - s + \frac{s}{M} + 2t \left( \frac{1}{M} - \frac{1}{M^2} \right) \right]} \\ &\leq \frac{\Delta R}{s - \frac{s}{M} - 2t \left( \frac{1}{M} - \frac{1}{M^2} \right)}. \end{aligned}$$

But for  $\tilde{o}(M^2) \stackrel{\text{def}}{=} \min \left\{ \frac{1}{4}, \frac{(1+2r)^2}{M^2} \right\}$ ,

$$\begin{aligned} \Delta R &= R_U - R_L = \log \frac{1 - s + \frac{s}{M} + 2t \left( \frac{1}{M} - \frac{1}{M^2} \right)}{1 - s + \frac{s}{M} + 2t \left( \frac{1}{M} - \tilde{o}(M^2) \right)} \\ &\leq \frac{2t \left( \tilde{o}(M^2) - \frac{1}{M^2} \right)}{1 - s + \frac{s}{M} + 2t \left( \frac{1}{M} - \tilde{o}(M^2) \right)} \leq 2Mt \left( \tilde{o}(M^2) - \frac{1}{M^2} \right). \end{aligned}$$

The last inequality holds because

$$\begin{aligned} 1 - s + \frac{s}{M} + 2t \left( \frac{1}{M} - \tilde{o}(M^2) \right) \\ \geq 1 - s + \frac{s}{M} = 1 - \left( 1 - \frac{1}{M} \right) s \geq \frac{1}{M}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\Delta R}{R} &\leq \frac{2Mt \left( \tilde{o}(M^2) - \frac{1}{M^2} \right)}{s - \frac{s}{M} - 2t \left( \frac{1}{M} - \frac{1}{M^2} \right)} \leq \frac{2Mr \left( \tilde{o}(M^2) - \frac{1}{M^2} \right)}{1 - \frac{1}{M} - 2r \left( \frac{1}{M} - \frac{1}{M^2} \right)} \\ &\leq \frac{2Mr \frac{4r(1+r)}{M^2}}{1 - \frac{1}{2} - \frac{r}{2}} = \frac{16r^2(1+r)}{M(1-r)} \leq \frac{12}{M}. \end{aligned}$$

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