Lemma 8: For any $P \in \overline{\Delta}^{(2)}$, we have

$$\tau(-\infty) \le \tau(P) \le \tau(\infty). \tag{65}$$

Proof: For $P \in \Delta^{(2)}$, let us define a function $\xi(t)$ by

$$\xi(t) = D(P\|P_t), -\infty < t < \infty. \tag{66}$$

We have, by (28),

$$\frac{d\xi(t)}{dt} = \tau(t) - \tau(P) \tag{67}$$

and

$$\frac{d^2\xi(t)}{dt^2} = \frac{d^2}{dt^2}\log \zeta_t. \tag{68}$$

Hence, by Lemma 3, $\xi(t)$ is a convex function of t, $-\infty < t < \infty$. From (14) and (15), we see that some of $P_t(i'|i)$, $i, i' \in \Omega$, tend to zero as t goes to infinity or minus infinity. Thus, by the definition of divergence, we have

$$\lim_{t\to\infty} \xi(t) = \lim_{t\to-\infty} \xi(t) = \infty. \tag{69}$$

Therefore, from (69) and the convexity of $\xi(t)$, there exists a unique t with $(d\xi(t)/dt) = 0$, or, by (67), $\tau(t) = \tau(P)$. Consequently, from (35), we have (65) for $P \in \Delta^{(2)}$. By the continuity of $\tau(P)$, $P \in \overline{\Delta}^{(2)}$, (65) holds also for $P \in \overline{\Delta}^{(2)}$. □

Proof of Theorem 3: Let us define a test function $\tilde{o}_n$ by

$$\tilde{o}_n(\omega^n) = \begin{cases} 1, & \text{if } \tau(P_{\omega^n}) < \tau(\infty), \\ \tilde{\delta}, & \text{if } \tau(P_{\omega^n}) = \tau(\infty), \\ 0, & \text{if } \tau(P_{\omega^n}) > \tau(\infty), \end{cases} \tag{70}$$

with

$$\tilde{\delta} = 1 - \frac{\exp(-rn)}{P_0\{\omega^n|\tau(P_{\omega^n}) = \tau(\infty)\}}. \tag{71}$$

By Lemma 8, we see that the set $\{\omega^n|\tau(P_{\omega^n}) > \tau(\infty)\}$ is empty. Hence, we have $\alpha(\tilde{o}_n) = \exp(-rn)$. Since $-\frac{1}{n}\log P_0\{\omega^n|\tau(P_{\omega^n}) = \tau(\infty)\}$ converges to $\sigma_0(\infty)$ $(< r)$ by Lemma 6, we have $0 < \tilde{\delta} < 1$ for sufficiently large n's. Therefore, for the most powerful test function o_n^*, we have

$$\begin{aligned}\beta(o_n^*) &\le \beta(\tilde{o}_n) \\ &= 1 - \exp(-rn)\frac{P_1\{\omega^n|\tau(P_{\omega^n}) = \tau(\infty)\}}{P_0\{\omega^n|\tau(P_{\omega^n}) = \tau(\infty)\}}.\end{aligned}$$

which implies $\overline{\rho}_3(r) \le r - \tau(\infty)$ by Lemmas 2 and 6.

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A Note on the Asymptotics of Distributed Detection with Feedback

Hossam M. H. Shalaby, *Member, IEEE,* and
Adrian Papamarcou, *Member, IEEE*

***Abstract*—The effect of feedback on the performance of a distributed Neyman–Pearson detection system consisting of n sensors, two-stage local quantizers, and a global detector is investigated. Two feedback schemes are discussed, only one of which yields an asymptotic gain in performance, as measured by an appropriate error exponent.**

***Index Terms*— Distributed detection, error exponents, feedback, hypothesis testing, Kullback–Leibler divergence, quantization.**

I. INTRODUCTION

A distributed (or decentralized) detection system is a network of n sensors which, together with a global detector (or data fusion center), cooperatively undertake the task of identifying a random signal source. Typically, the sensors compress their observations into low-rate data streams, which are then transmitted to the global detector for processing and decision making. Compared to a conventional detection system with passive sensors, the distributed setup offers the advantages of reduced communication bandwidth, shared processing and increased reliability, albeit at some expense of performance.

Most distributed detection models employed in the literature are feedforward systems, in which the information flow from the sensors to the global detector is unidirectional. Recently, there has been some interest [1], [2], [9] in models with bidirectional flow, in which feedback from the global detector to the sensors is allowed. In such systems, the process of local data compression and transmission

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H. M. H. Shalaby was with the Electrical Engineering Department and the Institute for Systems Research, University of Maryland, College Park, MD 20742. He is now with the Department of Electrical Engineering, Faculty of Engineering, University of Alexandria, Alexandria 21544, Egypt.

A. Papamarcou is with the Electrical Engineering Department and the Institute for Systems Research, University of Maryland, College Park, MD 20742.

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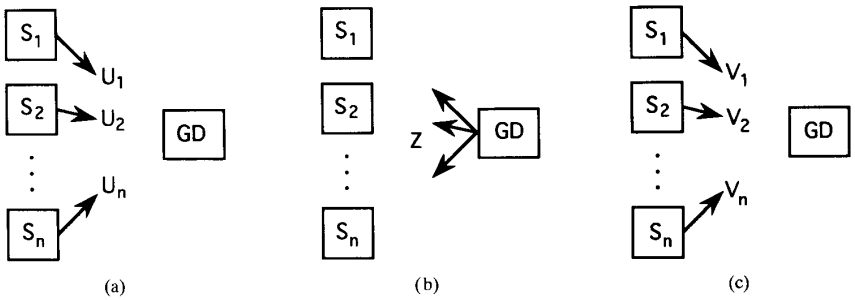

Fig. 1. Schematic representation of FB1. (a) First stage of feedforward transmission to global detector (GD). (b) Feedback from global detector. (c) Second stage of feedforward transmission.

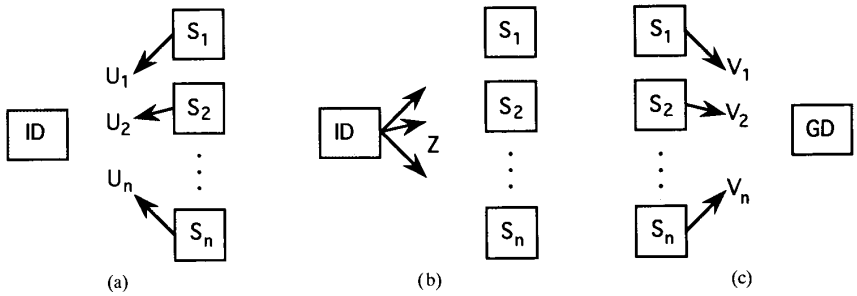

Fig. 2. Schematic representation of FB2. (a) Feedforward transmission to intermediate detector (ID). (b) Feedback from intermediate detector. (c) Feedforward transmission to global detector.

evolves in several stages, which are punctuated by the broadcast of feedback information from the global detector. In this manner, each individual sensor receives partial information about the observations of the other sensors in the network, and can use this information to improve on its data processing. Other things being equal, the introduction of feedback and the resulting increase in available information are beneficial; this was established through performance evaluations of different feedback schemes in [1], [2], and [9]. On the negative side, feedback requires a higher communication bandwidth and entails a longer delay in making a final decision.

In this correspondence, we investigate the asymptotic gain in performance resulting from introducing two distinct types of feedback in a distributed detection system consisting of n sensors. We undertake an asymptotic (in the number of sensors n) analysis because fixed-sample optimization and performance evaluation is analytically intractable for a majority of distributed detection systems (even in the absence of feedback), whereas asymptotically optimal schemes have been successfully developed in many instances [5]–[8], [10]. Although the feedback configurations discussed here are fairly general, they are by no means exhaustive. In particular, we limit ourselves to models in which the data sample size (per sensor) is *fixed* as the number n of sensors varies, and all data are collected *before* sensor-to-global detector communication commences (unlike in [2] and [9]). The data are represented by $X_1, \cdots, X_n$, where X_i denotes the observation of sensor S_i.

The first feedback scheme considered here (denoted by FB1 and shown in Fig. 1) is perhaps the simplest in the class of all feedback schemes. It involves data compression and transmission in two stages, with feedback from the global detector after the first stage. Thus, initially, each sensor S_i quantizes its observations into a message U_i, and transmits this message to the global detector. Upon receipt of $\boldsymbol{U} = (U_1, \cdots, U_n)$, the global detector broadcasts a feedback message Z. Each sensor then generates and transmits a second message V_i, and the global detector uses the two vectors $\boldsymbol{U}$ and $\boldsymbol{V}$ to make the final decision.

The second feedback scheme (denoted by FB2 and shown in Fig. 2) is markedly different, in that the global detector does not participate in the feedback process. In essence, here we have partial information exchange between sensors *before* transmitting data to the global detector. This exchange is affected via an *intermediate detector*, which is distinct from—and not linked with—the global detector. The process again evolves in two stages. Initially, the sensors transmit messages U_i to the intermediate detector. In response, the intermediate detector generates a message Z and broadcasts it to the sensors. The sensors then requantize their observations into messages V_i and transmit them to the global detector, which makes a decision based *solely* on the vector $\boldsymbol{V}$.

In our analysis, we assume that the n sensor observations are independent and identically distributed, and we consider binary detection ($H_0 : X \sim P_X$ versus $H_1 : X \sim Q_X$) under the classical (Neyman–Pearson) criterion. The problem is to minimize the type II error probability q_n subject to an upper bound (or *level*) ϵ on the type I error probability p_n. This is a joint minimization over all allowable local (sensor) quantizers, feedback maps and global decision rules. The resulting minimum value of q_n is denoted by $\beta_n(\epsilon)$. The *optimal error exponent* of the detection problem is defined as the quantity

$$\theta(\epsilon) = -\lim_n \frac{1}{n} \log \beta_n(\epsilon),$$

provided the limit exists. Whenever the error exponent is well defined, we say that a sequence of detection schemes is *asymptotically*, or *exponentially, optimal* if the associated type II error probability q_n satisfies

$$-\lim_n \frac{1}{n} \log q_n = \theta(\epsilon).$$

We proceed to present complete results on the optimal error exponents associated with the feedback schemes introduced previously, under conditions which are fairly general for FB1 and somewhat more restrictive for FB2. These results indicate that, in an asymptotic sense, *only one* of the two feedback schemes is useful. More precisely, the error exponent of FB1 is no better than what is attainable by a purely feedforward scheme in which the feedback stage (transmission of Z) is altogether eliminated. In FB2, on the other hand, the preliminary information exchange between sensors is quite valuable, resulting in a net performance gain compared with a single stage feedforward scheme in which only $\boldsymbol{V}$ is transmitted by the sensors to the global detector. The results are obtained under the following four assumptions:

1) the i.i.d. observations X_i have finite alphabet;
2) the alphabets of the feedforward messages are fixed in n, and are binary in the case of FB2;
3) the same compression scheme is used by all n sensors; and
4) the alphabets of the feedback messages can grow at most subexponentially in n.

As we argue briefly in Section V, assumptions 1–3 can be relaxed in the case of FB1.

II. Preliminaries

Error exponents for detection systems are customarily expressed in terms of the Kullback–Leibler informational divergence, or simply *divergence*. Given two distributions P_X and Q_X on a measurable space $(\mathcal{X}, \mathcal{B})$, the divergence of P_X relative to Q_X is given

$$D(P_X||Q_X) = \sup_{U:U=f(X),\ |\mathcal{U}|<\infty} \sum_{u\in\mathcal{U}} P_U(u)\log\frac{P_U(u)}{Q_U(u)}.$$

Here, $\mathcal{U}$ represents the alphabet of $U = f(X)$, and $|\cdot|$ denotes cardinality. We define P_U by $P_U(u) = P_X(f^{-1}\{u\})$, and Q_U similarly. The mapping f is assumed measurable.

If P_X is absolutely continuous with respect to Q_X (written as $P_X \ll Q_X$), then

$$D(P_X||Q_X) = E_P\left[\log\frac{dP_X}{dQ_X}\right].$$

where dP_X/dQ_X is the Radon–Nikodym derivative. If $P_X \not\ll Q_X$, then $D(P_X||Q_X) = +\infty$; the converse is also true if $\mathcal{B}$ is finite. In the discrete case, where without loss of generality, $\mathcal{B} = 2^{\mathcal{X}}$, the divergence has the simple expression

$$D(P_X||Q_X) = \sum_{x\in\mathcal{X}} P_X(x)\log\frac{P_X(x)}{Q_X(x)}.$$

The divergence functional plays an important role in hypothesis testing. A key result is Stein's lemma [3], which states that $D(P_X||Q_X)$ is the optimal Neyman–Pearson error exponent for testing a null hypothesis $H_0 : X \sim P_X$ versus an alternative $H_1 : X \sim Q_X$ at any level $\epsilon \in (0,1)$ on the basis of an i.i.d. sample $\boldsymbol{X} = (X_1, \cdots, X_n)$. A characterization of the optimal Bayesian error exponent in terms of divergence functionals is also possible, but is not of direct relevance to the problems presented here.

In deriving error exponents for hypothesis testing, one can generally apply results from the theory of large deviations. In this work, we follow an alternative approach based on typicality arguments [4, ch. 1, Section 2] which is suitable for finite sample spaces. The underlying idea is that sequences of length n from a finite space $\mathcal{X}$ can be divided into polynomially (in n) many equivalence classes $\hat{T}_X^n$, each characterized by a fixed *type* (composition, empirical distribution). The probability of each $\hat{T}_X^n$ under the n-fold i.i.d. extension of a measure P_X (resp., Q_X) on $\mathcal{X}$ decays exponentially in n, with rate (exponent) equal to the divergence of the associated type relative to P_X (resp., Q_X). Thus, in the context of testing P_X against Q_X on the basis of an i.i.d. sample size n, each set $\hat{T}_X^n$ has two exponents associated with it. For Neyman–Pearson testing at level ϵ, the optimal null acceptance region $\mathcal{A}_n$ can be approximated by the union of all classes $\hat{T}_X^n$ whose P_X-exponent is smaller than an arbitrarily small positive δ; while the resulting error exponent $\theta(\epsilon)$ is approximately given by the maximum of the corresponding Q_X-exponents. The precise facts needed for our analysis are as follows.

Let $\mathcal{P}_n(\mathcal{X})$ denote the set of distributions on $\mathcal{X}$ assigning probability values that are multiples of $1/n$. Every sequence $\boldsymbol{x} \in \mathcal{X}^n$ induces an empirical distribution on $\mathcal{X}$ that coincides with a $\hat{P}_X$ in $\mathcal{P}_n(\mathcal{X})$. We then say that $\hat{P}_X$ is the type of $\boldsymbol{x}$, or equivalently, that $\hat{P}_{\boldsymbol{X}}$ is $\hat{P}_X$—*typical*. Given $\hat{P}_X$ in $\mathcal{P}_n(\mathcal{X})$, we represent the set of all $\hat{P}_X$-typical sequences in $\mathcal{X}^n$ by $\hat{T}_X^n$.

We also use two concepts of approximate typicality defined with reference to an arbitrary distribution $\tilde{P}_X$ on $\mathcal{X}$. We say that $\boldsymbol{x}$ is $(\tilde{P}_X, \eta)$-*typical* if the type of $\boldsymbol{x}$ is absolutely continuous with respect to $\tilde{P}_X$ and also lies in a ball of sup-norm radius η centered at $\tilde{P}_X$. We denote the set of all $(\tilde{P}_X, \eta)$-*typical* sequences by $\tilde{T}_{X,\eta}^n$.

The second concept of approximate typicality is borrowed from [6]. We say that $\boldsymbol{x}$ is $(\tilde{P}_X, \eta)$-*divergent* if the type $\hat{P}_X$ of $\boldsymbol{x}$ satisfies

$$D\left(\hat{P}_X||\tilde{P}_X\right) \le \eta.$$

We denote the set of all $(\tilde{P}_X, \eta)$-*divergent* sequences by $\tilde{S}_{X,\eta}^n$.

We will use the notation P^n for the n-fold i.i.d. extension of a measure P. The following lemma contains results which are essential to our analysis.

Lemma 1: Let $\mathcal{X}$ be a finite space and $\hat{P}_X, \tilde{P}_X$ and Q_X be distributions on $\mathcal{X}$. Then

a) $|\mathcal{P}_n(\mathcal{X})| \le (n+1)^{|\mathcal{X}|}$;

b) $$(n+1)^{-|\mathcal{X}|}\exp\left[-nD\left(\hat{P}_X||Q_X\right)\right] \le Q_X^n\left(\hat{T}_X^n\right) \le \exp\left[-nD\left(\hat{P}_X||Q_X\right)\right].$$

c) If $\eta > 0$, then

$$\tilde{P}_X^n\left(\tilde{T}_{X,\eta}^n\right) \ge 1 - \frac{|\mathcal{X}|}{4n\eta^2}.$$

$$\tilde{P}_X^n\left(\tilde{S}_{X,\eta}^n\right) \ge 1 - (n+1)^{|\mathcal{X}|}\exp[-n\eta].$$

Proof: For a) and b) and the first inequality in c), see [4], Lemmas 2.2, 2.6 and 2.12. For the second inequality in c), see, [6], Lemma 2a).

Throughout the correspondence, we will assume that the sensor observations $X_1, \cdots, X_n$ take values in a *finite* alphabet $\mathcal{X}$, i.e., $|\mathcal{X}| < \infty$. This assumption is not critical for the analysis of FB1, and can be relaxed—as will be outlined in Section V—to a boundedness condition on a suitable moment generating function. It is, however, needed for the analysis of FB2. The assumption that $\mathcal{X}$ is finite implies that the set $\mathcal{Q}$ of possible quantizers is also finite.

III. Two-Stage Compression with Feedback from Global Detector

In this section, we evaluate the optimal error exponent for FBI under the simplifying assumption that all n sensors are identical, i.e., use the same compression scheme. We shall see in Section V that this constraint is not restrictive as long as optimality is assessed in terms of the error exponent, which is the case considered here. The common compression scheme used by the n sensors *is* allowed to vary with the value of n. This should be borne in mind in what follows, because

the notation used for the local quantizers and feedback maps suppresses n.

In the first stage of the compression process, each sensor uses a quantizer $f : \mathcal{X} \to \mathcal{U}$ to produce a message

$$U_i = f(X_i),$$

which is then transmitted to the global detector. It is assumed that $|\mathcal{U}| = M_U$, where M_U is finite and independent of the number of sensors n.

The global detector uses a mapping $\phi : \mathcal{U}^n \to \mathcal{Z}_n$ to produce a feedback message

$$Z = \phi(\boldsymbol{U}),$$

which it then broadcasts to the sensors. For simplicity we let $\mathcal{Z}_n = \{0, \cdots, L_n - 1\}$, and assume that L_n grows with n at a subexponential rate, i.e.,

$$\lim_n \frac{1}{n} \log L_n \to 0. \tag{3.1}$$

To motivate this choice of constraint on the size of $\mathcal{Z}_n$, we argue as follows. Clearly (3.1) makes it impossible to encode the sequence $\boldsymbol{U}$ into Z, yet it allows Z to represent the type of $\boldsymbol{U}$ unambiguously: the number of possible types is at most $(n+1)^{|\mathcal{U}|}$ by Lemma 1a) and thus satisfies (3.1). Knowledge of the type of $\boldsymbol{U}$ would be as good as knowledge of $\boldsymbol{U}$ itself, if each sensor S_i were to perform an optimal hypothesis test based solely on its observation X_i and the first-stage outputs $U_1, \cdots, U_{i-1}, U_{i+1}, \cdots, U_n$ of the remaining sensors. Indeed, since the U_j's are i.i.d., a sufficient statistic for this test would consist of X_i and the type of $\boldsymbol{U}$. We conjecture that this sufficiency extends to the overall detection scheme FB1 under the assumption of identical sensors. That is, if the size of $\mathcal{Z}_n$ is unconstrained (without loss of generality, $L_n = |\mathcal{U}|^n$), then the optimal second-stage quantizer depends on $\boldsymbol{U}$ only through its type. This conjecture leads us to adopt (3.1) as a reasonable constraint on L_n.

In the second stage of the compression process, each sensor S_i quantizes X_i according to the received value of Z, and transmits the resulting message $V_i \in \mathcal{V}$ to the global detector. The size of the alphabet $\mathcal{V}$ equals $M_V < \infty$ and is independent of n. It is convenient to regard V_i as one of L_n messages $W_i^{(0)}, \cdots, W_i^{(L_n-1)}$, which are generated during the first stage by means of quantizers $g^{(0)}, \cdots, g^{(L_n-1)}$, respectively. Thus,

$$(0 \le l \le L_n - 1) \qquad W_i^{(l)} = g^{(l)}(X_i),$$

and upon receipt of Z,

$$V_i = W_i^{(Z)} = \sum_{l=0}^{L_n-1} W_i^{(l)} \boldsymbol{I}\{Z = l\},$$

where $\boldsymbol{I}$ denotes the indicator function.

The global detector declares H_0 to be true, if and only if the pair $(\boldsymbol{U}, \boldsymbol{V})$ lies in the *null acceptance region* $\mathcal{A}_n \subset \mathcal{U}^n \times \mathcal{V}^n$. In evaluating the probabilities p_n and q_n of the error events, it is helpful to note that the feedback mapping ϕ induces a partition of $\mathcal{U}^n$ into L_n sets $\mathcal{F}_n^{(0)}, \cdots, \mathcal{F}_n^{(L_n-1)}$, where

$$(0 \le l \le L_n - 1) \qquad \mathcal{F}_n^l = \phi^{-1}\{l\}.$$

Under each hypothesis, the probability of accepting H_0 is then given by

$$\begin{aligned}\Pr\{(\boldsymbol{U}, \boldsymbol{V}) \in \mathcal{A}_n\} &= \sum_{l=0}^{L_n-1} \Pr\left\{\left(\boldsymbol{U}, \boldsymbol{W}^{(l)}\right) \in \mathcal{A}_n, Z = l\right\} \\ &= \sum_{l=0}^{L_n-1} \Pr\left\{\left(\boldsymbol{U}, \boldsymbol{W}^{(l)}\right) \in \mathcal{A}_n, \boldsymbol{U} \in \mathcal{F}_n^{(l)}\right\} \\ &= \sum_{l=0}^{L_n-1} \Pr\left\{\left(\boldsymbol{U}, \boldsymbol{W}^{(l)}\right) \in \mathcal{A}_n \cap \left(\mathcal{F}_n^{(l)} \times \mathcal{V}^n\right)\right\}.\end{aligned} \tag{3.2}$$

The probability equals $1 - p_n$ (resp., q_n) when Pr is the measure corresponding to H_0 (resp., H_1).

An optimal detection scheme for n sensores is a quadruple $(f, \{g^{(l)}\}, \phi, \mathcal{A}_n)$ that minimizes q_n subject to $p_n \le \epsilon$. Our main result is that when the feedback message alphabet is constrained by (3.1), it is always possible to find an exponentially optimal sequence of schemes in which the feedback message is degenerate, i.e., Z is identically equal to 0. Thus feedback in FB1 does not result in an improvement on the error exponent.

Theorem 1: If the sequence $\{L_n\}$ satisfies (3.1), then the optimal error exponent of FB1 for testing at level $\epsilon \in (0, 1)$ is given by

$$\theta^{(1)}(M_U, M_V, \epsilon) = \max_{\substack{U = f(X_1), \\ |f(\mathcal{X})| = M_U,}} \max_{\substack{V = g(X_1) \\ |g(\mathcal{X})| = M_V}} D(P_{UV} \| Q_{UV}). \tag{3.3}$$

Furthermore, this exponent is achieved by a sequence of schemes employing no feedback.

Proof: To prove that the error exponent in (3.3) is achievable without feedback, we consider two mappings f and g that achieve the maximum in (3.3). We then let, for every n and every $i \le n$,

$$U_i = f(X_i), \qquad W_i^{(0)} = g(X_i),$$

and $\phi \equiv 0$. Thus, $V_i = g(X_i)$, and the pairs $(U_1, V_1), \cdots, (U_n, V_n)$ are i.i.d., each with distribution P_{UV} (under H_0) and Q_{UV} (under H_1). By Stein's lemma [3], the optimal error exponent for testing on the basis of this i.i.d. sample exists, and is equal to $D(P_{UV} \| Q_{UV})$. This concludes the proof of the achievability of $\theta^{(1)}(M_U, M_V, \epsilon)$.

For the converse results, we need to show that the optimal type II error probability of FB1 satisfies

$$\beta_n(\epsilon) \ge \exp[-n(D(P_{UV} \| Q_{UV}) + \zeta_n)], \tag{3.4}$$

where U (resp., V) is a function of X_1 with alphabet size M_U (resp., M_V), and $\zeta_n \to 0$. The quantizers generating U and V from X_1 may vary with n, in which case the distributions P_{UV} and Q_{UV} will also vary with n.

Let the optimal detection scheme for n sensors consist of quantizers f and $\{g^{(l)}\}$, feedback map ϕ and null acceptance region $\mathcal{A}_n$. From (3.2) and the type I error constraint

$$P\{(\boldsymbol{U}, \boldsymbol{V}) \in \mathcal{A}_n\} \ge 1 - \epsilon,$$

we deduce the existence of l^* such that

$$P\left\{\left(\boldsymbol{U}, \boldsymbol{W}^{(l^*)}\right) \in \mathcal{A}_n \cap \left(\mathcal{F}_n^{(l^*)} \times \mathcal{V}^n\right)\right\} \ge \frac{1 - \epsilon}{L_n}. \tag{3.5}$$

For simplicity, we let $W_i = W_i^{(l^*)}$ and $g = g^{(l^*)}$. Then, $U_i = f(X_i)$ and $W_i = g(X_i)$, hence the pairs $(U_1, W_1), \cdots, (U_n, W_n)$ are i.i.d. This allows us to rewrite (3.5) as

$$P_{UW}^n(\mathcal{A}_n') \ge 2 \exp[-n\delta_n], \tag{3.6}$$

where $\mathcal{A}_n' = \mathcal{A}_n \cap (\mathcal{F}_n^{(l^*)} \times \mathcal{V}^n)$ and by (3.1), $\delta_n \to 0$ as $n \to \infty$. We emphasize that the encoders f and g, as well as the resulting distributions P_{UW} and Q_{UW}, are implicitly dependent on n.

We, thus, suppress the presence of feedback by isolating a subset $\mathcal{A}_n'$ of the null acceptance region whose probability under the n-fold

i.i.d. extension of P_{UW} decays subexponentially in n. The converse result (3.4) will follow from (3.2) if we find $\zeta_n \to 0$ such that

$$Q_{UW}^n(\mathcal{A}_n') \geq \exp[-n(D(P_{UW}\|Q_{UW}) + \zeta_n)]. \tag{3.7}$$

Since $\mathcal{X}$ is finite, for every n there are at most finitely many choices for the pair (P_{UW}, Q_{UW}). Thus, we obtain a finite partition of the positive integers $\boldsymbol{N}$ into sets $\boldsymbol{N}_i$ with the property that (P_{UW}, Q_{UW}) is fixed for $n \in \boldsymbol{N}_i$. It clearly suffices to establish (3.7) for n ranging over each separate index set $\boldsymbol{N}_i$ that is infinite. The case $D(P_{UW}\|Q_{UW}) = \infty$ is trivial, hence we will assume that $D(P_{UW}\|Q_{UW}) < \infty$, or equivalently, $P_{UW} \ll Q_{UW}$.

In what follows, we let $n \in \boldsymbol{N}_i$ for some i, and use "$\to$" to denote convergence as n approaches infinity along $\boldsymbol{N}_i$. From Lemma 1c), the set of (P_{UW}, η)-divergent sequence pairs $(\boldsymbol{u}, \boldsymbol{v})$ in $\mathcal{U}^n \times \mathcal{V}^n$ satisfies

$$P_{UW}^n(S_{UW,\eta}^n) \geq 1 - (n+1)^{M_U M_V} \exp[-n\eta].$$

If we choose

$$\eta = \eta_n = 2\delta_n + \frac{M_U M_V}{n} \log(n+1),$$

then $\eta_n \to 0$ and

$$\begin{aligned} P_{UW}^n(\mathcal{A}_n' \cap S_{UW,\eta}^n) &\geq P_{UW}^n(\mathcal{A}_n') + P_{UW}^n(S_{UW,\eta}^n) - 1 \\ &\geq 2\exp[-n\delta_n] - \exp[-2n\delta_n] \\ &\geq \exp[-n\delta_n] \cdot P_{UW}^n(S_{UW,\eta}^n). \end{aligned}$$

By the definition of $S_{UW,\eta}$, this implies that

$$P_{UW}^n\left(\mathcal{A}_n' \cap \hat{T}_{UW}^n\right) \geq \exp[-n\delta_n] \cdot P_{UW}^n\left(\hat{T}_{UW}^n\right). \tag{3.8}$$

for some type $\hat{P}_{UW} \in \mathcal{P}_n(\mathcal{U} \times \mathcal{V})$ that satisfies

$$D\left(\hat{P}_{UW}\|P_{UW}\right) \leq \eta_n$$

and clearly also

$$\hat{P}_{UW} \ll P_{UW} \ll Q_{UW}.$$

As sequences of the same type are equiprobable under any i.i.d. measure, (3.8) implies that

$$\frac{\left|\mathcal{A}_n' \cap \hat{T}_{UW}^n\right|}{\left|\hat{T}_{UW}^n\right|} \geq \exp[-n\delta_n].$$

and thus also

$$Q_{UW}^n\left(\mathcal{A}_n' \cap \hat{T}_{UW}^n\right) \geq \exp[-n\delta_n] \cdot Q_{UW}^n\left(\hat{T}_{UW}^n\right).$$

Lemma 1b) then gives

$$\begin{aligned} Q_{UW}^n\left(\mathcal{A}_n' \cap \hat{T}_{UW}^n\right) \geq &(n+1)^{-M_U M_V} \\ &\cdot \exp\left[-n\left(D\left(\hat{P}_{UW}\|Q_{UW}\right) + \delta_n\right)\right]. \end{aligned}$$

To establish (3.7), it is sufficient to show that if $D\left(\hat{P}_{UW}\|P_{UW}\right) \leq \eta_n \to 0$, then the difference $D\left(\hat{P}_{UW}\|Q_{UW}\right) - D(P_{UW}\|Q_{UW})$ goes to zero as n tends to infinity. This is easily accomplished using the inequality

$$\begin{aligned} \left[D\left(\hat{P}_{UW}\|P_{UW}\right)\right]^{1/2} \geq &\frac{1}{2\ln 2} \\ &\cdot \sum_{(u,v)\in\mathcal{U}\times\mathcal{V}} \left|\hat{P}_{UW}(u,v) - P_{UW}(u,v)\right| \end{aligned}$$

(cf. [4, p. 58]) together with the fact that for *fixed* Q_{UW}, the divergence $D(\hat{P}_{UW}\|Q_{UW})$ is continuous in the space of distributions $\hat{P}_{UW}$ such that $\hat{P}_{UW} \ll Q_{UW}$. □

Remark: In the characterization (3.3) of the error exponent $\theta^{(1)}(M_U, M_V, \epsilon)$, it is clearly legitimate to replace the pair (f, g) by a single quantizer with $M_U \cdot M_V$ output levels.

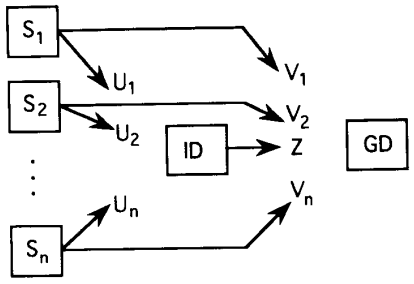

Fig. 3. Schematic representation of FF2.

IV. Two-Stage Compression with Feedback from Intermediate Detector

We now turn to the asymptotic analysis of FB2, using a similar formulation. After observing X_i, sensor S_i generates U_i and $W_i^{(0)}, \cdots, W_i^{(L_n-1)}$, defined as in previous section. It then transmits U_i to the intermediate detector. The intermediate detector generates a feedback message $Z = \phi(\boldsymbol{U})$ and communicates it to the sensors. Each sensor S_i then transmits $V_i = W_i^{(Z)}$ directly to the global detector. The alphabet sizes of U_i, V_i and Z are constrained as previously, and all n sensors employ a common compression scheme which is allowed to vary with n.

It is important to realize that the only information available to the global detector is the vector $\boldsymbol{V}$, and thus the null acceptance region $\mathcal{A}_n$ is a subset of $\mathcal{V}^n$ (*not* of $\mathcal{U}^n \times \mathcal{V}^n$). Using the partition of $\mathcal{U}_n$ into sets $\mathcal{F}_n^{(l)}$ introduced in the previous section, we now have

$$\begin{aligned} \Pr\{\boldsymbol{V} \in \mathcal{A}_n\} &= \sum_{l=0}^{L_n-1} \Pr\left\{Z = l, \boldsymbol{W}^{(l)} \in \mathcal{A}_n\right\} \\ &= \sum_{l=0}^{L_n-1} \Pr\left\{\boldsymbol{U} \in \mathcal{F}_n^{(l)}, \boldsymbol{W}^{(l)} \in \mathcal{A}_n\right\}. \end{aligned} \tag{4.1}$$

The main result of our analysis is that the asymptotic performance of FB2 is the same as that of a two-stage feedforward scheme in which the output Z of the intermediate detector is sent forward to the global detector, rather than back to the sensors. In this scheme (denoted by FF2 and shown in Fig. 3), the messages $U_i = f(X_i)$ and $V_i = g(X_i)$ are generated simultaneously. Thereafter:

1) the U_i's are sent to the intermediate detector, which relays a message Z to the global detector (*only*);
2) the V_i's are transmitted directly to the global detector.

The exponential equivalence of the two schemes is not immediately obvious, nor is a direct comparison of the two feasible: in the feedforward scheme, the global detector receives an extra message Z, while in the feedback scheme, each sensor receives extra information (again in the form of Z) about the observations of the remaining sensors. Yet, as we shall see in the discussion that follows, these differences have an asymptotically vanishing effect.

The feedforward scheme is closely related to—and is, in a certain sense, the space time dual of—the distributed detection schemes considered in [5] and [7]. Results from the latter body of work can readily be applied in the analysis of FF2. Thus, if the quantizer pair (f, g) is fixed and such that $Q_{UV} > 0$, then the best feedforward error exponent attainable by varying the global decision rule is given in Theorem 2 of [7]. It would be a simple matter to obtain the absolute optimum by varying (f, g) over the finite domain $\mathcal{Q}^2$, were it not for the difficulty that some choices of (f, g) yield $Q_{UV}(u^*, v^*) = 0$ for one or more pairs (u^*, v^*) in $\mathcal{U} \times \mathcal{V}$, in which case the error exponent is not known. Fortunately, this obstacle is not encountered

with binary quantizers ($M_U = M_V = 2$), and thus, we will restrict our attention to this case. The following lemma, adapted from [7], will be needed in the converse part of Theorem 2.

Lemma 2: Let $\mathcal{U}$ and $\mathcal{V}$ be binary sets, and P_{UV}, Q_{UV} be fixed distributions on $\mathcal{U} \times \mathcal{V}$ such that $D(P_{UV}\|Q_{UV}) < \infty$. If $B_n \subset \mathcal{U}^n$ and $C_n \subset \mathcal{V}^n$ satisfy

$$\lim_n \frac{1}{n} \log P_{UV}^n(B_n \times C_n) = 0. \tag{4.2}$$

then

$$\liminf_n \frac{1}{n} \log Q_{UV}^n(B_n \times C_n) \geq - \min_{\substack{\tilde{P}_{UV} \\ \tilde{P}_U = P_U, \tilde{P}_V = P_V}} D\left(\tilde{P}_{UV}\|Q_{UV}\right). \tag{4.3}$$

Proof: If $Q_{UV}(u, v) > 0$ for all $(u, v) \in \mathcal{U} \times \mathcal{V}$, then the statement of the lemma is a special case of Theorem 3 in [7].

If $Q_{UV}(u^*, v^*) = 0$ for some pair $(u^*, v^*) \in \mathcal{U} \times \mathcal{V}$, then we encounter the difficulty mentioned earlier, i.e., that the results in [7] do not apply. We note, however, that the assumption $D(P_{UV}\|Q_{UV}) < \infty$ also forces $P_{UV}(u^*, v^*)$ to be zero. It is then straightforward to verify that for $\mathcal{U}$ and $\mathcal{V}$ binary, there is only one distribution $\tilde{P}_{UV}$ on $\mathcal{U} \times \mathcal{V}$ such that $\tilde{P}_U = P_U$ and $\tilde{P}_V = P_V$, namely $\tilde{P}_{UV} = P_{UV}$. Thus,

$$\min_{\substack{\tilde{P}_{UV} \\ \tilde{P}_U = P_U, \tilde{P}_V = P_V}} D\left(\tilde{P}_{UV}\|Q_{UV}\right) = D(P_{UV}\|Q_{UV}),$$

and (4.3) follows from (4.2) in the same way as (3.7) was obtained from (3.6) in the proof of Theorem 1. □

We are now ready to state our results on the error exponent of FB2.

Theorem 2: If $M_U = M_V = 2$ and $\{L_n\}$ satisfies (3.1), then the optimal error exponent of FB2 for testing at level $\epsilon \in (0, 1)$ is given by

$$\theta^{(2)}(2, 2, \epsilon) = \max_{\substack{U = f(X_1), V = g(X_1) \\ |f(\mathcal{X})| = 2, |g(\mathcal{X})| = 2}} d(P_U, P_V\|Q_{UV}), \tag{4.4}$$

where

$$d(P_U, P_V\|Q_{UV}) = \min_{\substack{\tilde{P}_{UV} \\ \tilde{P}_U = P_U, \tilde{P}_V = P_V}} D\left(\tilde{P}\|\tilde{Q}_{UV}\right).$$

Furthermore, this exponent is achieved by a sequence of schemes in which the feedback message is binary.

Proof: To establish that the error exponent of (4.4) is achievable by binary feedback, we consider once again binary quantizers f and g that achieve the maximum in (4.4), and let, for every n,

$$U_i = f(X_i), \qquad W_i = g(X_i).$$

Upon receipt of the vector $\boldsymbol{U}$, the intermediate detector conveys to the sensors (via Z) whether or not $\boldsymbol{U}$ is (P_U, η)-*typical*. In other words,

$$Z = \phi(\boldsymbol{U}) = \begin{cases} 0, & \text{if } \boldsymbol{U} \in T_{U,\eta}^n, \\ 1, & \text{otherwise.} \end{cases}$$

(ϕ is also an exponentially optimal test for H_0 versus H_1 on the basis of $\boldsymbol{U}$ alone.) Each sensor S_i then sets

$$V_i = \begin{cases} W_i, & \text{if } Z = 0, \\ v^*, & \text{if } Z = 1, \end{cases}$$

where v^* is an element of $\mathcal{V}$ which is fixed across the sensors, and which will be specified soon. In the notation developed earlier, $W_i^{(0)} = W_i$ and $W_i^{(1)} = v^*$ with probability 1.

After receiving $\boldsymbol{V}$, the global detector declares that H_0 is true, if and only if the received sequence $\boldsymbol{V}$ is η-typical with respect to P_W. Thus, by (4.1), we have

$$\Pr\{\boldsymbol{V} \in \mathcal{A}_n\} = \Pr\{\boldsymbol{U} \in T_{U,\eta}^n, \boldsymbol{W} \in T_{W,\eta}^n\} + \Pr\{\boldsymbol{U} \notin T_{U,\eta}^n, v^* \cdot \mathbf{1} \in T_{W,\eta}^n\}.$$

The second term can be easily made equal to zero by choosing v^* such that $v^* \cdot \mathbf{1} \notin T_{W,\eta}$. This is possible for sufficiently small η, since at most one of the two constant $\boldsymbol{v}$-sequences can lie in $T_{W,\eta}^n$. Then

$$\Pr\{\boldsymbol{V} \in \mathcal{A}_n\} = \Pr\{\boldsymbol{U} \in T_{U,\eta}^n, \boldsymbol{W} \in T_{W,\eta}^n\}.$$

The pairs (U_i, W_i) are now i.i.d., and the probability of $\mathcal{A}_n$ under both hypotheses can be evaluated in a standard fashion (for details, see [5] or [7]). Briefly, if $\eta = n^{-1/3}$, then by Lemma 1c), both $P_U^n(T_{U,\eta}^n)$ and $P_W^n(T_{W,\eta}^n)$ approach 1 as n tends to infinity. Thus, $P_{UW}^n(T_{U,\eta}^n \times T_{W,\eta}^n) = 1 - p_n$ also approaches unity, and the type I error constraint is met for any $\varepsilon \in (0, 1)$. For the evaluation of $q_n = Q_{UW}^n(T_{U,\eta}^n \times T_{W,\eta}^n)$, the set $T_{U,\eta} \times T_{W,\eta}$ is expressed as a union of sets $\hat{T}_{UV}$, where $\hat{P}_U \approx P_U$ and $\hat{P}_V \approx P_V$. Application of Lemma 1a), b) together with a continuity argument then yields

$$-\lim_n \frac{1}{n} \log q_n = \min_{\substack{\hat{P}_{UV} \\ \hat{P}_U = P_U, \hat{P}_V = P_V}} D\left(\hat{P}_{UV}\|Q_{UV}\right) = d(P_U, P_V\|Q_{UV}).$$

This completes the proof of the achievability of the error exponent. Note that in the special case where P_X is not absolutely continuous with respect to Q_X, i.e., there exists $A \subset \mathcal{X}$ with $P_X(A) > 0$ and $Q_X(A) = 0$, the maximum in (4.4) equals infinity and can be achieved without feedback. This is easily verified by letting f be constant and g be such that it partitions $\mathcal{X}$ into A and A^c.

The proof of the converse result proceeds as for Theorem 1. Given an optimal acceptance region $\mathcal{A}_n$, we deduce from (4.1) and the type I error constraint the existence of l^* such that

$$P\left\{\boldsymbol{U} \in \mathcal{F}_n^{(l^*)}, \boldsymbol{W}^{(l^*)} \in \mathcal{A}_n\right\} \geq \frac{1 - \epsilon}{L_n}. \tag{4.5}$$

Letting $W_i = W_i^{(l^*)}$, $B_n = \mathcal{F}_n^{(l^*)}$ and $C_n = \mathcal{A}_n$, we deduce from (3.1) and (4.5) that

$$\lim_n \frac{1}{n} \log P_{UW}^n(B_n \times C_n) = 0.$$

By the remark at the end of the previous paragraph, we only need to consider the case $P_X \ll Q_X$, and thus we may assume $D(P_{UW}\|Q_{UW}) \leq D(P_X\|Q_X) < \infty$. Once again, the distribution pair (P_{UW}, Q_{UW}) will depend on n. As in the proof of Theorem 1, we let n approach infinity over each finite index set $\boldsymbol{N}_i$ defined by the property that (P_{UW}, Q_{UW}) is fixed over $\boldsymbol{N}_i$. We then apply Lemma 2 to obtain

$$\lim_{n \in \boldsymbol{N}_i} \inf \frac{1}{n} \log Q_{UW}^n(B_n \times C_n) \geq -d(P_U, P_W\|Q_{UW}),$$

and consequently,

$$\liminf_{n \to \boldsymbol{N}_i} \frac{1}{n} \log Q_{UW}^n(B_n \times C_n) \geq - \max_{\substack{U = f(X_1), W = g(X_1) \\ |f(\mathcal{X})| = 2, |g(\mathcal{X})| = 2}} d(P_U, P_W\|Q_{UW}).$$ □

In general, $D(P_{UV}\|Q_{UV}) > D(P_V\|Q_V)$, which implies that in most cases of interest, $d(P_U, P_V\|Q_{UV}) > D(P_V\|Q_V)$. Taking maxima on both sides of the inequality, we see that the error exponent of FB2 is greater than that of a single-stage feedforward scheme (denoted by FF1 and shown in Fig. 4) in which each sensor transmits

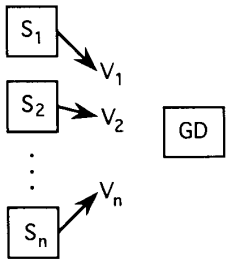

Fig. 4. Single-stage feedforward scheme FF1.

a single binary message V_i to the global detector. It should be emphasized that the same conclusion can be drawn for nonbinary messages ($(M_U, M_V) \neq (2,2)$) using the lower bound established in the direct (achievability) part of the previous proof:

$$\theta^{(2)}(M_U, M_V, \epsilon) \geq \max_{\substack{U=f(X_1), V=g(X_1) \\ |f(\mathcal{X})|=M_U, |g(\mathcal{X})|=M_V}} d(P_{UV}, P_V \| Q_{UV}).$$

Whether the opposite inequality holds, however, is not known for $(M_U, M_V) \neq (2,2)$.

The superior performance of FB2 compared to FF1 can be also explained heuristically. In FF1, each sensor transmits $V_i = W_i$, yet the global detector only needs to know whether $\boldsymbol{W}$ is (P_W, η)-typical or not. Thus in essence, the available sensor-to-global detector bandwidth is underutilized. In the exponentially optimal FB2 constructed in the previous proof, part of the bandwidth is cooperatively used by the sensors to encode the feedback message Z. Thus in the end, the global detector learns quite a bit more from $\boldsymbol{V}$, namely whether or not *both* $\boldsymbol{U}$ and $\boldsymbol{W}$ are η-typical under the null hypothesis.

As for the relative merits of FB2 and FF2, we conclude from the previous theorem and from Theorem 2 in [7] that both schemes have the same optimal error exponent, and are thus exponentially equivalent. One may wonder what happens if the two schemes are combined, i.e., Z is sent *both* forward (to the global detector) and back (to the sensors). In that case, the decision of the global detector is based on $(Z, \boldsymbol{V})$, and thus the null acceptance region $\mathcal{A}_n$ is a subset of $\mathcal{Z}_n \times \mathcal{V}^n$. The probability of deciding in favor of H_0 is then given by

$$\begin{aligned} \Pr\{(Z, \boldsymbol{V}) \in \mathcal{A}_n\} &= \sum_{l=0}^{L_n - 1} \Pr\left\{Z = l, \left(Z, \boldsymbol{W}^{(l)}\right) \in \mathcal{A}_n\right\} \\ &= \sum_{l=0}^{L_n - 1} \Pr\left\{\boldsymbol{U} \in \mathcal{F}_n^{(l)}, \boldsymbol{W}^{(l)} \in \mathcal{C}_n^{(l)}\right\}. \quad (4.6) \end{aligned}$$

where $\mathcal{C}_n^{(l)} = \mathcal{A}_n \cap (\{l\} \times \mathcal{V})^n$. Using (4.6) instead of (4.1) in the proof of the converse part of Theorem 2, we conclude by an identical argument that the error exponent of this scheme cannot be greater than $\theta^{(2)}(2,2,\epsilon)$. Thus the combination of FB2 and FF2 is exponentially equivalent to either of the two individual schemes.

Finally, we compare the performance of the two feedback schemes FB1 and FB2 (without reference to feedfoward schemes) in the case where the message alphabets $\mathcal{U}$ and $\mathcal{V}$ are binary. Upon minimizing $D(\tilde{P}_{UV} \| Q_{UV})$ under the constraints $\tilde{P}_U = P_U$ and $\tilde{P}_V = P_V$ (in the binary case this involves only one free parameter), we obtain $d(P_U, P_V \| Q_{UV}) \leq D(P_{UV} \| Q_{UV})$. Taking a suitable maximum on both sides of the inequality, we conclude that $\theta^{(2)}(2,2,\epsilon) \leq \theta^{(1)}(2,2,\epsilon)$, i.e., FB1 outperforms FB2. This is not surprising, since FB2 can also be realized by merging the intermediate and global detector, and requiring that the latter discard all information about $\boldsymbol{U}$ after transmitting the feedback message Z. In other words, FB2 is just a special case of FB1 in which the global decision is based only on the second-stage transmission $\boldsymbol{V}$.

V. Extensions

1) Nonidentical Sensors: Our results for FB1 and FB2 were obtained under the assumption that the sensors are identical, i.e., all use the same quantizer in a given stage. A natural question to ask is whether nonidentical sensors can yield an improvement in the error exponent. The answer is known to be negative (as long as $\mathcal{X}$ is finite) for Neyman–Pearson and Bayesian distributed detection problems using the single-stage feedforward scheme FF1 [10].

As it turns out, the same answer can be obtained for FB1. We give a sketch of the converse argument. Since the sensors are nonidentical, the independent pairs (U_i, W_i) in (3.5) have different distributions, and thus the results of Lemma 1 are not directly applicable. However, by considering

$$S_{\boldsymbol{UW},\eta} = \left\{(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{U}^n \times \mathcal{V}^n : \left|\log \frac{P_{\boldsymbol{UW}}(\boldsymbol{u}, \boldsymbol{v})}{Q_{\boldsymbol{UW}}(\boldsymbol{u}, \boldsymbol{v})} - \sum_{i=1}^{n} D(P_{U_i W_i} \| Q_{U_i W_i})\right| \leq n\eta_n\right\},$$

(instead of $S^n_{U,W,\eta}$) and using moment generating functions, one can show that

$$1 - P_{\boldsymbol{UW}}\left(S_{\boldsymbol{UW},\eta}\right) \leq \exp[-2n\delta_n] \quad (5.1)$$

for suitable $\eta_n = \eta_n(\delta_n) \to 0$. An inequality similar to (3.8) follows, and the converse result is readily obtained from the definition of $S_{\boldsymbol{UW},\eta}$.

The exponential optimality of identical sensors in FB2 is a much more difficult problem and remains open. It should be added that once nonidentical sensors are allowed in either FB1 or FB2, our conjecture as to the sufficiency of feedback messages with alphabet sizes constrained by (3.1) is no longer defensible.

2) Infinite Observation Spaces: In this case, the set of possible quantizer pairs (f, g) is also infinite, and the same holds for the set of distribution pairs (P_{UW}, Q_{UW}) encountered in the converse arguments for Theormes 1 and 2. As a result, these arguments are not applicable to this case.

An alternative approach for FB1 is based on moment generating functions as outlined in a). One can establish the exponential optimality of identical sensors and obtain the result of Theorem 1 (with supremum replacing the maximum in (3.3)) under the assumption that, in a fixed neighborhood of the origin, the second derivative of the moment generating function (under P) of the log-likelihood ratio of (U, V) is uniformly bounded for all choices of $U = f(X)$ and $V = g(X)$. An analogous condition was used in [10] for Neyman–Pearson detection employing FF1. This method works well for FB1, but is not suited for FB2, whose optimum performance in the case of infinite $(\mathcal{X}, \mathcal{B})$ remains open.

3) Randomization: When randomized quantizers are allowed, the optimal error exponents for FB1 and FB2 are expressed in terms of channels (conditional distributions) $\Delta_{U,V|X}$ instead of deterministic mappings f and g. By invoking convexity properties of the divergence functional, it is straightforward to show that the best channels are, in fact, deterministic. Thus, randomization is not useful asymptotically.

4) Bayesian Detection: Although our emphasis has been on Neyman–Pearson (classical) detection, we believe that the bulk of our results can be extended to the Bayesian framework to yield anologous conclusions. The analysis should be fairly straightforward for FB1, but rather less so for FB2 because of unresolved issues related to the Bayesian error exponent of FF2 [6].

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Avoiding Decoder Malfunction in the Peterson–Gorenstein–Zierler Decoder

Arne Dür, *Member, IEEE*

***Abstract*— BCH decoders based on the Peterson–Gorenstein–Zierler algorithm can malfunction and produce output vectors that are no codewords. To avoid this malfunction a simple additional check is proposed that can be performed before the computation of the error locations. The additional check consists of the minimum number of algebraic equations in the syndrome components that are necessary over a general field to obtain a bounded-distance decoder.**

***Index Terms*— BCH coding, Reed–Solomon coding, Peterson–Gorenstein–Zierler decoder, bounded-distance decoding, decoder malfunction.**

I. INTRODUCTION

Let C be a linear code with minimum distance d^*, and let t be a nonnegative integer such that $2t < d^*$. The decoding sphere of radius t around a codeword c is the set of all vectors at Hamming distance at most t from c. As $2t < d^*$, the decoding spheres around the codewords of C are disjoint but, in general, do not cover the whole space. An t-error-correcting bounded-distance decoder for C outputs the codeword c if the received vector lies in the decoding sphere of radius t around c, and announces a decoding failure if no such codeword exists.

In a recent correspondence [1], Sarwate and Morrison have pointed out that the Peterson–Gorenstein–Zierler (PGZ) decoder is no bounded-distance decoder, and they have described a situation where the output of the PGZ decoder is not a codeword but the decoder does not announce a decoding failure. To avoid malfunction the authors recommend to check whether the decoder output is a codeword which, for a BCH code of designed distance d, in general requires the computation of $d-1$ syndrome values. In Section II, we show that this malfunction can be detected already at the beginning of the PGZ algorithm by checking a few syndrome determinants, and we prove that no other kind of malfunction is possible. This leads to an extended version of the PGZ decoder that is a bounded-distance decoder and is attractive for small designed distances because of its explicit formulas. Answering another question posed in [1] we remark that the Berlekamp–Massey decoder and the variation of the Berlekamp–Massey decoder proposed in [2] are bounded-distance decoders, and we refer to [2] for a proof and a discussion of the decoding problem in terms of invariant theory of binary forms.

II. THE EXTENDED PGZ ALGORITHM

Let C be a BCH code over $K = GF(q)$ of length n and designed distance d where $2 \le d \le n$. Then the generator polynomial of C has $\zeta^b, \zeta^{b+1}, \cdots, \zeta^{b+d-2}$ as roots where ζ is a primitive nth root of unity in some extension field of K. By the BCH bound, the minimum distance of C is at least d. Let $t = \lfloor (d-1)/2 \rfloor$. Then a t-error-correcting bounded-distance decoder can based on the following steps:

Let $(R_0, R_1, \cdots, R_{n-1})$ be the received vector, and let $R(x) = R_0 + R_1 x + \cdots + R_{n-1}x^{n-1}$ denote the corresponding polynomial.

Step 1) Compute the syndrome sequence $S_0, S_1, \cdots, S_{d-2}$ where $S_i = R(\zeta^{b+i})$, $0 \le i \le d-2$.

Step 2) Compute the coefficients $\Lambda_1, \Lambda_2, \cdots, \Lambda_\nu, \nu \le t$, of the error locator polynomial $\Lambda(x) = 1 + \Lambda_1 x + \cdots + \Lambda_\nu x^\nu$.

Step 3) Determine the reciprocals of the roots of $\Lambda(x)$ to get the estimated error locations $X_1, X_2, \cdots, X_\nu$.

Step 4) Determine the estimated error values $Y_1, Y_2, \cdots, Y_\nu$.

Step 5) Correct the errors in the received vector: For $i = 1, 2, \cdots, \nu$ subtract Y_i from R_j where j is determined by $\zeta^j = X_i$.

A decoding failure is declared whenever one of the following events occurs:

1) the decoder is unable to compute the coefficients of the error locator polynomial in Step 2);
2) the error locations determined in Step 3) are not ν distinct nth roots of unity;
3) the decoder is unable to compute the error values in Step 4), or some of the error values are either zero or do not belong to the symbol field.

Note that the steps 2) and 4) are still to be specified in detail, and care has to be taken in order to obtain a bounded-distance decoder. In [1], the authors show that, contrary to the impression from the literature, the conventional PGZ algorithm does not yield a bounded-distance decoder whereas the Sugiyama–Kasahara–Hirasawa–Namekawa Euclidean algorithm does

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The author is with the Institut für Mathematik, Universität Innsbruck, Technikerstraße 25, 6020 Innsbruck, Austria.

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